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DEA Network Structures Sensitivity to Non-Archimedean Epsilon

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Abstract: The role of ε value in Data Envelopment Analysis (DEA) is obvious for DEA researchers and improperly determining this value leads to unfeasibility/infinity of multiplier form/envelop in DEA problems. Thus, Ali and Seiford in 1993 suggesting a bound for ε claimed that it would prevent unfeasibility/infinity of multiplier forms/envelope of DEA problems. Mehrabian in 2000 rejected the validity of the boundary and offered another boundary for ε in CCR and BCC Models of DEA and further, determined assurance value for under ε special terms. Assigning ε value in network structures may cause encountering unfeasibility/infinity of multiplier forms/envelope challenge. Therefore, the purpose of this study is to determine ε boundary in network structure and to define ε assurance value.

Key words: Network Data Envelopment Analysis (NDEA), Decision Making Units (DMU), epsilon, assurance interval, assurance value

INTRODUCTION

DEA initially introduced by Charnes *et al.* (1978) is an important tool to evaluation demos known as CCR (constant returns to scale). Banker *et al.* (1984) extended it to variable returns to scale known as BCC. These methods evaluate a number of decision-making units in which units consume similar inputs to produce similar outputs. In traditional DEA to measure the relative performance of the unit under evaluation, the system considered as a black box ignoring its internal processes. Such models faced with the challenge of zero weights or close to zero weights; thus, remove some inputs or outputs or the little effect on units' assessment. There are several methods dealing with such a challenge one of them is identifying lower boundary as weights' Epsilon non-Archimedes. However, improper allocation of ε leads to unfeasibility of multiplier form and then the infinity of envelopment form in DEA. In this regard, Mehrabian *et al.* (2000) and Toloo (2014) proposed a boundary for ε in traditional DEA. In real world, most of the evaluated decision-making units include internal processes. The models that stress on internal structures for assessing DMUs called network structure-based DEA models represented by NDEA symbol (Fare and Grosskopf, 1996). Initially used the term network structures. The overall state of the network DEA models are classified into series, parallel and public groups. Structures connected in series are known to model series network structure. One of the most widely used network structures are the two-stage networks of which Kao and Hwang (2008), Fukuyama and Weber (2010) and Tone and Tustsui (2009) research can be pointed out. Parallel network models suggest parallel structures behavior in which divisions connected in parallel mode, Kao (2009) works can be noted in this regard. In real world, most public decision-making units have public network structure, Tone and Tustsui (2009) and Lozano (2011) work on envelopment form and Kao (2009) research on multiplier form can be noted. It seems that in the NDEA models, particular work has not been done to determine ε assurance interval.

The purpose of this study is to introduce a model determining ε value range and ensure the feasibility of multiplier form and consequently the bounded envelope form of public network structures. In what follows, Chapter 2 reviews the literature to determine ε assurance interval and value in traditional DEA model and multiplier form with networked structure. Chapter 3 reviews Linear Programming mathematical model to determine the value of ε assurance interval and assurance value in NDEA. A numerical example is given in Chapter 4 determining the value of ε assurance interval in NDEA. Finally, last chapter concludes the results.

Literature review: As mentioned in introduction, CCR and BCC models face with the unfeasibility/infinite multiplier forms/envelope challenge of DEA by assigning inappropriate values for lower boundary of weights (ε). In this study, to deal with this challenge, methods for determining ε assurance interval and the maximum assurance interval are expressed in traditional DEA. Moreover, multiplier procedures for the public network structures are explained.

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Determining ε assurance interval and confident value in DEA model traditional: In many studies on CCR and BCC in DEA models, both models include non-Archimedes infinitesimal ε. Theoretically, ε introduced as lower boundary preventing zero weights influence assessing all inputs and output units. Some problems in showing infinitesimals arise due to finite tolerance of computer calculations. To solve the problem of zero weights, Ali and Seiford (1993) offered an upper boundary for feasibility of multiplier models and bounded envelope models CCR and BCC as follows:

$$\begin{array}{ll} \text{CCR} & \text{BCC} \\ \text{Max}\, UY_0 & \text{Max}\, UY_0 + U_0 \\ VX_0 = 1 & VX_0 = 1 \\ UY_j - VX_j \leq 0, j = 1,\ldots,J & UY_j - VX_j + U_0 \leq 0, j = 1,\ldots,J \\ U \geq \varepsilon.1, V \geq \varepsilon.1 & U \geq \varepsilon.1, V \geq \varepsilon.1 \end{array} \quad (1)$$

where, 1 is a vector with all 1 components. He provided the following theorem for the upper boundary of ε.

Theorem: For CCR and BCC models when: $(x_0, x_0) = (x_j, x_j)$ $j = 1, \ldots, J$:

- For $\varepsilon < 1/\min_j \{\sum_{i=1}^m x_{ij}\}$, CCR and BCC models have feasible multiplier form and optimized value
- For $\varepsilon > 1/\max_j \{\sum_{i=1}^m x_{ij}\}$, CCR and BCC models are infinite

However, Mehrabian *et al*. (2000) offering a counterexample rejected his claim. They also showed that ε depends on the output viewed values. Then, a model was also provided for finding assurance interval of non-Archimedes ε. To do this, they considered CCR Model (Eq. 1) for DMU_0 evaluation and presented the following model to determine the maximum amount that ε can adopt to ensure the feasibility of multiplier form of CCR:

$$\begin{array}{l} P_0: \text{Max}\, \varepsilon \\ \text{s.t. } VX_0 = 1, \quad UY_j - VX_j \leq 0, j = 1,\ldots,J \\ U \geq \varepsilon \cdot 1, \quad V \geq \varepsilon.1 \end{array} \quad (2)$$

Mehrabian *et al*. (2000) showed that model 2 is feasible and has a finite optimum solution. Besides, they introduced assurance interval for feasibility/bounded multiplier form/envelope form of CCR Model for assessing DMU_0 in term of $(0, \varepsilon_0^*]$ where, ε_0^* is the optimum solution of Eq. 2 introducing sharing of all assurance intervals to evaluate all DMUs as overall assurance interval $\varepsilon^* = \min\{\varepsilon_1^*, \ldots, \varepsilon_J^*\}$; $(0, \varepsilon^*]$. Mehrabian *et al*. (2000) demonstrated that the optimum solution of the following model equals optimum solution of model 2:

$$\begin{array}{l} \overline{P}_0: \text{Max}\, \varepsilon \\ \text{s.t. } VX_0 \leq 1, \quad UY_j - VX_j \leq 0, j = 1,\ldots,J \\ U \geq \varepsilon.1, V \geq \varepsilon.1 \end{array} \quad (3)$$

In addition, ε assurance value obtained from the following model:

$$\begin{array}{l} P': \text{Max}\, \varepsilon \\ \text{s.t. } VX_j \leq 1, \quad j = 1,\ldots,J \\ UY_j - VX_j \leq 0, j = 1,\ldots J \\ U \geq \varepsilon.1, V \geq \varepsilon.1 \end{array} \quad (4)$$

They showed that the absolute amount of ε in model 4 is no larger than the upper boundary of the assurance interval. Moreover, in BCC, the assurance interval and assurance value of non-Archimedes ε was introduced by Toloo (2014). He showed that ε of BCC model is the upper boundary of CCR model ε.

NDEA multiplier models: Many researchers presented multiplier model for network structures, as mentioned in the introduction. Kao (2009) introduced the multiplier model of DEA for network structure extensively for series, parallel and general states and showed how to convert the system with an overall network structure to a series structure in which each process has a parallel structure. Matin and Azizi (2015) introduced the integrated NDEA Model to measure the production systems' performance and showed that the model presented by Kao (2009) is a special case of the presented model. Kao introduced the public network structure by a particular example so that each unit consists of the third divisions (Fig. 1).

In Kao example, system main inputs and outputs are X_1 and X_2 and Y_1, Y_2 and Y_3, respectively. Division 1 consumes some of X_1 and X_2 values for producing Y_1 so that a part of Y_1 remains for division 3. Division 2 consumes a specific value of X_1 and X_2 similar to division 1 and a part of Y_2 for division 3.

Division 3 consumes residual of X_1 and X_2 along side the parts produced Y_1 and Y_2 resulting from divisions 1 and 2 for producing Y_3. Assume that $X_{ij}^{(k)}$ indicates the ith input of division k (k = 1, 2, 3) from DMU. Particularly, total inputs of three Divisions $(X_{ij}^{(1)} + X_{ij}^{(2)} + X_{ij}^{(3)})$ for system input are x_{ij} (j = 1, ..., j, i = 1, 2). It means that $(X_{ij}^{(1)} + X_{ij}^{(2)} + X_{ij}^{(3)})$. The output of division 1 is separated as $Y^{(I)}_1$, $Y^{(0)}_1$ where, $Y^{(0)}_1$ is the system final output and $Y^{(I)}_1$ is the value consumed by division 3 as an input.

Similarly, output of division 2 is $Y^{(I)}_1$, $Y^{(O)}_2$, where, $Y_2^{(o)}$ is the system final output and the consumed amount by division 3. Accordingly, $Y^{(I)}_{rj}+Y^{(0)}_{rj} = y_{rj}$ r = 1, 2; J = 1,..., j. Multiplier model of public network structure are shown in Fig. 1.

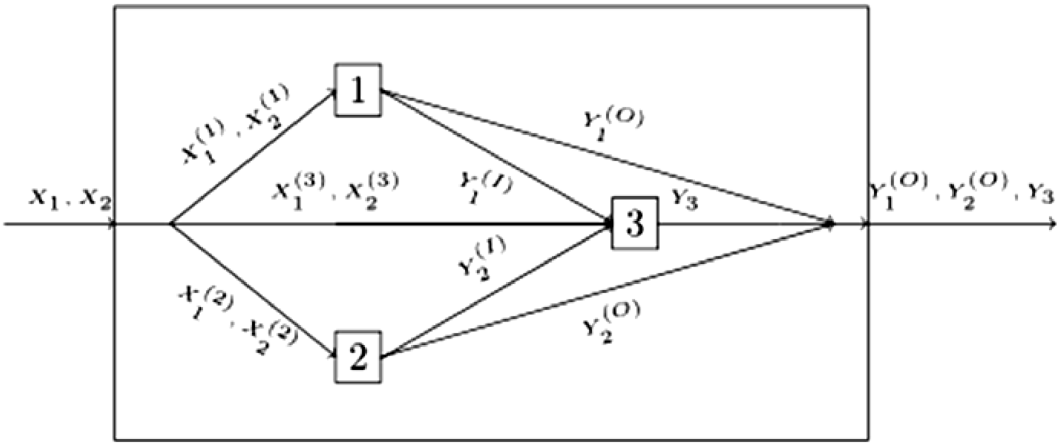

Fig. 1: Network system with three processes

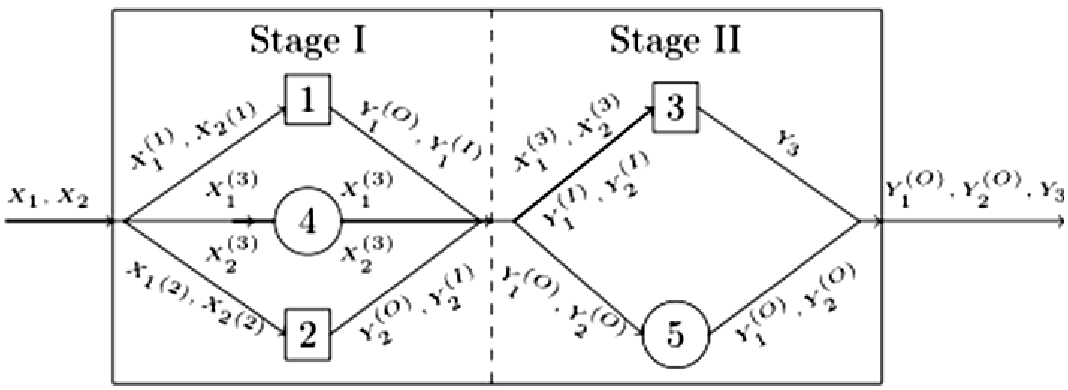

Fig. 2: A two-step network structure, each stage has a parallel structure

$$
\begin{aligned}
&E_0 = \text{Max}\, u_1 Y_{10}^{(0)} + u_2 Y_{20}^{(0)} + u_3 Y_{3o}\\
&\text{s.t. } v_1 x_{10} + v_2 x_{20} = 1\\
&\left(u_1 Y_{1j}^{(o)} + u_2 Y_{2j}^{(o)} + u_3 Y_{3j}\right) - \left(v_1 X_{1j} + v_2 X_{2j}\right) \le 0,\ \forall j\\
&u_1 Y_{1j} - \left(v_1 X_{1j}^{(1)} + v_2 X_{2j}^{(1)}\right) \le 0,\ \forall j \qquad\qquad 5)\\
&u_2 Y_{2j} - \left(v_1 X_{1j}^{(2)} + v_2 X_{2j}^{(2)}\right) \le 0,\ \forall j\\
&u_3 Y_{3j} - \left(v_1 X_{1j}^{(3)} + v_2 X_{2j}^{(3)} + u_1 Y_{1j}^{(1)} + u_2 Y_{2j}^{(1)}\right) \le 0,\ \forall j\\
&u_1, u_2, u_3, v_1, v_2 \ge \varepsilon
\end{aligned}
$$

where, u_r indicates the allocated weight to rth output (r = 1, 2, 3) and v_i is the allocated weight to the ith input (i = 1, 2) used for measuring system performance DMU_0 of each process. As observed in model 5, X_1 input weight is always v_1 no matter to be used by division 1 for $X^{(1j)}_{ij}$ input, division 2 as $X^{(2)}_{1j}$ or division 3 as $X^{(3)}_{1j}$ or that Y_1 output weight is always u_1 a no matter to be used by division 3 as input or to be the final output of the system. Other indices are in a similar condition. Kao also revealed that every public network structure could be converted into a two-step network structure through introducing dummy Divisions, where each stage has a parallel structure. When synthetic Divisions 4 and 5 presented as Fig. 1 structure, we will have (Fig. 2).

According to Fig. 1 and 2, division performance point and procedures obtained from following relationships:

$$
\begin{aligned}
&E_o^{(1)} = \frac{u_1^* Y_{10}}{v_1^* X_{1o}^{(1)} + v_2^* X_{2o}^{(1)}}\\
&E_o^{(2)} = \frac{u_2^* Y_{20}}{v_1^* X_{1o}^{(2)} + v_2^* X_{2o}^{(2)}}\\
&E_o^{(3)} = u_3^* Y_{3o} \,/\, \left(v_1^* X_{1o}^{(3)} + v_2^* X_{2o}^{(3)} + u_1^* Y_{1o}^{(1)} + u_2^* Y_{2o}^{(1)}\right) \qquad (6)\\
&E_o^{I} = \frac{u_1^* Y_{10} + \left(v_1^* X_{10}^{(3)} + v_2^* X_{20}^{(3)}\right) + u_2^* Y_{20}}{v_1^* X_{10} + v_2^* X_{20}}\\
&E_o^{II} = \frac{u_1^* Y_{10}^{(0)} + u_2^* Y_{20}^{0} + u_3^* Y_{30}}{u_1^* Y_{10} + (v_1^* X_{10}^{(3)} + v_{20}^{*3}) + u_2^* Y_{20}}
\end{aligned}
$$

So that according to Eq. 6, the total performance is equal to the product functionality of two stages, in other words:

$$E_O = E_O^{I} \times E_O^{II}$$

MATERIALS AND METHODS

Determining the assurance interval and an assurance value of ε in network DEA models (NDEA): As mentioned in introduction, presenting inappropriate ε unfeasibility/infinite multiplier forms/envelope create challenges in CCR and BCC models of traditional DEA. Similar to traditional DEA states, it is also possible to face with this challenge in NDEA models. The introduction of the feasibility ε assurance interval for multiple models NDEA-CCR which covers bounded form NDAE_CCR, seems necessary. Therefore, it is necessary to introduce assurance interval ε for feasibility of multiplier models of NDEA-CCR and subsequently bounded envelope form NDAE _CCR.

$$\begin{aligned}
&P_o = \text{Max } \varepsilon \\
&\text{s.t. } v_1 X_{1o} + v_2 X_{2o} = 1 \\
&\left(u_1 Y_{1j}^{(0)} + u_2 Y_{2i}^{(0)} + u_3 Y_{3j}\right) - \left(v_1 X_{1j} + v_2 X_{2j}\right) \le 0, \ \forall j \\
&u_1 Y_{1j} - \left(v_1 X_{1j}^{(1)} + v_2 X_{2j}^{(1)}\right) \le 0, \ \forall j \\
&u_2 Y_{2j} - \left(v_1 X_{1j}^{(2)} + v_2 X_{2j}^{(2)}\right) \le 0, \ \forall j \\
&u_3 Y_{3j} - \left(v_1 X_{1j}^{(3)} + v_2 X_{2j}^{(3)} + u_1 Y_{1j}^{(1)} + u_2 Y_{2j}^{(1)}\right) \le 0, \ \forall j \\
&u_1, u_2, u_3, v_1, v_2 \ge \varepsilon
\end{aligned} \tag{7}$$

Therefore, this study seeks to find assurance interval and its assurance value for ε to ensure the feasibility of multiplier form and bounded envelope form NDAE_CCR. To find ε upper boundary (assurance interval) in model NDEA-CCT for DMU under evaluation, the following model is applied. On the other hand:

$$(\varepsilon, U, V) = \left(0, 0, \frac{1}{\sum_i X_{i0}} \cdot 1\right)$$

is a feasible solution for model 7 in which 1 is the vector with all scalar components, meaning that model 7 is feasible. Now, we show that model 7 has finite optimum solution.

Lemma 1: Prove the optimum value of model 7 is positive and finite.

Proof: First, prove that model 7 is finite. For this purpose, assume that (ε, u, v) is a feasible solution for model 7.

Then, $v \ge \varepsilon.1$. On the other hand, non-negative inputs values results in $vX_0 \ge \varepsilon 1 X_0$. If the first constraint of model 7 is applied, we will have $\varepsilon \le 1/(X_{10}+X_{20})$. Besides, non-negative outputs and constraint $U \ge \varepsilon 1$ will result $UY_0 \ge \varepsilon 1 Y_0$ Moreover, according to constraints of the second series in model 7, inequality obtained and according to constraints of the second series of model 7, we get inequality $\varepsilon \le 1/(Y_{10}^{(o)}+Y_{20}^{(o)}+Y_{30})$. Similarly, the inequalities $\varepsilon \le 1/Y_{10}$ and $\varepsilon \le 1/Y_{20}$ achieved considering divisions constraints. Therefore:

$$\varepsilon \le \min\left\{\frac{1}{X_{10} + X_{20}}, \frac{1}{Y_{10}^{o} + Y_{20}^{o} + Y_{30}}, \frac{1}{Y_{10}}, \frac{1}{Y_{20}}\right\}$$

Now, we show that model 7 has an optimum positive solution. Considering that:

$$(\varepsilon, U, V) = \left(0, 0, \frac{1}{\sum_i X_{i0}} \cdot 1\right)$$

is a feasible solution for model 7, optimum solution of model 7 is non-negative. Therefore, it required to show that the optimum value of model 7 could not be zero. Assume that (assumption by contradiction) the optimum value of model 7 is zero; then, consider dual model 7:

$$\begin{aligned}
&D_o: \min \theta \\
&\text{s.t. } w_1 + w_2 + z_1 + z_2 + z_3 = 1 \\
&X_{10}\theta - \sum_{j=1}^{J} X_{1j}\alpha_j - \sum_{j=1}^{J} X_{1j}^{(1)}\beta_j - \sum_{j=1}^{J} X_{1j}^{(2)}\gamma_j - \sum_{j=1}^{J} X_{1j}^{(3)}\rho_j - w_1 = 0 \\
&X_{20}\theta - \sum_{j=1}^{J} X_{2j}\alpha_j - \sum_{j=1}^{J} X_{2j}^{(1)}\beta_j - \sum_{j=1}^{J} X_{2j}^{(2)}\gamma_j - \sum_{j=1}^{J} X_{2j}^{(3)}\rho_j - w_2 = 0 \\
&\sum_{j=1}^{J} Y_{1j}^{(0)}\alpha_j + \sum_{j=1}^{J} Y_{1j}\beta_j - z_1 = 0 \\
&\sum_{j=1}^{J} Y_{2j}^{(0)}\alpha_j + \sum_{j=1}^{J} Y_{2j}\gamma_j - z_2 = 0 \\
&\sum_{j=1}^{J} Y_{3j}^{(0)}\alpha_j + \sum_{j=1}^{J} Y_{3j}\rho_j - z_3 = 0 \\
&\alpha_j, \beta_j, \gamma_j, \rho_j, w_1, w_2, z_1, z_2, z_3 \ge 0, \theta_{\text{free}}, j = 1, \ldots, J
\end{aligned} \tag{8}$$

Considering the optimum values of primal and dual models, we have: θ = 0. Substituting θ in the constraints of the second and third constraints of dual models, we have $\alpha_j = \beta_j = y_j = w_1 = w_2 = 0$; further, substituting results in the next dual constraints $z_1 = z_2 = z_3 = 0$ achieved. The obtained results are on the contrary to the first constraint of the dual problem. This proves that contradiction assumption is invalid. Thus, the optimum value of primal problem is positive. Now, the following definitions determine assurance interval and assurance value in NDEA-CCR.

Definition 1: Assurance interval for feasibility of multiplier form NDEA-CCR to evaluation DMU0 is defined as $(0, \varepsilon^*]$ where ε^* is the optimal value of model 7.

Definition 2: The intersection of all assurance interval for feasibility of multiplier form NDEA-CCRfor the evaluation of all DMUs defines the overall assurance interval $(0, \varepsilon^*]$ with $\varepsilon^* = \min\{\varepsilon_1^*, \varepsilon_J^*\}$. According to Definition 2, to determine the overall assurance interval, J linear programming model should be resolved. If the number of inputs, outputs and DMUs were large, the number of operations for finding the above assurance interval would be extremely high. To fix this problem, linear programming will be introduced. Introducing the proposed model to determine the wide assurance interval, only a linear programming model is solved that is important in terms of computing. Introducing proposed model, following cases are necessary:

Theorem 1: The optimum solution of model 7 is similar to the following optimum one:

$$\begin{aligned}
&\bar{P}_o = \text{Max } \varepsilon, \quad \text{s.t. } v_1X_{10} + v_2X_{20} \le 1,\\
&\left(u_1Y_{1j}^{(o)} + u_2Y_{2j}^{(0)} + u_3Y_{3j}\right) - \left(v_1X_{1j} + v_2X_{2j}\right) \le 0,\ \forall j\\
&u_1Y_{1j} - \left(v_1X_{1j}^{(1)} + v_2X_{2j}^{(1)}\right) \le 0,\ \forall j\\
&u_2Y_{2j} - \left(v_1X_{1j}^{(2)} + v_2X_{2j}^{(2)}\right) \le 0,\ \forall j\\
&u_3Y_{3j} - \begin{pmatrix} v_1X_{1j}^{(3)} + v_2X_{2j}^{(3)} + \\ u_1Y_{1j}^{(1)} + u_2Y_{2j}^{(1)} \end{pmatrix} \le 0,\ \forall j\\
&u_1, u_2, u_3, v_1, v_2 \ge \varepsilon
\end{aligned} \tag{9}$$

Proof: Consider the dual of model 9 as follows:

$$\begin{aligned}
&D_o : \min \theta\\
&\text{s.t. } w_1 + w_2 + z_1 + z_2 + z_3 = 1\\
&X_{10}\theta - \sum_{j=1}^{J} X_{1j}\alpha_j - \sum_{j=1}^{J} X_{1j}^{(1)}\beta_j - \sum_{j=1}^{J} X_{1j}^{(2)}\gamma_j - \sum_{j=1}^{J} X_{1j}^{(3)}\rho_j - w_1 = 0\\
&X_{20}\theta - \sum_{j=1}^{J} X_{2j}\alpha_j - \sum_{j=1}^{J} X_{2j}^{(1)}\beta_j - \sum_{j=1}^{J} X_{2j}^{(2)}\gamma_j - \sum_{j=1}^{J} X_{2j}^{(3)}\rho_j - w_2 = 0\\
&\sum_{j=1}^{J} Y_{1j}^{(0)}\alpha_j + \sum_{j=j1}^{J} Y_{1j}\beta_j - z_1 = 0, \qquad \sum_{j=1}^{J} Y_{2j}^{(0)}\alpha_j + \sum_{j=j1}^{J} Y_{2j}\gamma_j - z_2 = 0\\
&\sum_{j=1}^{J} Y_{3j}^{(0)}\alpha_j + \sum_{j=j1}^{J} Y_{3j}\rho_j - z_3 = 0\\
&\alpha_j, \beta_j, \gamma_j, \rho_j, w_1, w_2, z_1, z_2, z_3, \theta \ge 0,\ j = 1, \dots, J
\end{aligned} \tag{10}$$

According to Lemma 1, we have $\theta \ge 0$. Then, the feasible space of model 8 is as feasible space of model 10. Therefore, models 8 and 10 have optimal solutions and identical optimal values; and the identical optimum solutions and optimum values result in models 7 and 9.

The proposed model: When $(\varepsilon_0^*, u_0^*, v_0^*)$ is the optimum solution of model, to specify the overall assurance interval according to definition 2, $\varepsilon^* = \min\{\varepsilon_0^*, 0 = 1, \dots, J\}$ J models solved requiring higher computations. In this part, the following model introduced for determining the overall assurance interval that eliminates the problem of much computation:

$$\begin{aligned}
&\dot{P}_o = \text{Max } \varepsilon\\
&\text{s.t. } v_1X_{1j} + v_2X_{2j} \le 1,\ \forall j\\
&\left(u_1Y_{1j}^{(o)} + u_2Y_{2j}^{(0)} + u_3Y_{3j}\right) - \left(v_1X_{1j} + v_2X_{2j}\right) \le 0,\ \forall j\\
&u_1Y_{1j} - \left(v_1X_{1j}^{(1)} + v_2X_{2j}^{(1)}\right) \le 0,\ \forall j\\
&u_2Y_{2j} - \left(v_1X_{1j}^{(2)} + v_2X_{2j}^{(2)}\right) \le 0,\ \forall j\\
&u_3Y_{3j} - \begin{pmatrix} v_1X_{1j}^{(3)} + v_2X_{2j}^{(3)} + \\ u_1Y_{1j}^{(1)} + u_2Y_{2j}^{(1)} \end{pmatrix} \le 0,\ \forall j\\
&u_1, u_2, u_3, v_1, v_2 \ge \varepsilon
\end{aligned} \tag{11}$$

In what follows, the above proposed method is represented with NDEA_PZ. Assume that $(\varepsilon^{**}, U^{**}, V^{**})$, is the optimum solution for model NDEA_PZ; then, we have $\varepsilon^{**} \le \varepsilon^*$.

Theorem 2: The optimum value for model NDEA_PZ is always positive.

Proof: Proof using reductio ad absurdum is performed on the dual problem of the model NDEA_PZ similar to Lemma1.

Theorem 3: Considering the presented symbols of ε^* and ε^{**} and single-member set as $\bar{J} = \{j : v_1X_{1j} + v_2X_{2j} = 1\} = \{t\}$, we have $\varepsilon^{**} = \varepsilon^*$.

Proof: Consider the model NDEA_PZ dual as follows:

$$\begin{aligned}
&\bar{D}_o : \min \sum_{j=1}^{J} \theta_j, \quad \text{s.t. } w_1 + w_2 + z_1 + z_2 + z_3 = 1\\
&X_{10}\theta - \sum_{j=1}^{J} X_{1j}\alpha_j - \sum_{j=j1}^{J} X_{1j}^{(1)}\beta_j - \sum_{j=j1}^{J} X_{1j}^{(2)}\gamma_j - \sum_{j=j1}^{J} X_{1j}^{(3)}\rho_j - w_1 = 0\\
&X_{20}\theta - \sum_{j=1}^{J} X_{2j}\alpha_j - \sum_{j=1}^{J} X_{2j}^{(1)}\beta_j - \sum_{j=j1}^{J} X_{2j}^{(2)}\gamma_j - \sum_{j=j1}^{J} X_{2j}^{(3)}\rho_j - w_2 = 0\\
&\sum_{j=1}^{J} Y_{1j}^{(0)}\alpha_j + \sum_{j=j1}^{J} Y_{1j}\beta_j - z_1 = 0\\
&\sum_{j=1}^{J} Y_{2j}^{(0)}\alpha_j + \sum_{j=j1}^{J} Y_{2j}\gamma_j - z_2 = 0\\
&\sum_{j=1}^{J} Y_{3j}\alpha_j + \sum_{j=1}^{J} Y_{3j}\rho_j - z_3 = 0\\
&\alpha_j, \beta_j, \gamma_j, \rho_j, w_1, w_2, z_1, z_2, z_3, \theta_j \ge 0,\ j = 1, \dots, J
\end{aligned} \tag{12}$$

Assume that $j \neq t$, $v_1X_{1j}+v_2X_{2j}<1$, $j = 1,..., J$, according to theorem of complementary slackness in the optimumsolution of model $j \neq t$, $\theta_j^0 = 0$, $j = 1,..., j$, $j \neq t$. Thus, θ_t^o is the optimum value of the following problem:

$$D_o : \min \theta_t, \text{ s.t. } w_1 + w_2 + z_1 + z_2 + z_3 = 1$$
$$X_{10}\theta - \sum_{j=1}^{J} X_{1j}\alpha_j - \sum_{j=1}^{J} X_{1j}^{(1)}\beta_j - \sum_{j=1}^{J} X_{1j}^{(2)}\gamma_j - \sum_{j=1}^{J} X_{1j}^{(3)}\rho_j - w_1 = 0$$
$$X_{20}\theta - \sum_{j=1}^{J} X_{2j}\alpha_j - \sum_{j=1}^{J} X_{2j}^{(1)}\beta_j - \sum_{j=1}^{J} X_{2j}^{(2)}\gamma_j - \sum_{j=1}^{J} X_{2j}^{(3)}\rho_j - w_2 = 0$$
$$\sum_{j=1}^{J} Y_{1j}^{(0)}\alpha_j + \sum_{j=1}^{J} Y_{1j}\beta_j - z_1 = 0$$
$$\sum_{j=1}^{J} Y_{2j}^{(0)}\alpha_j + \sum_{j=j1}^{J} Y_{2j}\gamma_j - z_2 = 0$$
$$\sum_{j=1}^{J} Y_{3j}\alpha_j + \sum_{j=1}^{J} Y_{3j}\rho_j - z_3 = 0$$
$$\alpha_j, \beta_j, \gamma_j, \rho_j, w_1, w_2, z_1, z_2, z_3, \theta_t \geq 0, \; j = 1,...,J \tag{13}$$

Moreover, this problem is the dual of model 9 in assessing DMU_t with the optimum value ε_t^*. Therefore, $\varepsilon^{**} = \varepsilon^*$.

For finding the greatest non-Archimedes, ε value for the feasibility of multiplier model or in other words, model NDEA_PZ is applied for the overall assurance interval for the problem NDEA-CCR.

RESULTS AND DISCUSSION

This study presents numerical examples of the overall assurance value impact on the performance of decision-making units. For this purpose, ε value calculated using NDEA_PZ. Then, Kao's model implemented in two cases without value and with the overall assurance value on decision-making units data. Next, the performance results of these two cases compared with CCR performance value. Finally, the results of the two-state performance compared to CCR traditional performance.

Example 1: Consider five decision-making units A, B, C, D, E with three divisions related to Kao example (Kao, 2009), the structure of each decision-making unit shown in Fig. 1. The corresponding inputs/outputs are listed in Table 1.

According to the data shown in Table 1, implementing model NDEA_PZ the overall assurance value will be equal . Thus, the overall assurance interval is (0, 0.0344828). Table 2 shows the results of Kao (2009) model in two modes without value and with overall assurance value alongside the traditional CCR Model

As Table 2 shows, using ε assurance value in CCR Model, Unit A converts from efficient mode to inefficient mode and the performance scores of units C, B dropped. In addition, using the assurance value ε in NDEA-CCR Model, the performance scores of all units reduced; though, ratings of units are still constant.

Example 2: Consider the example of 24 Taiwanese insurance companies extracted from Kao (2008) study in which the structure of each is similar to Fig. 3. Inputs/outputs of insurance companies are listed in Table 2.

Implementing model NDEA_PZ on data of Table 3 gives us the overall assurance interval $\varepsilon^{**} = 1.04573\times10^{-8}$ Thus, the overall assurance interval is (0, 0.0000000104573] Kao implementation results are listed (2008) in the following two modes:

Table 1: Input/output data of Kao example in the year 2009

DMU	Process	X_1	X_2	$Y_1^{(0)}$	$Y_1^{(1)}$	$Y_2^{(0)}$	$Y_2^{(1)}$	Y_3
A		11	14	2	-	2	-	1
	1	3	5	*2*	*2*	-	-	-
	2	*4*	*3*	-	-	*2*	*1*	-
	3	*4*	*6*	-	*2*	-	*1*	*1*
B		7	7	1	-	1	-	1
	1	2	3	1	1	-	-	-
	2	2	3	-	-	1	1	-
	3	3	1	-	1	-	1	1
C		11	14	1	-	1	-	2
	1	3	4	1	1	-	-	-
	2	5	3	-	-	1	1	-
	3	3	7	-	1	-	1	2
D		14	14	2	-	3	-	1
	1	4	6	2	1	-	-	-
	2	5	5	-	-	3	1	-
	3	5	3	-	1	-	1	1
E		14	15	3	-	2	-	3
	1	5	6	3	1	-	-	-
	2	5	4	-	-	2	2	-
	3	4	5	-	1	-	2	3

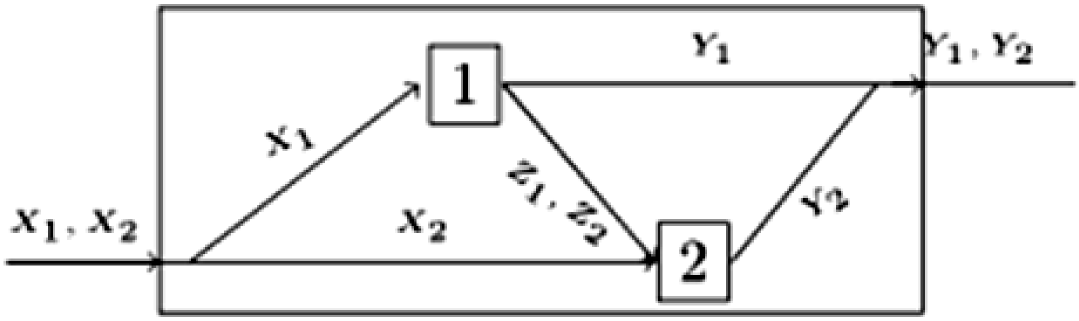

Fig. 3: Network structure of the insuarance operation system

Table 2: Comparing 5- DMU system performances independently calculated through ordinary Model CCR, Kao model and the model presented here

DMU	E CCR	E CCR $^{\epsilon}$	EN kao	EN $^{\epsilon}$
A	1.0000	0.9266	0.5227	0.4744
B	0.8980	0.8832	0.5952	0.5895
C	0.8485	0.7377	0.5682	0.5209
D	1.0000	1.0000	0.4821	0.4706
E	1.0000	1.0000	0.8000	0.7931

Table 3: Input/output table of Kao, case study: Taiwanese insurance companies in 2008

DMU	Operation expenses (X1)	Insurance expenses (X2)	Direct writtenpremiums (Z1)	Reinsurancepremiums (Z2)	Underwritingprofit (Y1)	Investmentprofit (Y2)
DMU1	1,178,744	673,512	7,451,757	856,735	984,143	681,687
DMU2	1,381,822	1,352,755	10,020,274	1,812,894	1,228,502	834,754
DMU3	1,177,494	592,790	4,776,548	560,244	293,613	658,428
DMU4	601,320	594,259	3,174,851	371,863	248,709	177,331
DMU5	6,699,063	3,531,614	37,392,862	1,753,794	7,851,229	3,925,272
DMU6	2,627,707	668,363	9,747,908	952,326	1,713,598	415,058
DMU7	1,942,833	1,443,100	10,685,457	643,412	2,239,593	439,039
DMU8	3,789,001	1,873,530	17,267,266	1,134,600	3,899,530	622,868
DMU9	1,567,746	950,432	11,473,162	546,337	1,043,778	264,098
DMU10	1,303,249	1,298,470	8,210,389	504,528	1,697,941	554,806
DMU11	1,962,448	672,414	7,222,378	643,178	1,486,014	18,259
DMU12	2,592,790	650,952	9,434,406	1,118,489	1,574,191	909,295
DMU13	2,609,941	1,368,802	13,921,464	811,343	3,609,236	223,047
DMU14	1,396,002	988,888	7,396,396	465,509	1,401,200	332,283
DMU15	2,184,944	651,063	10,422,297	749,893	3,355,197	555,482
DMU16	1,211,716	415,071	5,606,013	402,881	854,054	197,947
DMU17	1,453,797	1,085,019	7,695,461	342,489	3,144,484	371,984
DMU18	757,515	547,997	3,631,484	995,620	692,731	163,927
DMU19	159,422	182,338	1,141,950	483,291	519,121	46,857
DMU20	145,442	53,518	316,829	131,920	355,624	26,537
DMU21	84,171	26,224	225,888	40,542	51,950	6491
DMU22	15,993	10,502	52,063	14,574	82,141	4181
DMU23	54,693	28,408	245,910	49,864	0.1	18,980
DMU24	163,297	235,094	476,419	644,816	142,370	16,976
Mean	1,544,215	828,963	7,832,893	667,964	1,602,873	477,733

Table 4: Comparing the efficiencies of 24 insurance companies in Thailand independently calculated through ordinary CCR, Kao Models and the model presented here

DMU	E CCR	E CCR $^{\epsilon}$	EN kao	Rank Kao	EN $^{\epsilon}$	Rank NDEA PZ
DMU1	0.984	0.978	0.962	4.0	0.913	2
DMU2	1.000	1.000	1.000	1.5	0.805	5
DMU3	0.988	0.970	0.936	5.0	0.894	3
DMU4	0.488	0.488	0.488	11.0	0.450	12
DMU5	1.000	1.000	0.979	3.0	0.599	8
DMU6	0.594	0.588	0.390	15.0	0.403	14
DMU7	0.470	0.467	0.374	17.0	0.325	17
DMU8	0.415	0.415	0.295	20.0	0.293	20
DMU9	0.327	0.327	0.280	22.0	0.262	23
DMU10	0.781	0.772	0.705	9.0	0.582	9

Table 4: Continue

DMU	E CCR	E CCR $^{\varepsilon}$	EN kao	Rank Kao	EN $^{\varepsilon}$	Rank NDEA PZ
DMU11	0.283	0.277	0.283	21.0	0.266	22
DMU12	1.000	1.000	0.714	8.0	0.711	7
DMU13	0.353	0.351	0.337	18.0	0.320	18
DMU14	0.470	0.468	0.394	14.0	0.362	15
DMU15	0.979	0.972	0.737	7.0	0.729	6
DMU16	0.472	0.472	0.321	19.0	0.320	19
DMU17	0.635	0.633	0.427	13.0	0.420	13
DMU18	0.427	0.426	0.385	16.0	0.345	16
DMU19	0.822	0.821	0.487	12.0	0.480	11
DMU20	0.935	0.934	0.850	6.0	0.848	4
DMU21	0.333	0.333	0.268	23.0	0.268	21
DMU22	1.000	1.000	1.000	1.5	1.000	1
DMU23	0.599	0.598	0.580	10.0	0.579	10
DMU24	0.257	0.256	0.172	24.0	0.167	24

- Without any value
- With overall assurance value

In this table, the second, third, fourth and sixth columns are results of implementing traditional CCR_ ε models without overall assurance value, traditional CCR_ ε with overall assurance value, lattice without an overall assurance value and lattice with the overall assurance value obtained from model NDEA_PZ. The fifth and seventh columns are units' ratings in lattice models without overall assurance value and with overall assurance value.

The results of Table 4 show that applying an overall assurance value changes the efficiency of decision-making units. According to the fifth column, Kaomodel without overall assurance value of decision-making units will not be thoroughly rated. While, the seventh column suggests that applying an overall assurance value led to full rating of decision-making units.

CONCLUSION

Network DEA models are sensitive to selecting the amount of ε. Therefore, inappropriate selection of ε leads to unfeasibility of multiplier form NDEA-CCR or underestimating the inputs/outputs. Thus, it requires precisely selecting ε value in network DEA models. Besides, to avoid finite tolerance in computerized calculations, it is recommended that in addition to using double precision, ε value calculated by model NDEA-PZ.

REFERENCES

Ali, A.I. and L.M. Seiford, 1993. Computational accuracy and infinitesimals in data envelopment analysis. INFOR: Inf. Syst. Oper. Res., 31: 290-297.

Banker, R.D., A. Charnes and W.W. Cooper, 1984. Some models for estimating technical and scale Inefficiencies in data envelopment analysis. Manage. Sci., 30: 1078-1092.

Charnes, A., W.W. Cooper and E. Rhodes, 1978. Measuring the efficiency of decision making units. Eur. J. Operat. Res., 2: 429-444.

Fare, R. and S. Grosskopf, 1996. Productivity and intermediate products: A frontier approach. Econ. Lett., 50: 65-70.

Fukuyama, H. and W.L. Weber, 2010. A slacks-based inefficiency measure for a two-stage system with bad outputs. Omega, 38: 398-409.

Kao, C. and S.N. Hwang, 2008. Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan. Eur. J. Operat. Res., 185: 418-429.

Kao, C., 2009. Efficiency decomposition in network data envelopment analysis: A relational model. Eur. J. Oper. Res., 192: 949-962.

Lozano, S., 2011. Scale and cost efficiency analysis of networks of processes. Expert Syst. Appl., 38: 6612-6617.

Matin, R.K. and R. Azizi, 2015. A unified network-DEA model for performance measurement of production systems. Meas., 60: 86-193.

Mehrabian, S., G.R. Jahanshahloo, M.R. Alirezaee and G.R. Amin, 2000. An assurance interval for the non-archimedean epsilon in DEA models. Oper. Res., 48: 344-347.

Toloo, M., 2014. An epsilon-free approach for finding the most efficient unit in DEA. Appl. Math. Modell., 38: 3182-3192.

Tone, K. and M. Tsutsui, 2009. Network DEA: A slacks-based measure approach. Eur. J. Oper. Res., 197: 243-252.