Research Journal of Applied Sciences 14 (2): 61-66, 2019
ISSN: 1815-932X

New Mathematical Studies for Surface Waves on Multi-Layered Liquid Films

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Abstract: We consider a coupled system of mathematical problem given in the form of two partial differential equations in viscoelasticities describing the propagation of surface waves on multi-layered liquid films. We establish the existence of solutions for initial-value problem (1.3) in the linear case by using a priori energy estimates.

Key words: Surface waves, existence, multi-layered liquid film, infinite memory, mathematical problem, viscoelasticities

INTRODUCTION

In recent years, modern technology has seen more interest in physical sciences and a rapid increase, especially in areas that are exploited and dependent on multiple physical connections. In this research, we present a broad view, new perspective and a fruitful study of recent and serious issues in the form of time differential equations that represent important physical phenomena in many fields of application in modern sciences. We will consider a mathematical problem given in the form of partial differential equations related to the appropriate time for some important physical phenomena in various fields of application in engineering and modern technology and try to propose and develop some new mathematical methods for a new study of the output of interactions between some associated effects. The Kuramoto-Sivashinsky (KS) equation in the form, $\alpha, \gamma>0$:

$$u'+uu_x+\alpha u_{xx}-\gamma u_{xxxx}=0 \tag{1}$$

is a well known model of 1D turbulence which was derived in different physical contexts, including chemical reaction waves, propagation of combustion fronts in gases, surface waves in a film of a viscous liquid owing along a diagonal level, patterns in thermal convection, rapid solidification and others.

Recently, a linear coupled Kuramoto-Sivashinsky KdV equation with an extra linear dissipative equation, was studies by many researchers Malomed *et al*. (2011), Feng *et al*. (2003) and Cai *et al*. (2013). The model had the form:

$$\begin{cases} u'+\alpha u_{xx}+u_{xxx}+uu_x+\beta u_{xxxx}=\upsilon_x \\ \upsilon'+a_1\upsilon_x-\gamma\upsilon_{xx}=u_x \end{cases} \tag{2}$$

and a studies in wider way are made. Viscoelastic substances exhibit behavior between exible solids and Newtonian liquids. In fact, pressures in these media depend on the entire history of their distortion, not only on current state of deformation or the state of their current movement. This is why they are called materials with memory. Many researchers have studied viscous systems with faded memory in a specific area. In this study, we propose and develop in-depth and useful mathematical studies related to a new class of Kuramoto-Sivashinsky system, along with an additional linear equation and of course we will extend the studies to the viscoelastic system. These proposed models apply to the description of surface waves of layer liquid films in different fields of applied science and modern technology modeled. Let $(x, y)\in\Omega = R^2$, let us consider the system:

$$\begin{cases} u'+\alpha u_{xx}+\Delta u_x+uu_x+\beta\Delta^2 u+\int_{-\infty}^{t}\mu_1(t-s)u_{xx}(s)ds=\upsilon_x \\ \upsilon'+a_1\upsilon_x-\gamma\Delta\upsilon+\int_{-\infty}^{t}\mu_2(t-s)\upsilon(s)ds=u_x \end{cases} \tag{3}$$

where, u and υ are the two real wave fields, the dissipative parameter $\gamma>0$ accounts for the stabilization and a_1 is a group-velocity mismatch between the two wave modes. The coefficients α and β are all positive constants. The given functions μ_1, μ_2 are specified later. The terms $\int_{-\infty}^{t}\mu_i(t-s)u_i(s)ds = \int_0^{\infty}\mu_i(s)u_i(t-s)ds$, i = 1, 2 represent the infinite memories. To deal with infinite history, we assume that the kernel functions μ_1, μ_2 satisfy the following hypothesis, $\mu_1, \mu_2: R^+\to R^+$ are a non-increasing C^1 functions such that:

$$1-\int_0^{\infty}\mu_1(s)ds = l>0,\ \mu_1(0)>0 \tag{4}$$

And:

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$$1-\int_0^{\infty}\mu_2(s)ds=\tilde{1}>0,\ \mu_2(0)>0 \quad (5)$$

Let, H·(Ω) usual Sobolev space defined by the norm:

$$\|w\|_m^2=\int_\Omega\sum_{|j|\le m}|D^j w|^2\,dxdy \quad (6)$$

Where, H·(Ω) = L·(Ω) and ||w|| = ||w||·, where:

$$D^j w=\frac{\partial^{|j|}w}{\partial_x^{j1}\partial_y^{j2}}$$

The main problem is the quantitative studies of surface waves on multilayered liquid films. In particular, the fundamental cause of the matter is under consideration by many mathematicians to answer physicist's questions to achieve a complex and new physical structure by merging several phenomena into one side and considering their effectiveness. It is therefore, natural to ask whether a comprehensive presence of strong solutions can arise when the dissipation changes.

It is very important to address some scientific issues through the theories of functional analysis and then provide numerical simulation of the theoretical study to obtain a useful and stable convergence. Thus, the goal here is to develop some of the recent results obtained in research work near to our subject. In this study we are studying the mathematical question of the Kuramoto-Sivashinsky-Corteug-de-Fries equation in a multidimensional field. This model was put into study, to describe surface waves on multilayered liquid films by the theory of perturbation where the researchers studied dissipation and acquired acquisition of instability in the model as small disturbances. Later, two-dimensional model was proposed and developed by Feng *et al.* (2003). In fact, the problem was originally proposed by Benney (1966), attention was focused in particular on the existence and uniqueness of the solution. These studies were later significantly known by Christov and Velarde (1995) and Elphick *et al.* (1991).

MATERIALS AND METHODS

Linear stability: The proposed system (1.3) describes the propagation of surface waves in a two-layer liquid with a single layer dominated by viscosity and infinite memory. Let $(\tilde{u},\tilde{\upsilon})$ be a small perturbation of a bounded C· solution (u·, ·) of Eq. 3:

$$\begin{aligned} u &\square\ u_0+\varepsilon\tilde{u}\\ \upsilon &\square\ \upsilon_0+\varepsilon\tilde{\upsilon},\quad \varepsilon<1\end{aligned} \quad (7)$$

To linearize system for $(\tilde{u},\tilde{\upsilon})$, substituting Eq. 7 in Eq. 3, we get:

$$\begin{cases}\tilde{u}'+\alpha\tilde{u}_{xx}+\Delta\tilde{u}_x+u_0\tilde{u}_x+\beta\Delta^2\tilde{u}+\int_0^{\infty}\mu_1(t-s)\tilde{u}_{xx}(s)ds=\tilde{\upsilon}_x+f,\\ \tilde{\upsilon}'+a_1\tilde{\upsilon}_x-\gamma\Delta\tilde{\upsilon}+\int_0^{\infty}\mu_2(t-s)\Delta\tilde{\upsilon}(s)ds=\tilde{u}_x+g\end{cases} \quad (8)$$

by omitting the higher order terms of ·. Under small initial perturbation, the stability of solution (u·, ·) is determined by the energy estimate for $(\tilde{u},\tilde{\upsilon})$. Let, u· be a given bounded smooth function and · be a given bounded smooth function. We will consider the following linearized system with $(\tilde{u},\tilde{\upsilon})$:

$$\begin{cases}\tilde{u}'+\alpha\tilde{u}_{xx}+\Delta\tilde{u}_x+u_0\tilde{u}_x+\beta\Delta^2\tilde{u}+\int_{-\infty}^{t}\mu_1(t-s)\tilde{u}_{xx}(s)ds=\tilde{\upsilon}_x+f,\\ \tilde{\upsilon}'+a_1\tilde{\upsilon}_x-\gamma\Delta\tilde{\upsilon}+\int_{-\infty}^{t}\mu_2(t-s)\Delta\tilde{\upsilon}(s)ds=\tilde{u}_x+g\\ u(x,y,0)=u_0(x,y),\ \upsilon(x,y,0)=\upsilon_0(x,y)\end{cases} \quad (9)$$

RESULTS AND DISCUSSION

Theorem 2.1: For any solution $(\tilde{u},\tilde{\upsilon})$ of linearized system (Eq. 9), the Schwartz rapidly decaying function space S(R·), (Introduced by Renardy and Rogers (2004), satisfies the estimate:

$$\begin{aligned}&\int_\Omega|\tilde{u}'|^2dxdy+\int_\Omega|\tilde{\upsilon}'|^2dxdy+\|\tilde{u}\|_H+\|\tilde{\upsilon}\|_H\le\\ &c\left(\int_\Omega\left(|\tilde{u}|^2+|\tilde{\upsilon}|^2\right)dxdy+\int_\Omega|f|^2dxdy+\int_\Omega|g|^2dxdy\right)\end{aligned} \quad (10)$$

And:

$$\begin{aligned}&\sup_{0\le t\le T}\int_\Omega|\tilde{u}|^2+|\tilde{\upsilon}|dxdy+\int_0^T\|\tilde{u}\|_H+\|\tilde{\upsilon}\|_H ds\le\\ &c\int_\Omega\left(|\tilde{u}_0|^2+|\tilde{\upsilon}_0|^2dxd+y\int_0^T\int_\Omega|f|^2dxdy+\int_\Omega|g|^2dxdyds\right)\end{aligned}$$

Where: (12)

$$\begin{aligned}&\|\tilde{u}\|_H+\|\tilde{\upsilon}\|_H=\int_\Omega\left(|\tilde{u}|^2+|\tilde{u}_x|^2\right)dxdy+c(\mu_1\circ\tilde{u}_x)(t)+\\ &c\int_\Omega\left(|\tilde{\upsilon}|^2+|\nabla\tilde{\upsilon}|^2\right)dxdy+c(\mu_2\circ\nabla\tilde{\upsilon})(t)\end{aligned}$$

Proof; Multiplying (2.3).: By $\tilde{u}$ and (2:3). by $\tilde{\upsilon}$ integrating over , we have:

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\int_\Omega |\tilde{u}|^2 dxdy - \alpha\int_\Omega |\tilde{u}_x|^2 dxdy - \int_\Omega \nabla\tilde{u}_x \nabla\tilde{u} dxdy + \int_\Omega u_0 \tilde{u}_x \tilde{u} dxdy - \\ \beta\int_\Omega \Delta\tilde{u}\Delta\tilde{u} dxdy - \int_\Omega\int_0^\infty \mu_1(s)\tilde{u}_x(t-s)\tilde{u}_x dsdxdy = \int_\Omega \tilde{\upsilon}_x \tilde{u} dxdy + \int_\Omega f\tilde{u} dxdy, \frac{1}{2}\frac{d}{dt}\int_\Omega |\tilde{\upsilon}|^2 dxdy + \\ a_1\int_\Omega \tilde{\upsilon}_x \tilde{\upsilon} dx + \gamma\int_\Omega |\nabla\tilde{\upsilon}|^2 dxdy - \int_\Omega\int_0^\infty \mu_2(s)\nabla\tilde{\upsilon}(t-s)\nabla\tilde{\upsilon} dsdxdy = \int_\Omega \tilde{u}_x \tilde{\upsilon} dxdy + \int_\Omega g\tilde{\upsilon} dxdy \end{cases}$$

Summing to get:

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\int_\Omega \left(|\tilde{u}|^2 + |\tilde{\upsilon}|^2\right) dxdy - \alpha\int_\Omega |\tilde{u}_x|^2 dxdy - \int_\Omega \left(\beta|\nabla\tilde{u}|^2 - \gamma|\nabla\tilde{\upsilon}|^2\right) dxdy - \int_\Omega \nabla\tilde{u}_x \nabla\tilde{u} dxdy + \int_\Omega u_0\tilde{u}_x\tilde{u} dxdy + \\ a_1\int_\Omega \tilde{\upsilon}_x\tilde{\upsilon} dxdy - \int_\Omega\int_0^\infty \mu_1(s)\tilde{u}_x(t-s)\tilde{u}_x dsdxdy - \int_\Omega\int_0^\infty \mu_2(s)\nabla\tilde{\upsilon}(t-s)\nabla\tilde{\upsilon} dsdxdy = \\ \int_\Omega \tilde{\upsilon}_x\tilde{u} dxdy + \int_\Omega \tilde{u}_x\tilde{\upsilon} dxdy, \int_\Omega f\tilde{u} dxdy + \int_\Omega g\tilde{\upsilon} dxdy \end{cases} \tag{13}$$

For any >0, we have:

$$\int_\Omega \left(\beta|\nabla\tilde{u}|^2 - \gamma|\nabla\tilde{\upsilon}|^2\right) dxdy \le -\upsilon\int_\Omega \left(|\nabla\tilde{u}|^2 - |\nabla\tilde{v}|^2\right) dxdy$$

And:

$$\left|\int_\Omega \alpha|\tilde{u}_x|^2 + \nabla\tilde{u}_x\nabla\tilde{u} + u_0\tilde{u}_x\tilde{u} + a_1\tilde{\upsilon}_x\tilde{\upsilon} dxdy + \tilde{\upsilon}_x\tilde{u} dxdy + \tilde{u}_x\tilde{\upsilon} dxdy\right| \le \nu\int_\Omega \left(|\Delta\tilde{u}|^2 + |\nabla\tilde{\upsilon}|^2\right) dxdy + c\int_\Omega \left(|\tilde{u}|^2 + |\tilde{\upsilon}|^2\right) dxdy \tag{14}$$

And:

$$\int_\Omega f\tilde{u} dxdy + \int_\Omega g\tilde{\upsilon} dxdy \le \int_\Omega \left(|f|^2 + |g|^2\right) dxdy + \int_\Omega \left(|\tilde{u}|^2 + |\tilde{\upsilon}|^2\right) dxdy \tag{15}$$

We have:

$$\int_\Omega \tilde{u}_x \int_0^\infty \mu_1(s)\tilde{u}_x(t-s) dsdxdy \le \frac{1}{2}\int_\Omega \tilde{u}_x^2 dxdy + \frac{1}{2}\int_\Omega \left(\int_0^\infty \mu_1(s)\tilde{u}_x(t-s) ds\right)^2 dx \le$$
$$\frac{1}{2}\int_\Omega \tilde{u}_x^2 dxdy + \frac{1}{2}\int_\Omega \left(\int_0^\infty \mu_1(s)|\tilde{u}_x(t-s) - \tilde{u}_x| + |\tilde{u}_x| ds\right) dxdy$$

By Cauchy-Schwarz and Young inequalities, we obtain, for some >0:

$$\int_\Omega \left(\int_0^\infty \mu_1(s)|\tilde{u}_x(t-s) - \tilde{u}_x| + |\tilde{u}_x| ds\right)^2 dxdy \le \int_\Omega \left(\int_0^\infty \mu_1(s)|\tilde{u}_x(t-s) - \tilde{u}_x| ds\right)^2 dxdy + \int_\Omega \left(\int_0^\infty \mu_1(s)|\tilde{u}_x| ds\right)^2 dxdy +$$
$$2\int_\Omega \left(\int_0^\infty \mu_1(s)|\tilde{u}_x(t-s) - \tilde{u}_x| ds\right)\left(\int_0^\infty \mu_1(s)|\tilde{u}_x| ds\right) dxdy \le \left(1 + \frac{1}{\nu}\right)\int_\Omega \left(\int_0^\infty \mu_1(s)|\tilde{u}_x(t-s) - \tilde{u}_x| ds\right)^2 dxdy +$$
$$(1+\nu)\int_\Omega \left(\int_0^\infty \mu_1(s) ds|u_x|\right)^2 dxdy \le \left(1 + \frac{1}{\nu}\right)(1-1)\int_\Omega\int_0^\infty \mu_1(s)|\tilde{u}_x(t-s) - \tilde{u}_x|^2 dsdxdy + (1+\nu)(1-1)^2\int_\Omega |\tilde{u}_x|^2 dxdy$$

Then:

$$\int_\Omega \tilde{u}_x \int_0^\infty \mu_1(s)\tilde{u}_x(t-s) dsdxdy \le \frac{1}{2}\left(1 + (1+\nu)(1-1)^2\right)\int_\Omega \tilde{u}_x^2 dxdy + \frac{1}{2}\left(1 + \frac{1}{\nu}\right)(1-1)\int_\Omega\int_0^\infty \mu_2(s)|\tilde{u}_x(t-s) - \tilde{u}_x|^2 dsdxdy$$

And similarly:

$$\int_\Omega \nabla\tilde{\upsilon}\int_0^\infty \mu_2(s)\nabla\tilde{\upsilon}(t-s) dsdxdy \le \frac{1}{2}\left(1 + (1+\nu)(1-\tilde{1})^2\right)\int_\Omega |\nabla\tilde{\upsilon}|^2 dxdy + \frac{1}{2}\left(1 + \frac{1}{\nu}\right)(1-\tilde{1})\int_\Omega\int_0^\infty \mu_2(s)|\nabla\tilde{\upsilon}(t-s) - \nabla\tilde{\upsilon}|^2 dsdxdy$$

Then, Eq. 13 becomes:

$$\begin{cases}\frac{1}{2}\frac{d}{dt}\int_\Omega\left(|\tilde{u}|^2+|\tilde{\upsilon}|^2\right)dxdy + c\int_\Omega\left(|\tilde{u}|^2+|\tilde{\upsilon}|^2\right)dxdy + \frac{1}{2}\left(1+(1+\nu)(1-1)^2\right)\int_\Omega \tilde{u}_x^2dxdy + \frac{1}{2}\left(1+\frac{1}{\nu}\right)(1-1)(\mu_1\circ\tilde{u}_x)(t)+\\ \frac{1}{2}\left(1+(1+\nu)(1-\tilde{1})^2\right)\int_\Omega|\nabla\tilde{\upsilon}|^2dxdy+\frac{1}{2}\left(1+\frac{1}{\nu}\right)(1-\tilde{1})(\mu_2\circ\nabla\tilde{\upsilon})(t)\le\int_\Omega\left(|f|^2+|g|^2\right)dxdy+\int_\Omega\left(|\tilde{u}|^2+|\tilde{\upsilon}|^2\right)dxdy\end{cases}$$

Where:

$$(\phi\circ\psi)(t)=\int_\Omega\int_0^\infty\phi(s)|\psi(t-s)-\psi|^2\,dsdxdy$$

Thus:

$$P'(t)-cP(t)\le M(t) \tag{17}$$

Where:

$$P(t)=\int_\Omega\left(|\tilde{u}|^2+|\tilde{\upsilon}|^2\right)dxdy+\int_0^t\int_\Omega\left(|\tilde{u}|_H+|\tilde{\upsilon}|_H\right)dxdyds$$

And:

$$M(t)=\int_\Omega\left(|f|^2+|g|^2\right)dxdy$$

Therefore, we have:

$$\exp(-ct)P(t)-P(0)\le\int_0^t\exp(-cs)M(s)ds$$

And:

$$c(t)\left(P(0)+\int_0^t M(s)ds\right)\ge P(t) \tag{18}$$

this proves Eq. 10 and 11 (Cai *et al.*, 2013).

Theorem 2.2 (Cai *et al.*, 2013): Any solution $(\tilde{u},\tilde{\upsilon})$ of linearized system Eq. 9, satisfies the estimate, for k.0:

$$\sup_{0\le t\le T}\left[\|\tilde{u}\|_{\kappa+2}^2+\|\tilde{\upsilon}\|_{\kappa+1}^2\right]+\int_0^T\|\tilde{u}\|_{H\kappa}+\|\tilde{\upsilon}\|_{H\kappa}+\|\tilde{u}'\|_\kappa^2+\|\tilde{\upsilon}'\|_\kappa^2ds\le c\left(\|\tilde{u}_0\|_{\kappa+2}^2+\|\tilde{\upsilon}_0\|_{\kappa+1}^2+\int_0^T\|f\|_\kappa^2+\|g\|_\kappa^2ds\right) \tag{19}$$

Where:

$$\|\tilde{u}\|_{H\kappa}+\|\tilde{\upsilon}\|_{H\kappa}=\left(\|\tilde{u}\|_{\kappa+4}^2+\|\tilde{u}_x\|_2^2\right)+c(\mu_1\circ\tilde{u}_x)+c\left(\|\tilde{\upsilon}\|_{\kappa+2}^2+\|\nabla\tilde{\upsilon}\|^2\right)+c(\mu_2\circ\nabla\tilde{\upsilon})$$

Existence for linearized problem: To prove the existence and uniqueness results for related problem (Eq. 9), we use the well known continuation method. We assume that:

$$(1)u_0\in H^{\kappa+2}(\Omega)\ \text{and}\ \upsilon_0\in H^{\kappa+1}(\Omega)$$
$$f,g\in L^2([0,T],\ H^\kappa(\Omega))$$

Let us define the Banach space:

$$Y=\begin{Bmatrix}(\tilde{u},\tilde{\upsilon}):\tilde{u}\in C([0,T],H^{\kappa+2}(\Omega))\cap L^2([0,T],H^{\kappa+4}(\Omega))\cap\\ H^1([0,T],H^\kappa(\Omega))\tilde{\upsilon}\in C([0,T],H^{\kappa+1}(\Omega))\cap\\ L^2([0,T],H^{\kappa+2}(\Omega))\cap H^1([0,T],H^\kappa(\Omega))\end{Bmatrix}$$

Equipped with the norm:

$$\sup_{0\le t\le T}\left[\|\tilde{u}\|_{\kappa+2}^2+\|\tilde{\upsilon}\|_{\kappa+1}^2\right]+\int_0^T\|\tilde{u}\|_{H\kappa}+\|\tilde{\upsilon}\|_{H\kappa}+\|\tilde{u}'\|_\kappa^2+\|\tilde{\upsilon}'\|_\kappa^2ds$$

Theorem 3.1: Let, k.0 be any integer and under the assumption (1), system (Eq. 9) has a unique solution (u, .) in the Banach space Y satisfying estimate in Theorem 2.2.

Proof: We rewrite (Eq. 9) as:

$$\begin{cases}\tilde{u}'+L_1(\tilde{u},\tilde{\upsilon})=f\\ \tilde{\upsilon}'+L_2(\tilde{u},\tilde{\upsilon})=g\\ \tilde{u}(x,y,0)=u_0(x,y),\quad\tilde{\upsilon}(x,y,0)=\upsilon_0(x,y)\end{cases} \tag{20}$$

Where:

$$L_1(\tilde{u},\tilde{\upsilon})=\left[\alpha\tilde{u}+\int_0^\infty\mu_1(s)\tilde{u}(t-s)ds\right]_{xx}+\Delta\tilde{u}_x+u_0\tilde{u}_x+\beta\Delta^2\tilde{u}-\tilde{\upsilon}_x$$

$$L_2(\tilde{u},\tilde{\upsilon})=a_1\tilde{\upsilon}_x-\Delta\left[\gamma\tilde{\upsilon}-\int_0^\infty\mu_2(s)\tilde{\upsilon}(t-s)ds\right]-\tilde{u}_x$$

For .[0; 1], we define:

$$\begin{cases}\tilde{u}'+\lambda L_1(\tilde{u},\tilde{\upsilon})+(1-\lambda)\Delta^2\tilde{u}=f\\ \tilde{\upsilon}'+\lambda L_2(\tilde{u},\tilde{\upsilon})-(1-\lambda)\Delta\tilde{\upsilon}=g\\ \tilde{u}(x,y,0)=u_0(x,y),\quad\tilde{\upsilon}(x,y,0)=\upsilon_0(x,y)\end{cases} \tag{21}$$

In order to prove our result, let us consider a subset B.[0, 1] such that .B. We will show that B is not empty and it is both closed and open.

B is not empty: At least, 0.B. Since, for . = 0, problem (21) takes the form:

$$\begin{cases}\tilde{u}'+\Delta^2\tilde{u}=f\\ \tilde{\upsilon}'+\Delta\tilde{\upsilon}=g\\ \tilde{u}(x,y,0)=u_0(x,y),\quad\tilde{\upsilon}(x,y,0)=\upsilon_0(x,y)\end{cases} \tag{22}$$

It is not hard to see that the Cauchy problem of general parabolic Eq. 22, admits a solutions $(\tilde{u},\tilde{\upsilon})$ (Renardy and Rogers, 2004).

B is closed in [0, 1]: Let $\lambda \in B$ and let $(\tilde{u},\tilde{v})$ be the solution of the following initial value problem:

$$\begin{cases} \tilde{u}_j' + \lambda_j L_1(\tilde{u}_j,\tilde{\upsilon}_j) + (1-\lambda_J)\Delta^2\tilde{u} = f \\ \tilde{\upsilon}_j' + \lambda_j L_2(\tilde{u}_J,\tilde{\upsilon}_J) - (1-\lambda_j)\Delta^2\tilde{\upsilon}_J = g \\ \tilde{u}_j(x,y,0) = u_0(x,y),\ \tilde{\upsilon}_j(x,y,0) = \upsilon_0(x,y) \end{cases} \tag{23}$$

Bu Theorem 2.2: We have (u_j,υ_j) is uniformly bounded in Y. Let $(\overline{\tilde{u}_j},\overline{\tilde{\upsilon}_j}) = (\tilde{u}_j - \tilde{u}_{j-1} - \tilde{\upsilon}_j - \tilde{\upsilon}_{j-1})$ with satisfies:

$$\begin{cases} \overline{\tilde{u}'_j} + \lambda_j L_1(\overline{\tilde{u}_j},\overline{\tilde{\upsilon}_j}) + (1-\lambda)\Delta^2\overline{\tilde{u}_j} = -(\lambda_j-\lambda_{j-1})\left[L_1(\tilde{u}_{j-1},\tilde{\upsilon}_{j-1}) + (1-\lambda)\Delta^2\tilde{u}_{j-1}\right] \\ \overline{\tilde{\upsilon}'_j} + \lambda_j L_2(\overline{\tilde{u}_j},\overline{\tilde{u}_j}) - (1-\lambda_j)\Delta\overline{\tilde{u}_j} = -(\lambda_j-\lambda_{j-1})\left[L_2(\tilde{u}_{j-1},\tilde{\upsilon}_{j-1}) + (1-\lambda)\Delta^2\tilde{u}_{j-1}\right] \\ \overline{\tilde{u}_j}(x,y,0) = 0,\ \ \overline{\tilde{\upsilon}_j}(x,y,0) = 0 \end{cases}$$

By Theorem 2.2, we have:

$$\sup_{0\le t\le T}\left[\left\|\overline{\tilde{u}_j}\right\|^2_{\kappa+2} + \left\|\overline{\tilde{\upsilon}_j}\right\|^2_{\kappa+1}\right] + \int_0^T \left\|\overline{\tilde{u}_j}\right\|_{H\kappa} + \left\|\overline{\tilde{\upsilon}_j}\right\|_{H\kappa} ds \le c\left|(\lambda_J-\lambda_{j-1})\right|^2 \int_0^T \left\|\overline{\tilde{u}_{j-1}}\right\|_{H\kappa} + \left\|\overline{\tilde{\upsilon}_{j-1}}\right\|_{H\kappa} ds \le c\left|(\lambda_J-\lambda_{j-1})\right|^2 K \tag{24}$$

Where:

$$\|\tilde{u}\|_{H\kappa} + \|\tilde{\upsilon}\|_{H\kappa} = \left(\|\tilde{u}\|^2_{\kappa+4} + \|\tilde{u}_x\|^2_2\right) + c(\mu_1 \circ \tilde{u}_x) + c\left(\|\tilde{\upsilon}\|^2_{\kappa+2} + \|\nabla\tilde{\upsilon}\|^2\right) + c(\mu_2 \circ \nabla\tilde{\upsilon})$$

It follows that $(\tilde{u}_j,\tilde{\upsilon}_J)$ is a Cauchy sequence in Y and its limit (u_j,υ_j) is obviously the solution of (Eq. 23). This shows that B is closed in [0, 1].

B is open in [0, 1]: Let $\lambda_0 \in B$ and $\lambda \in [0, 1]$ with $|\lambda-\lambda_0|<\varepsilon$. Let $(\tilde{u}_1,\tilde{v}_1)$ be the solution of system:

$$\begin{cases} \tilde{u}_1' + \lambda_0 L_1(\tilde{u}_1,\tilde{\upsilon}_1) + (1-\lambda_0)\Delta^2\tilde{u}_1 = f \\ \tilde{\upsilon}_1' + \lambda_0 L_2(\tilde{u}_1,\tilde{\upsilon}_1) + (1-\lambda_0)\Delta\tilde{\upsilon}_1 = g \\ \tilde{u}_1(x,y,0) = u_0(x,y),\ \ \tilde{\upsilon}_1(x,y,0) = \upsilon_0(x,y) \end{cases} \tag{25}$$

We now constract a sequence of solutions for the system:

$$\begin{cases} \tilde{u}'_j + \lambda_0 L_1(\tilde{u}_j,\tilde{\upsilon}_j) + (1-\lambda_0)\Delta^2\tilde{u}_j = f + (\lambda_0-\lambda)\left[L_1(\tilde{u}_{j-1},\tilde{\upsilon}_{j-1}) - \Delta^2\tilde{u}_{j-1}\right] \\ \tilde{\upsilon}'_j + \lambda_0 L_2(\tilde{u}_j,\tilde{\upsilon}_j) - (1-\lambda_0)\Delta\tilde{\upsilon}_j = g + (\lambda_0-\lambda)\left[L_2(\tilde{u}_{j-1},\tilde{\upsilon}_{j-1}) - \Delta\tilde{\upsilon}_{j-1}\right] \\ \tilde{u}_j(x,y,0) = u_0(x,y),\ \ \tilde{\upsilon}_j(x,y,0) = \tilde{\upsilon}_0(x,y) \end{cases} \tag{26}$$

As Cai *et al*. (2013) by Theorem 2.2, we have:

$$\sup_{0\le t\le T}\left[\left\|\overline{\tilde{u}_j}\right\|^2_{\kappa+2} + \left\|\overline{\tilde{\upsilon}_j}\right\|^2_{\kappa+1}\right] + \int_0^T \left\|\overline{\tilde{u}_j}\right\|_{H\kappa} + \left\|\overline{\tilde{\upsilon}_j}\right\|_{H\kappa} ds \le c\left|(\lambda-\lambda_0)\right|^2 \int_0^T \left\|\overline{\tilde{u}_{j-1}}\right\|_{H\kappa} + \left\|\overline{\tilde{\upsilon}_{j-1}}\right\|_{H\kappa} ds \le c\varepsilon^2 K \tag{27}$$

Choosing ε smal enought, so that, $c\varepsilon K<1/2$ and $(\tilde{u}_j,\tilde{\upsilon}_J)$ is a Cauchy sequence with limit $(\tilde{u},\tilde{\upsilon})$ being the solution of Eq. 21. Hence, B is open. This completes the proof of Theorem 3.1.

CONCLUSION

Our research methodology or plan includes the following parts. The main theme is to give more informations about the solution for surface waves on multi-layer liquid films. Interactions between externally applied power and dissipation by infinite memory term lead to the associated systems which also makes us use a new mathematical methods in the linear case.

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