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An Efficient Two Step Hybrid Block Approach for Solving Third Order Initial Value Problems Directly

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Abstract: Solving Ordinary value problems (ODEs) numerically is an area that has drawn much consideration in latest literature and hence, the general object of this study. In this study, new hybrid block method is proposed using collocation and interpolation approach and then is employed to approximate third order initial value problems directly. Certain conditions of the block method established include it being of step number k = 2 with two off-step points. This hybrid block method involves of mathematical expressions that will concurrently supply results at the step and off-step points. Numerical properties which involve zero stability, order of the method, consistency and convergence are established. After that some third order initial value problems are considered and the new hybrid block method is employed to solve it numerically and comparison was made with existing methods in terms of error. The generated numerical results show superiority of the new method in terms of error. Therefore, this developed hybrid block method can be adopted a more appropriate numerical approach to approximate the third order IVPs numerically.

Key words: Hybrid method, block method, third order differential equation, power series, two off step points, numerical

INTRODUCTION

In this study, our main focus is to improve the numerical approximation solution of the initial value problems of the form:

$$y''' = f(x, y, y', y''),\ x \in [a, b] \tag{1}$$

With three initial conditions y (a) = η_0, y′ (a) = η_1, y″ (a) = η_2. Such these problems occur in numerous fields of engineering and applied sciences such as celestial mechanics, theoretical physics dynamics, nuclear physics, chemistry, electronics and so on.

In the last decades, methods for the numerical solution of the third order initial value problems in (1) have attracted the importance of many researchers. Yap *et al*. (2014) proposed three step block method with two off step points for the solution of third order Ordinary Differential Equations (ODEs). More recently, by Awoyemi (2003) by Abdelrahim and Omar (2015, 2016), Omar and Abdelrahim (2016) introduced one step hybrid block method for the solutions of these equations. Similarly, Omar and Kuboye (2015) by Awoyemi *et al*. (2006) constructed a seven step block method which solves certain third order initial value problems. Motivated by the research by Yap *et al*. (2014), we construct a new accurate six order two step hy-brid block method with two off step points. The remaining part of this study is built as follows.

Derivation of the method: In this study, we will derive two-step six-order implicit hybrid block method with two off step $x_{n+1/3}$ and $x_{n+5/3}$ using interpolation and collocation technique. Let us consider the power series given by:

$$y(x) = \sum_{i=0}^{\upsilon+m-1} a_i \left(\frac{x-x_n}{h}\right)^i,\ x \in [x_n, x_{n+1}] \tag{2}$$

In order to approximate (Eq. 1) where, υ is the number of interpolation points, m is the number of collocation points, $h = x_n - x_{n-1}$ is constant step size and n = 0, 1, 2, ..., N, differentiating (2) three times gives:

$$y'''(x) = f(x, y, y', y'') = \sum_{i=3}^{\upsilon+m-1} \frac{i(i-1)(i-2)}{h^3} a_i \left(\frac{x-x_n}{h}\right)^{i-3} \tag{3}$$

Equation 2 is interpolated at x_n, $x_{n+1/3}$ and x_{n+1} while Eq. 3 is collocated at all points, i.e., x_n, $x_{n+1/3}$, x_{n+1}, $x_{n+5/3}$, x_{n+2} to obtain the following equations which can be written in matrix form:

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$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} & \frac{1}{243} & \frac{1}{729} & \frac{1}{2187} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{6}{h^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{8}{h^3} & \frac{20}{3h^3} & \frac{40}{9h^3} & \frac{70}{27h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{24}{h^3} & \frac{60}{h^3} & \frac{120}{h^3} & \frac{210}{h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{40}{h^3} & \frac{500}{3h^3} & \frac{5000}{9h^3} & \frac{43750}{27h^3} \\ 0 & 0 & 0 & \frac{6}{h^3} & \frac{48}{h^3} & \frac{240}{h^3} & \frac{960}{h^3} & \frac{3360}{h^3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{1}{3}} \\ y_{n+1} \\ f_n \\ f_{n+\frac{1}{3}} \\ f_{n+1} \\ f_{n+\frac{5}{3}} \\ f_{n+2} \end{pmatrix} \quad (4)$$

Gaussian elimination method is applied in (Eq. 4) to find the values of a_i, i = 0 (1) 6. Then, these values are substituted back into Eq. 2 to give a continuous implicit scheme of the form:

$$y^{(m)}(x) = \sum_{i=0,\frac{1}{3},1} \frac{d^{(m)}}{dx^{(m)}} \alpha_i(x)\, y_{n+i} + \sum_{i=0,\frac{1}{3},1,\frac{5}{3},2} \frac{d^{(m)}}{dx^{(m)}} \beta_i(x)\, f_{n+i},\ m = 0, 1, 2 \quad (5)$$

Where:

$$\alpha_0 = \frac{\left(3(x-x_n)^2\right)}{h^2} - \frac{(4x-4x_n)}{h} + 1$$

$$\alpha_s = \frac{\left((9x-9x_n)\right)}{(2h)} - \frac{\left(9(x-x_n)^3\right)}{\left(2h^2\right)}$$

$$\alpha_1 = \frac{\left(3(x-x_n)^2\right)}{\left(2h^2\right)} - \frac{(x-x_n)}{(2h)}$$

$$\beta_0 = \frac{(x-x_n)^3}{6} - \frac{17(x-x_n)^4}{80h} + \frac{77(x-x_n)^5}{600h^2} - \frac{3(x-x_n)^6}{80h^3} + \frac{3(x-x_n)^7}{700h^4} - \frac{113h(x-x_n)^2}{2025} + \frac{739h^2(x-x_n)}{113400}$$

$$\beta_{\frac{1}{3}} = \frac{9(x-x_n)^4}{32h} - \frac{189(x-x_n)^5}{800h^2} + \frac{63(x-x_n)^6}{800h^3} - \frac{27(x-x_n)^7}{2800h^4} - \frac{287h(x-x_n)^2}{1800} + \frac{457h^2(x-x_n)}{10080}$$

$$\beta_1 = -\frac{5(x-x_n)^4}{48h} + \frac{41(x-x_n)^5}{240h^2} - \frac{3(x-x_n)^6}{40h^3} + \frac{3(x-x_n)^7}{280h^4} - \frac{43h(x-x_n)^2}{6480} + \frac{193h^2(x-x_n)}{45360}$$

$$\beta_{\frac{5}{3}} = -\frac{81(x-x_n)^5}{800h^2} + \frac{9(x-x_n)^4}{160h} + \frac{9(x-x_n)^6}{160h^3} - \frac{27(x-x_n)^7}{2800h^4} - \frac{h(x-x_n)^2}{1200} - \frac{13h^2(x-x_n)}{16800}$$

$$\beta_2 = \frac{23(x-x_n)^5}{600h^2} - \frac{(x-x_n)^4}{48h} - \frac{9(x-x_n)^6}{400h^3} + \frac{3(x-x_n)^7}{700h^4} + \frac{h(x-x_n)^2}{2025} + \frac{h^2(x-x_n)}{4536}$$

Equation 5 is evaluated at the non-interpolating point, i.e., $x_{n+5/3}$ and x_{n+2} when m = 0 and at all points when m = 1, 2 to produce the discrete schemes and its derivatives. Next, the discrete scheme and its derivative at x_n are combined to obtained the following Eq. 6 in the matrix form:

$$AY_m = BR_1 + DR_2 + CR_3 \quad (6)$$

Where:

$$A = \begin{pmatrix} 5 & \frac{-10}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & -5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-9}{2h} & \frac{1}{2h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-3}{2h} & \frac{-1}{2h} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{9}{2h} & \frac{-5}{2h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{21}{2h} & \frac{-9}{2h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{27}{2h} & \frac{-11}{2h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{9}{h^2} & \frac{-3}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{9}{h^2} & \frac{-3}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{9}{h^2} & \frac{-3}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{9}{h^2} & \frac{-3}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{9}{h^2} & \frac{-3}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, R_1 = \begin{pmatrix} y''_n \\ y'_n \\ y_n \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{8}{3} & 0 & 0 \\ 5 & 0 & 0 \\ \frac{-4}{h} & -1 & 0 \\ \frac{-2}{h} & 0 & 0 \\ \frac{2}{h} & 0 & 0 \\ \frac{6}{h} & 0 & 0 \\ \frac{8}{h} & 0 & 0 \\ \frac{6}{h^2} & 0 & -1 \\ \frac{6}{h^2} & 0 & 0 \\ \frac{6}{h^2} & 0 & 0 \\ \frac{6}{h^2} & 0 & 0 \\ \frac{6}{h^2} & 0 & 0 \end{pmatrix}, D = \begin{pmatrix} -\frac{91h^3}{7290} \\ -\frac{7h^3}{324} \\ \frac{739h^2}{113400} \\ \frac{29h^2}{68040} \\ -\frac{191h^2}{22680} \\ -\frac{8563h^2}{340200} \\ -\frac{227h^2}{7560} \\ -\frac{226h}{2025} \\ \frac{329h}{16200} \\ -\frac{647h}{16200} \\ -\frac{61h}{5400} \\ -\frac{37h}{2025} \end{pmatrix} R_2 = (f_n)$$

$$C = \begin{pmatrix} \frac{35h^3}{324} & \frac{229h^3}{1458} & -\frac{17h^3}{1620} & \frac{7h^3}{1458} \\ \frac{143h^3}{720} & \frac{229h^3}{648} & \frac{h^3}{48} & \frac{7h^3}{1620} \\ \frac{457h^2}{10080} & \frac{193h^2}{45360} & -\frac{13h^2}{16800} & \frac{h^2}{4536} \\ -\frac{269h^2}{8400} & -\frac{463h^2}{68040} & \frac{31h^2}{15120} & -\frac{233h^2}{340200} \\ \frac{379h^2}{5040} & \frac{1213h^2}{22680} & -\frac{23h^2}{1680} & \frac{103h^2}{22680} \\ \frac{3587h^2}{15120} & \frac{3659985530504775h^2}{7881299347898368} & \frac{1919h^2}{75600} & \frac{127h^2}{68040} \\ \frac{5167h^2}{16800} & \frac{10751h^2}{15120} & \frac{197h^2}{1120} & \frac{83h^2}{37800} \\ -\frac{287h}{900} & -\frac{43h}{3240} & \frac{-h}{600} & \frac{2h}{2025} \\ -\frac{329h}{3600} & -\frac{167h}{3240} & \frac{7h}{400} & -\frac{97h}{16200} \\ \frac{1039h}{3600} & \frac{229h}{648} & -\frac{83h}{1200} & \frac{367h}{16200} \\ \frac{727h}{3600} & \frac{91h}{120} & \frac{373h}{1200} & -\frac{203h}{5400} \\ \frac{199h}{900} & \frac{2333h}{3240} & \frac{323h}{600} & \frac{191h}{2025} \end{pmatrix}, R_3 = \begin{pmatrix} f_{n+s} \\ f_{n+1} \\ f_{n+r} \\ f_{n+2} \end{pmatrix}$$

Multiplying Eq. 6 by A^{-1} gives hybrid block method of the form:

$$IY_m = \bar{B}R_1 + \bar{D}R_2 + \bar{C}R_3 \quad (7)$$

Where:

$$\bar{B} = \begin{pmatrix} 1 & \frac{h}{3} & \frac{h^2}{18} \\ 1 & h & \frac{h^2}{2} \\ 1 & \frac{5h}{3} & \frac{25h^2}{18} \\ 1 & 2h & 2h^2 \\ 0 & 1 & \frac{h}{3} \\ 0 & 1 & h \\ 0 & 1 & \frac{5h}{3} \\ 0 & 1 & 2h \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \bar{D} = \begin{pmatrix} 4111h^3/1020600 \\ 69h^3/1400 \\ 5375h^3/40824 \\ 33h^3/175 \\ 7h^2/225 \\ 29h^2/300 \\ 25h^2/162 \\ 14h^2/75 \\ \frac{2137h}{16200} \\ \frac{43h}{600} \\ \frac{65h}{648} \\ \frac{7h}{75} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1181h^3}{453600} & -\frac{139h^3}{204120} & \frac{53h^3}{151200} & -\frac{131h^3}{1020600} \\ \frac{639h^3}{5600} & \frac{h^3}{420} & \frac{9h^3}{5600} & \frac{-h^3}{1400} \\ \frac{2875h^3}{6048} & \frac{6875h^3}{40824} & -\frac{125h^3}{18144} & \frac{125h^3}{40824} \\ \frac{261h^3}{350} & \frac{13h^3}{35} & \frac{9h^3}{350} & \frac{h^3}{525} \\ \frac{25h^2}{864} & -\frac{43h^2}{6480} & \frac{73h^2}{21600} & \frac{-h^2}{810} \\ \frac{279h^2}{800} & \frac{h^2}{16} & -\frac{9h^2}{800} & \frac{h^2}{300} \\ \frac{625h^2}{864} & \frac{4617948836659202809h^2}{9575778707696517120} & \frac{25h^2}{864} & 0 \\ \frac{9h^2}{10} & \frac{11h^2}{15} & \frac{9h^2}{50} & 0 \\ \frac{91h}{400} & -\frac{31h}{810} & \frac{23h}{1200} & -\frac{113h}{16200} \\ \frac{243h}{400} & \frac{11h}{30} & -\frac{27h}{400} & \frac{13h}{600} \\ \frac{25h}{48} & \frac{125h}{162} & \frac{5h}{16} & -\frac{25h}{648} \\ \frac{27h}{50} & \frac{11h}{15} & \frac{27h}{50} & \frac{7h}{75} \end{pmatrix}$$

Properties of method

Order of method: The linear difference operator L associated with (Eq. 7) is defined as:

$$L\left[y(x); h\right] = IY_m - \bar{B}R_1 - \left[\bar{D}R_2 + \bar{C}R_3\right] \quad (8)$$

where, y (x) is an arbitrary test function continuously differentiable on [a, b]. Y_m and R_3 components are expanded in Taylors series, respectively and their terms are collected in powers of h to give:

$$L\left[y(x), h\right] = \bar{Q}_0 y(x) + \bar{Q}_1 h y'(x) + \bar{Q}_2 h^2 y''(x) + , \ldots \quad (9)$$

Definition 1: Hybrid block method (Eq. 7) and associated linear operator (8) are said to be of order p, if $\bar{Q}_0 = \bar{Q}_1 = \bar{Q}_2 = , \cdots, = \bar{Q}_{p+2} = 0$ with error vector constants $\bar{Q}_{p+3} \neq 0$. Expanding the functions of y and f- in (Eq. 7) gives:

$$\begin{bmatrix} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^{j_h j}}{j!} y_n^j - y_n - \left(\frac{h}{3}\right) y'_n - \frac{\left(h^2\right)}{18} y''_n - \frac{4111h^3}{1020600} y'''_n \\ -\frac{1181}{453600} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^j h^j + 3}{j!} y_n^{j+3} \\ + \frac{139}{204120} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \\ - \frac{53}{151200} \sum_{j=0}^{\infty} \frac{(r)^j h^{j+3}}{j!} y_n^{j+3} \\ + \frac{131}{1020600} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \\ \\ \sum_{j=0}^{\infty} \frac{h^j}{j!} y_n^j - y_n - (h) y'_n - \frac{h^2}{2} y''_n - \frac{69h^3}{1400} y'''_n \\ - \frac{639h^3}{5600} \sum_{j=0}^{\infty} \frac{(s)^j h^{j+3}}{j!} y_n^{j+3} \\ - \frac{h^3}{420} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \\ - \frac{9h^3}{5600} \sum_{j=0}^{\infty} \frac{(r)^j h^{j+3}}{j!} y_n^{j+3} \\ - \frac{1}{1400} \sum_{j=0}^{\infty} \frac{h^{j+3}}{j!} y_n^{j+3} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{\left(\frac{5}{3}\right)^{j}h^{j}}{j!}y_n^{j}-y_n-\left(\frac{5h}{3}\right)y_n'-\frac{25h^2}{18}y_n''-\frac{5375}{40824}y_n''' \\ -\frac{2875h^3}{6048}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}\right)^{j}h^{j+3}}{j!}y_n^{j+3} \\ -\frac{6875}{40824}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3} \\ +\frac{125h^3}{18144}\sum_{j=0}^{\infty}-\frac{\left(\frac{5}{3}\right)^{j}h^{j+3}}{j!}y_n^{j+3} \\ -\frac{125}{40824}\sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^{j+3} \\ \sum_{j=0}^{\infty}\frac{(2)^{j}h^{j}}{j!}y_n^{j}-y_n-(2h)y_n'-2h^2y_n''-\frac{33}{175}y_n''' \\ -\frac{261h^3}{350}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}\right)^{j}h^{j+2}}{j!}y_n^{j+2} \\ -\frac{13}{35}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\ -\frac{9}{350}\sum_{j=0}^{\infty}-\frac{\left(\frac{5}{3}\right)^{j}h^{j+2}}{j!}y_n^{j+2} \\ -\frac{1}{525}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Comparing the coefficients of h^j and y^j yields the order of the method which is $[6, 6, 6, 6]^T$.

Zero stability

Definition 2: The hybrid block method (Eq. 7) is said to be zero stable, if the multiplicity of non trivial roots of the first characteristic polynomial not >3.

In order to find the zero-stability of the hybrid block (Eq. 7), we only consider the first characteristic polynomial according to definition (2). Therefore, we have:

$$\Pi(z) = \left|zI-\bar{\bar{B}}\right| = \left| z\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = z^3(z-1)$$

Which implies z = 0, 0, 0, 1. Hence, our method is zero stable.

Consistency

Definition 3: Hybrid block method is said to be consistent, if it has order≥ 1.

Table 1: Comparison of the new method with some existing methods for solving problem 1

h	Methods	E (x_N) at x = 1	Steps
	New method	$2{:}08e^{-13}$	40
	Yap *et al.*(2014)	$4{:}23e^{-12}$	27
0.0125	Adams	$3{:}56e^{-8}$	80
	Awoyemi and Idowu (2005)	$7{:}60e^{-6}$	80
		E (x_N) at x = 4	
	New method	$7{:}88e^{-13}$	200
	Yap *et al.* (2014)	$3{:}50e^{-11}$	134
0.01	Adams	$2{:}91e^{-6}$	400
	Awoyemi and Idowu (2005)	$1{:}16e^{-3}$	400

Table 2: Comparison of the new method with some existing methods for solving problem 2

h	Methods	E (x_N) at x = 1	Steps
	New method	$1{:}28e^{-11}$	5
	Yap *et al.* (2014)	$1{:}11e^{-10}$	4
0.1	Adams	$2{:}76e^{-6}$	10
	Awoyemi *et al.* (2006)	$1{:}07e^{-6}$	10
		E (x_N) at x = 5	
	New method	$6{:}46e^{-15}$	100
	Yap *et al.* (2014)	$1{:}19e^{-12}$	67
0.025	Adams	$9{:}72e^{-8}$	200
	Awoyemi *et al.* (2006)	$3{:}53e^{-8}$	200

i.e., p≥1. Our hybrid block method (7) is consistent because it satisfies the condition stated in definition 3.

Convergence

Theorem 4 (Henrici, 1962): Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent following Henrici theorem, the new hybrid block method proposed is convergent, since, it is consistent and zero stable.

Numerical examples: Two numerical examples were used to ascertain the accuracy of the method. The new block methods solved the same problems that existing methods solved (Table1 and 2).

Problem 1:

$$y'''-2y''-3y'+10y-34xe^{-2x}+16 = 0,\ y(0) = 3,\ y'(0) = 0,\ y''(0) = 0$$

Exact solution: $y(x) = e^{-x}$

Problem 2:

$$y'''+y = 0,\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 1$$

Exact solution: $y(x) = e^{-x}$ with h = 0.1

CONCLUSION

A new two step hybrid block method with two off-step points for the direct solution of third order ordinary differential equation has been developed successfully. The developed method is consistent, zero-stable and also, convergent. When solving the same problems, the numerical results confirm that the new method produces better accuracy, if compared to the existing methods.

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