Journal of Modern Mathematics and Statistics 1 (1-4): 39-41, 2007
ISSN: 1994-5388

Generalized Forms of Riemann-Stieltjes Theorm

Ali Parsian
Department of Mathematics, University of Tafresh, Tafresh, Iran

Abstract: If α be a monotically increasing function on [a, b] and $\alpha(a)$, $\alpha(b)$ are Finite and f be a Bounded function defined on [a, b] such that $f \in R(\alpha)$, $m \le f \le M$, φ is continuous on [m, M] And $h(x) = \varphi \circ f(x)$ on [a, b], then $h \in R(\alpha)$. In this study, we generalize the above theorem in a suitable forms for both integrand and integrator

Key words: Bounded variation functions, riemann-stieltjes integral, uniformly continuous functions, 2000 Math, Subject classification, 26A45, 28B05, 46T20

INTRODUCTION

Let φ be a n variable continuous function and f_i $(1 \le i \le n)$ are all Riemann- Stieltjes integrable Functions with respect to a monotically increasing function α. First, as a generalization of Rudin (1976) we prove $h = \varphi \circ (f_1,...,f_n)$. In the next, we generalize the theorem for all n variable continuous functions φ and n bounded variation functions f_i, $(1 \le i \le n)$. At the end we show the essentiality of the bounded varational property of f_i in the above theorem by a counterexample.

GENERALIZED THEOREMS

In this study, we prove the generalized existence theorems on Riemann-Stieltjes integrability of functions.

Theorem: Suppose α is a monotically increasing function on [a, b] and $\alpha(\alpha)$, $\alpha(b)$ are finite and $f_i \in R(\alpha)$ on $[\alpha, b]$, $m_i < f_i < M_i$ for $1 \le i \le n$, φ is continuous on the compact subset of R^n, the n-cell $C = \prod_{i=1}^{n} [m_i, M_i]$ and $h = \varphi \circ (f_1,...f_n)$ On [a, b].Then $h \in R(\alpha)$ on [a, b].

Proof: Choose $\varepsilon > 0$. Since φ is uniformly continuous on C, there exists $\delta > 0$ such that $\delta < \varepsilon$ and $|\varphi(t)-\varphi(s)| < \varepsilon$ if $\|s-t\| < \delta$ for $s, t \in C$ (Dieudonne, 1960). Since $f_i \in R(\alpha)$, by Riemann criterion of integrability there exists a partition P_i of $[\alpha, b]$ such that the inequality $U(P_i, f_i, \alpha)-L(P_i, f_i, \alpha) < \delta^2$ is hold for $1 \le i \le n$.
Let

$$\cup P_i = P = \{a = x_0, x_1, \ldots, x_{m-1}, x_m = b\}$$

The $P \supseteq P_i$ and $U(P, f_i, \alpha)-L(P, f_i, \alpha) < \delta^2$ for $1 \le i \le n$. Let $1 \le j \le m$ and define:

$$M_{ij} = \sup\{f_i(x); x \in [x_{j-1}, x_j]\}$$
$$m_{ij} = \inf\{f_i(x); x \in [x_{j-1}, x_j]\}$$
$$M_j^* = \sup\{\varphi \circ (f_1, \ldots, f_n)(x); x \in [x_{j-1}, x_j]\}$$
$$m_j^* = \inf\{\varphi \circ (f_1, \ldots, f_n)(x); x \in [x_{j-1}, x_j]\}$$
$$m_j = (m_{1j}, \ldots, m_{nj})$$
$$M_j = (M_{1j}, \ldots, M_{nj})$$
$$A = \{j : 1 \le j \le m, \|M_j - m_j\| < \delta\}$$
$$B = \{j : 1 \le j \le m, \|M_j - m_j\| \ge \delta\}$$

Then $A \cap B = \phi$, $A \cup B = \{1,...,m\}$. Let $j \in A$ and y, $z \in [x_{j-1}, x_j]$, So $m_{ij} \le f_i(y) \le M_{ij}$ and $m_{ij} \le f_i(y) \le M_{ij}$, for $1 \le i \le n$ Moreover ,if

$$Y_j = (f_1(y), \ldots, f_n(y))$$
$$Z_j = (f_1(z), \ldots, f_n(z))$$

then:

$$\|Y_j - Z_j\| \le$$
$$\le \sqrt{(M_{1j} - m_{1j})^2 + \ldots + (M_{nj} - m_{nj})^2} =$$
$$= \|M_j - m_j\| < \delta$$

Thus $|\varphi(Y_j)-\varphi(Z_j)| < \varepsilon$ and $M^*_j - m^*_j \le \varepsilon$.

Let $j = B$, since the sum on n components is not less than then δ^2, there exists $1 \le k = k(j) < n$. Such that, so

$$M_{kj} - m_{kj} \ge \frac{\delta}{\sqrt{n}}$$

We have:

$$\frac{\delta}{\sqrt{n}}\sum_{j\in B}\Delta\alpha_j \le \sum_{j\in B}(M_{kj}-m_{kj})\Delta\alpha_j \le$$
$$\le \sum_{j\in B}(M_{1j}-m_{1j})\Delta\alpha_j + \ldots$$
$$\ldots + \sum_{j\in B}(M_{nj}-m_{nj})\Delta\alpha_j \le$$
$$\le \sum_{i=1}^{n}(U(P,f_i,\alpha)-L(P,f_i,\alpha)) \le n\delta^2$$

Therefore

$$\sum_{j\in B}\alpha_j \le n\sqrt{n}\delta.$$

Moreover M^*_j-m^*_j $\le$ 2M for 1 $\le$ j $\le$ m where M = sub $\{|\varphi(x)|; x \in [\alpha, b]\}$. It follows that:

$$= \sum_{j=1}^{m}(M^*_j - m^*_j)\Delta\alpha_j =$$
$$= \sum_{j=1}^{m}(M^*_j - m^*_j)\Delta\alpha_j =$$
$$= \sum_{j\in A}(M^*_j - m^*_j)\Delta\alpha_j + \sum_{j\in B}(M^*_j - m^*_j)\Delta\alpha_j =$$
$$\le \varepsilon\sum_{j\in A}\Delta\alpha_j + 2M\sum_{j\in B}\Delta\alpha_j \le$$
$$\le \varepsilon(\alpha(b)-\alpha(a)+2Mn\sqrt{n})$$

So h $\in$ R(α) on [α, b].

Theorem: Suppose α is a bounded variation function on [α, b], $f_i \in R(\alpha)$ on [α, b], $m_i \le f_i \le M_i$, for 1 $\le$ i $\le$ n and the n variable function φ is continuous on the n-cell

$$C = \prod\nolimits_{i=1}^{n}[m_i, M_i]$$

Let h = $\varphi\circ$ $(f_i,\ldots,f_n)$ then h $\in$ R(α) on [α, b].

Proof: Let V(x) = v_α (α, x), $\alpha \le x \le b$ is the total variation function of α on [α, b]. The functions V and V-α are monotically increasing functions on [α, b], $f_i \in$ R(V-α) and $f_i \in$ R(V-α) on [α, b] for 1 $\le$ i $\le$ n (Royden, 1968). Now the above theorem implies that h $\in$ R(V) and h $\in$ R(V-α) on [α, b]. Therefore, h $\in$ R(α) on [α, b].

Theorem: Let α_i, (1 $\le$ i $\le$ n) are bounded functions on [α, b], $m_i \le \alpha_i \le M_i$ and f $\in$ R(α) on [α, b] for a bounded variation function f and φ is continuous on

$$C = \prod\nolimits_{i=1}^{n}[m_i, M_i]$$

then f $\in$ R($\varphi\circ$ α_i) on [α, b].

Proof: The theorem of integration by parts (Graves, 1956) implies that $\alpha_i \in$ R(f) on [α, b]. So $\varphi\circ$ $\alpha_i \in$ R(f) by the first theorem and f $\in$ R($\varphi\circ$ α_i) by integration by parts theorem once more again.

Now we proceed to show that the bounded variational property of α is essentially in the second theorem.

Theorem: There exists a non bounded variation function α and continuous functions f and φ all defined on [0, 1] such that f $\in$ R(α) on [0, 1] but $\varphi\circ$ f $\notin$ R(α) on the same interval.

Proof: Let [x] denotes the greatest integer part of x and

$$\alpha(x) = \frac{1}{x} - [\frac{1}{x}]$$

for x $\in$(0, 1), α(0) = 0 and f(x) = x. Since the set of points of [0, 1] at which α is discontinuous is countable, f $\in$ R(α) by integration by parts theorem. Define

$$\varphi(x) = x\sin\frac{\pi}{x}$$

for x $\in$ (0, 1) and φ(0) = 0, then $\varphi\circ$ f = φ for x $\in$ [0, 1]. Suppose $\varphi \in$ R(α) and

$$\int_0^1 \varphi.d\alpha = I$$

then there exists a fixed partition:

$$P_0 = \{0 = x_0, x_1, \ldots, x_{n-1}, x_n = 1\}$$

Such that S(P, φ, α) < I + 1 for all partitions P of [0, 1] such that P $\supseteq$ P_0. Let m, n $\in$ N such that 1/2n < x_1 and consider the partition;

$$P_m = \left\{0, \frac{1}{2n+2m+2}, \frac{2}{4n+4m+1}, \ldots, \frac{1}{2n+2j+2}, \right.$$
$$\frac{2}{4n+4j+1}, \ldots, \frac{1}{2n+4}, \frac{2}{4n+5}, \frac{1}{2n+2}, \frac{1}{2n},$$
$$\left. x_1, x_2, \ldots, x_n = 1\right\}$$

Of [0, 1], obviously $P_m \supseteq P_n$ and $P_1 = P_0$- {0} is a partition of [x_1, 1]and

$$S(P_m, \varphi, \alpha) = S(P_1, \varphi, \alpha) +$$

$$\varphi(x_1)(\alpha(x_1) - \alpha(\frac{1}{2n})) + \sum\nolimits_{j=1}^{m}[\varphi(\frac{1}{2n+2j})$$

$$(\alpha(\frac{1}{2n+2j}) - \alpha(\frac{2}{4n+4j+1})) + \varphi(\frac{2}{4n+4j+1})$$
$$(\alpha(\frac{2}{4n+4j+1}) - \alpha(\frac{1}{2n+2j+2}))] + \varphi(\frac{1}{2n+2m+2})$$
$$(\alpha(\frac{1}{2n+2m+2}) - \alpha(0)) = S(P_1, \varphi, \alpha) + \varphi(x_1)\alpha(x_1) +$$
$$\sum_{j=1}^{m} \frac{1}{4n+4j+1} < I + 1$$

And consequently,

$$\sum_{j=1}^{m} \frac{1}{4n+4j+1} < I + 1 - S(P_1, \varphi, \alpha) - \varphi(x_1)\alpha(x_1)$$

For all $m \in N$ which is a contradiction (Knopp, 1956).

REFERENCES

Dieudonne, J., 1960. Foundations of modern analysis. Academic Press, Inc., New York, pp: 60.

Graves, L.M., 1956. The theory of functions of real Variables. (2nd Edn.), McGraw-Hill book Company, New York, pp: 202-262.

Knopp, K., 1956. Infinite sequences and series, tranle-Ted by Bagemihl, F. Dover Publications Inc, New York, pp: 70.

Royden, H.L., 1968. Real analysis. The Macmillan Company, pp: 100.

Rudin, W., 1976. Principles of Mathematical Ana-lysis. (3rd Edn.), McGraw-Hill Book Comp-N.Y., pp: 124-127.