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Analytical Methods for Solving Fuzzy Fredholm Integral Equation of the Second Kind

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Abstract: In this study some analytical methods, namely; the homotopy analysis method and the Adomain decomposition method have been investigated and implemented to find an approximate solutions to a fuzzy Fredholm integral equation of the second kind. Consequently, converting a linear fuzzy Fredholm integral equation of the second kind to a linear system of integral equations of the second kind in crisp case. To test the efficiency of these proposed methods we consider test example. The approximate solutions have shown to be in a closed agreement with the exact solutions.

Key words: Fuzzy function, fuzzy integral, homotopy analysis, adomain decomposition, kind

INTRODUCTION

Fuzzy integral equations have attracted the attention of many scientists and researchers in recent years, due to their wide range of applications such as fuzzy control, fuzzy finance, approximate reasoning and economic systems, etc.

The concept of integration of fuzzy functions was first introduced by Dubois and Prade (1982). Then alternative approaches were later suggested by Goetschel and Voxman (1986), Kaleva (1987), Nanda (1989) and others. While Goetschel and Voxman (1986) and later Matloka, preferred a Riemann integral type approach, Kaleva (1987), choose to define the integral of fuzzy function using the Lebesgue type concept for integration. Park *et al*. (1995) have considered the existence of solution of fuzzy integral equation in Banach space. Cong and Ming (1991) investigated the fuzzy Fredholm integral equation of the second kind which is one of the first applications of fuzzy integration. Recently, Babolian *et al*. (2005) have used the Adomain Decomposition Method (ADM) to solve linear Fredholm fuzzy integral equations of the second kind. Liao in 1992 introduced some basic ideas of homotopy in topology to propose general analytic method for nonlinear problems, namely, the method (HAM) (Liao, 2004; 2009). Since then, the homotopy analysis method has been used to obtain solutions to wide class of deterministic direct and inverse problems.

Due to the complexity of solving fuzzy Fredholm integral equations analytically, numerical methods have been proposed. For instance, Maleknejad *et al*. (2010), solved the first kind Fredholm integral equation by using the sinc function. Parandin and Araghi (2010), established a method to approximate the solution using finite and divided differences methods. Jafarzadeh (2012), solved linear fuzzy Fredholm integral equation with Upper-bound on error by Splinder's Interpolation. Altaie (2012), used Bernstein piecewise polynomial. Parandin and Araghi (2009), proposed the approximate solution by using an iterative interpolation.

Lotfi and Mahdiani (2011), used fuzzy Galerkin method with error analysis. Attari and Yazdani (2011), studied the application of Homotopy perturbation method. Mirzaee *et al*. (2012), presented direct method using triangular functions.

Goghary and Goghary (2006), found an approximate solution for a system of linear fuzzy Fredholm integral equation of the second kind with two variables which exploit hybrid Legendre and block-pulse functions and Legendre wavelets. Ziari *et al*. (2012), Abbasbandy and Shivanian (2011) used fuzzy Haar Wavelet. Ghanbari *et al*. (2009), presented a numerical method based on Block-Pulse Functions (BPFs). Amawi (2014) has investigated some analytical and numerical solutions for the fuzzy Fredholm integral equation of the second kind.

In this study some analytical methods, namely; the homotopy analysis method and the Adomain decomposition method have been investigated to solve fuzzy Fredholm integral equation of the second kind. Using the parametric form of fuzzy numbers, the fuzzy linear Fredholm integral equation of the second kind can be converted to a linear system of Fredholm integral equations of the second kind in the crisp case.

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MATERIALS AND METHODS

Fuzzy Fredholm integral equation: A standard form of the Fredholm integral equation of the second kind is given by Jafarian *et al*. (2012):

$$g(t) = f(t) + \lambda\int_a^b k(s,t)g(s)ds \quad (1)$$

Where:

- λ = A positive parameter
- $k(s, t)$ = A function called the kernel of the integral equation defined over the square G: $[a, b]\times[a, b]$
- $f(t)$ = A given function of $t\in[a, b]$

Now, if f(t) is a crisp function then (Eq. 1) possess crisp solution and the solution is fuzzy if f(t) is a fuzzy function. We introduce parametric form of a fuzzy Fredholm integral equation of the second kind. Let $(\underline{f}(t,r),\overline{f}(t,r))$ and $(\underline{g}(t,r),\overline{g}(t,r))$, $0\le r\le 1$ and $t\in[a, b]$ are parametric forms of f(t) and g(t), respectively then the parametric form of fuzzy Fredholm integral equation of the second kind is as follows:

$$\begin{aligned} \underline{g}(t,r) &= \underline{f}(t,r) + \lambda\int_a^b \underline{U}(s,r)ds \\ \overline{g}(t,r) &= \overline{f}(t,r) + \lambda\int_a^b\int_a^b \overline{U}(s,r)ds \end{aligned} \quad (2)$$

Where:

$$\underline{U}(s,r) = \begin{cases} k(s,t)\underline{g}(s,r), & k(s,t)\ge 0 \\ k(s,t)\overline{g}(s,r), & k(s,t)<0 \end{cases} \quad (3)$$

and:

$$\overline{U}(s,r) = \begin{cases} k(s,t)\overline{g}(s,r), & k(s,t)\ge 0 \\ k(s,t)\underline{g}(s,r), & k(s,t)<0 \end{cases} \quad (4)$$

for each $0\le r\le 1$ and $a\le s, t\le b$. We can see that (Eq. 2) is a crisp system of linear Fredholm integral equations for each $0\le r\le 1$ and $a\le t\le b$.

Definition (1) (Keyanpour and Akbarian, 2011): The fuzzy Fredholm integral equations system of the second kind is of the form:

$$g_i(t) = f_i(t) + \sum_{j=1}^{m}\left(\lambda_{ij}\int_a^b k_{ij}(s,t)g_j(s)ds\right), \quad i = 1, \ldots, m \quad (5)$$

where, s, t, λ are real constants and s, $t\in[a, b]$, $\lambda_{ij}\neq 0$ for i, j = 1, ..., m. In system (Eq. 5), $g(t) = [g_1(t), ..., g_m(t)]^T$ is a unknown function. Moreover, the fuzzy function $f_i(t)$ and kernel $k_{ij}(s, t)$ are known and assumed to be sufficiently differentiable functions with respect to all their arguments on the interval [a, b]. Now, let the parametric forms of $f_i(t)$ and $g_i(t)$ are $\underline{f}_i(t,r),\overline{f}_i(t,r)$ and $\underline{g}_i(t,r),\overline{g}_i(t,r)$, $0\le r\le 1$, $t\in[a, b]$, respectively. We write the parametric form of the given fuzzy Fredholm integral equations system as follows:

$$\begin{cases} \underline{g}_i(t,r) = \underline{f}_i(t,r) + \sum_{j=1}^{m}\left(\lambda_{ij}\int_a^b \underline{U}_{i,j}(s,r)ds\right) \\ \overline{g}_i(t,r) = \overline{f}_i(t,r) + \sum_{j=1}^{m}\left(\lambda_{ij}\int_a^b \overline{U}_{i,j}(s,r)ds\right) \end{cases}, \ i = 1, \ldots, m \quad (6)$$

Where:

$$\underline{U}_{i,j}(s,r) = \begin{cases} k_{i,j}(s,t)\underline{g}_j(s,r), & k_{i,j}(s,t)\ge 0 \\ k_{i,j}(s,t)\overline{g}_j(s,r), & k_{i,j}(s,t)< 0 \end{cases} \quad (7)$$

and:

$$\overline{U}_{i,j}(s,r) = \begin{cases} k_{i,j}(s,t)\underline{g}_j(s,r), \ k_{i,j}(s,t)\ge 0 \\ k_{i,j}(s,t)\overline{g}_j(s,r), \ k_{i,j}(s,t)< 0 \end{cases} \quad (8)$$

Homotopy Analysis Method (HAM): The homotopy analysis method is considered as an analytical approach in order to obtain solutions in series form and use it to solve various types of integral equations. This will ensure the convergence of the solution series using an auxiliary parameter. Moreover, homotopy analysis method provides a kind of freedom for choosing initial approximations and an auxiliary linear operator which helps us to simplify any problem.

Definition (9) (Liao, 2009): Let ϕ be a function of the homotopy parameter p from homotopy theory. Then the nth-order homotopy-derivative of is ϕ defined by:

$$D_n(\phi) = \frac{1}{n!}\frac{\partial^n\phi}{\partial p^n}\bigg|_{p=0} \quad (9)$$

where, $n\ge 0$ is an integer.

Lemma (Eq. 10) (Liao, 2004): Suppose $\phi = \sum_0^{\infty} u_n(t,r)p^n$ denote a homotopy series, where $p\in[0, 1]$ is the embedding homotopy parameter in the theory of topology, u_n is an unknown function, where t and r denote a spatial and temporal independent variables respectively. Let L denote an auxiliary linear operator and u_0 is an initial guess solution. It holds that:

$$D_n\{(1-p)\mathcal{L}[\phi-u_0]\} = \mathcal{L}[u_n(t,r)-\chi_n u_{n-1}(t,r)] \quad (10)$$

where, D_{n-1} is defined by Eq. 11 and x_n defined by:

$$\chi_n = \begin{cases} 0, & n \le 1 \\ 1, & n > 1 \end{cases} \quad (11)$$

Theorem (Eq. 9) (Liao, 2009): Let L be a linear operator independent of the homotopy-parameter p. For homotopy series:

$$\phi = \sum_{k=0}^{\infty} u_k p^k$$

it holds:

$$D_n\{L[\phi]\} = L[D_n(\phi)]$$

Theorem (Eq. 10) (Liao, 2004): Let $\phi = \sum_{n=0}^{\infty} u_n(t,r)p^n$ where $p\in[0, 1]$ is the homotopy parameter. Let L denote an auxiliary linear operator, N is a nonlinear operator, $u_0(t, r)$ an initial guess solution, h the convergence-control parameter and H(t, r)an auxiliary function both h and H(t, r) are independent of p. Then we define the zero-order deformation equation as follows:

$$(1-p)L[\phi - u_0] = phH(t,r)N[\phi]$$

The corresponding nth-order deformation equation ($n\ge1$):

$$L[u_n(t, r) - ?_n u_{n-1}(t, r)] = hH(t, r)D_{n-1}(N[\phi]) \quad (12)$$

where, D_{n-1} and χ_n are defined by Eq. 9 and 12, respectively. Finding the solution as a series form require its convergence in any region. Thus, we have the following theorem:

Theorem (Eq. 12) (Abbasbandy and Allahviranloo, 2011) (Convergence theorem): If the series $u_0(t,r) + \sum_{n=1}^{\infty} u_n(t,r)$ converges to function u(t, r), then u(t, r) must be the exact solution where $u_n(t, r)$ is governe by the nth-order deformation Eq. 12 under the definition (Eq. 9 and 11).

To ensure convergence of the series, we have to concentrate on choosing $u_0(t, r)$ the initial guess, the linear operator L the embedding parameter p, the auxiliary parameter h and finally the auxiliary function H(t, r). In this part, we rewrite the fuzzy Fredholm integral equations of the second kind and then solve them by homotopy analysis method. Also, we get the solution in series form. Now, we partition the interval [a, b] into two parts according to the sign of the kernel k(s, t), i.e., k(s, t)>0 on [a, c] and k(s, t)<0 on [c, b]. Therefore, we rewrite (Eq. 1) as follow:

$$\underline{g}(t,r) = \underline{f}(t,r) + \lambda\int_a^c k(s,t)\underline{g}(s,r)ds + \lambda\int_c^b k(s,t)\overline{g}(s,r)ds$$
$$\overline{g}(t,r) = \overline{f}(t,r) + \lambda\int_a^c k(s,t)\overline{g}(s,r)ds + \lambda\int_c^b k(s,t)\underline{g}(s,r)ds \quad (13)$$

From system (Eq. 13) we define the nonlinear operator N(t, p, r) as follows (Molabahrami *et al*., 2011):

$$\underline{N}(t,p,r) = \underline{U}(t,p,r) - \underline{f}(t,r) - \lambda\int_a^c k(s,t)\underline{U}(s,p,r)ds - \lambda\int_c^b k(s,t)\overline{U}(s,p,r)ds$$
$$\overline{N}(t,p,r) = \overline{U}(t,p,r) - \overline{f}(t,r) - \lambda\int_a^c k(s,t)\overline{U}(s,p,r)ds - \lambda\int_c^b k(s,t)\underline{U}(s,p,r)ds \quad (14)$$

We choose the auxiliary linear operator L with the following assumption:

$$\mathcal{L}[U(t,p,r)] = U(t,p,r) \quad (15)$$

Applying homotopy analysis method to solve system (Eq. 13), we consider the zero-order deformation equation:

$$(1-p)\mathcal{L}[\underline{U}(t,p,r) - \underline{u}_0(t,r)] = ph\underline{H}(t,r)\underline{N}(t,r)$$
$$(1-p)\mathcal{L}[\overline{U}(t,p,r) - \overline{u}_0(t,r)] = ph\overline{H}(t,r)\overline{N}(t,r) \quad (16)$$

Where:

$p\in[0, 1]$	=	An embedding parameter called the homotopy parameter
L	=	An auxiliary linear parameter
$\underline{u}_0(t,r), \overline{u}_0(t,r)$	=	The initial guess of $\underline{g}_0(t,r)$ and $\overline{g}_0(t,r)$, respectively
$\underline{U}(t,p,r), \overline{U}(t,p,r)$	=	Auxiliary functions
	=	The unknown functions on independent variable p
$h\neq0$	=	Denote convergence-controller parameter

In virtue of Eq. 15 and 16 yields:

$$(1-p)[\underline{U}(t,p,r) - \underline{u}_0(t,r)] = ph\underline{H}(t,r)\underline{N}(t,r)$$
$$(1-p)[\overline{U}(t,p,r) - \overline{u}_0(t,r)] = ph\overline{H}(t,r)\overline{N}(t,r) \quad (17)$$

Then substitute Eq. 14 and 17 with the assumption that H(t, r) =1, we get (Ghanbani, 2012):

$$(1-p)\left[\underline{U}(t,p,r)-\underline{u}_0(t,r)\right]=ph\begin{bmatrix}\underline{U}(t,p,r)-\underline{f}(t,r)-\\ \lambda\int_a^c k(s,t)\underline{U}(s,p,r)ds-\\ \lambda\int_c^b k(s,t)\overline{U}(s,p,r)ds\end{bmatrix}$$

$$(1-p)\left[\overline{U}(t,p,r)-\overline{u}_0(t,r)\right]=ph\begin{bmatrix}\overline{U}(t,p,r)-\overline{f}(t,r)-\\ \lambda\int_a^c k(s,t)\overline{U}(s,p,r)ds-\\ \lambda\int_c^b k(s,t)\overline{U}(s,p,r)ds\end{bmatrix} \tag{18}$$

when p = 0 the zero-order deformation Eq. 18 becomes

$$\begin{aligned}\underline{U}(t,0,r)&=\underline{u}_0(t,r)\\ \overline{U}(t,0,r)&=\overline{u}_0(t,r)\end{aligned} \tag{19}$$

when, p = 1the zero-order deformation (Eq. 18) becomes:

$$\begin{aligned}\underline{U}(t,1,r)&=\underline{f}(t,r)+\lambda\int_a^c k(s,t)\underline{U}(s,1,r)ds+\\ &\lambda\int_c^b k(s,t)\underline{U}(s,1,r)ds\\ \overline{U}(t,1,r)&=\overline{f}(t,r)+\lambda\int_a^c k(s,t)\overline{U}(s,1,r)ds+\\ &\lambda\int_c^b k(s,t)\overline{U}(s,1,r)ds\end{aligned} \tag{20}$$

Notice that Eq. 20 is exactly the same as Eq. 13. Now as the value of increases from 0-1 the analytical solution ($\underline{U}(t,p,r), \overline{U}(t,p,r)$) changes from the initial approximation guess $u_o(t,r), u_o(t,r)$ to the exact solution $\underline{g}(t,r), \overline{g}(t,r)$. We expand the functions $\underline{U}(t,p,r)$ and $\overline{U}(t,p,r)$ in a Taylor series with respect to the embedding parameter p. Thi expansion can be written as follows (Liao, 2009):

$$\begin{aligned}\underline{U}(t,p,r)&=\overline{u}_0(t,r)+\sum_{n=1}^{\infty}\overline{u}_n(t,r)p^n\\ \overline{U}(t,p,r)&=\overline{u}_0(t,r)+\sum_{n=1}^{\infty}\overline{u}_n(t,r)p^n\end{aligned} \tag{21}$$

Where:

$$\begin{aligned}\underline{u}_n(t,r)&=\frac{1}{n!}\frac{\partial^n\overline{U}(t,p,r)}{\partial p^n}\bigg|_{p=0}\\ \overline{u}_n(t,r)&=\frac{1}{n!}\frac{\partial^n\overline{U}(t,p,r)}{\partial p^n}\bigg|_{p=0}\end{aligned} \tag{22}$$

Differentiating the zero-order deformation Eq. 18 n-times with respect to p, we get:

$$\frac{\partial^n\underline{U}-(t,p,r)}{\partial p^n}-\frac{\partial^{n-1}\underline{U}(t,p,r)}{\partial p^{n-1}}=h\begin{bmatrix}\dfrac{\partial^{n-1}U(t,p,r)}{\partial p^{n-1}}-\underline{f}(t,r)-\\ \lambda\int_a^c k(s,t)\dfrac{\partial^{n-1}\underline{U}(s,p,r)}{\partial p^{n-1}}ds-\\ \lambda\int_c^b k(s,t)\dfrac{\partial^{n-1}\underline{U}(s,p,r)}{\partial p^{n-1}}ds\end{bmatrix}$$

$$\frac{\partial^n\overline{U}(t,p,r)}{\partial p^n}-\frac{\partial^{n-1}\overline{U}(t,p,r)}{\partial p^{n-1}}=h\begin{bmatrix}\dfrac{\partial^{n-1}\overline{U}(t,p,r)}{\partial p^{n-1}}-\overline{f}(t,r)-\\ \lambda\int_a^c k(s,t)\dfrac{\partial^{n-1}\overline{U}(s,p,r)}{\partial p^{n-1}}ds-\\ \lambda\int_c^b k(s,t)\dfrac{\partial^{n-1}\overline{U}(s,p,r)}{\partial p^{n-1}}ds\end{bmatrix} \tag{23}$$

Dividing (Eq. 21) by n!, then p = 0 set, we get the nth-order deformation equation (Molabahrami *et al*., 2011):

$$\underline{u}_n(t,r)=a_n\underline{u}_{n-1}(t,r)+h\begin{bmatrix}\underline{u}_{n-1}(t,r)-ß_n\underline{f}(t,r)-\\ \lambda\int_a^c k(s,t)\underline{u}_{n-1}(s,p,r)ds-\\ \lambda\int_c^b k(s,t)\underline{u}_{n-1}(s,p,r)ds\end{bmatrix}$$

$$\overline{u}_n(t,r)=a_n\overline{u}_{n-1}(t,r)+h\begin{bmatrix}\overline{u}_{n-1}(t,r)-ß_n\overline{f}(t,r)-\\ \lambda\int_a^c k(s,t)\overline{u}_{n-1}(s,p,r)ds-\\ \lambda\int_c^b k(s,t)\overline{u}_{n-1}(s,p,r)ds\end{bmatrix} \tag{14}$$

where, n≥1 and:

$$a_n=\begin{cases}0, & n=1\\ 1, & n\neq 1\end{cases},\ \beta_n=\begin{cases}0, & m\neq 1\\ 1, & n=1\end{cases}$$

If we let $\underline{u}_o(t,r)=\overline{u}_o(t,r)=\overline{0}$, then for $n\geq 2$ we have:

$$\underline{u}_n(t,r)=(1+h)\underline{u}_{n-1}(t,r)-h\lambda\left[\begin{array}{l}\int_a^c k(s,t)\underline{u}_{n-1}(s,p,r)ds+\\ \int_c^b k(s,t)\underline{u}_{n-1}(s,p,r)ds\end{array}\right]$$

$$\overline{u}_n(t,r)=(1+h)\overline{u}_{n-1}(t,r)-h\lambda\left[\begin{array}{l}\int_a^c k(s,t)\overline{u}_{n-1}(s,p,r)ds+\\ \int_c^b k(s,t)\overline{u}_{n-1}(s,p,r)ds\end{array}\right] \quad (15)$$

The solution of Eq. 3 in series form can be obtained as follows:

$$\underline{g}(t,r)=\lim_{p\to 1}\underline{U}(t,p,r)=\sum_{n=1}^{\infty}\underline{u}_n(t,r)$$
$$\overline{g}(t,r)=\lim_{p\to 1}\overline{U}(t,p,r)=\sum_{n=1}^{\infty}\overline{u}_n(t,r) \quad (16)$$

Now, we denote the mth-order approximation to solution $\underline{g}(t,r)$ with:

$$\underline{g}_m(t,r)=\sum_{n=1}^{m}\underline{u}_n(t,r)$$

and $\overline{g}(t,r)$ with:

$$\overline{g}_m(t,r)=\sum_{n=1}^{m}\overline{u}_n(t,r) \quad (17)$$

Adomain Decomposition Method (ADM): Since, the beginning of 1980s, scientists and engineers did apply the Adomain decomposition method to functional equations in order to calculate the solutions as an infinite series which usually converges to the exact solution. However, the Adomain decomposition method is a special case of homotopy analysis method.

Theorem (4.1) (Liao, 2009): If we set the convergence-controller parameter h = -1 in the frame of homotopy analysis method when it is applied on integral equations, the method will be converted to Adomain decomposition method. To solve the linear system (2.6), we rewrite it as follows (Abbasbandy and Allahviran, 2006):

$$\underline{g}_i(t,r)=\underline{f}_i(t,r)+N_i(\underline{g}_1,\ldots,\underline{g}_n)(t,r)$$
$$\overline{g}_i(t,r)=\overline{f}_i(t,r)+N_i(\overline{g}_1,\ldots,\overline{g}_n)(t,r) \quad (18)$$

Where:

$$N_i(\underline{g}_1,\ldots,\underline{g}_n)(t,r)=\int_a^b\sum_{j=1}^{n}K_{ij}(s,t)\underline{g}_j(s,r)ds$$
$$N_i(\overline{g}_1,\ldots,\overline{g}_n)(t,r)=\int_a^b\sum_{j=1}^{n}K_{ij}(s,t)\overline{g}_j(s,r)ds \quad (19)$$

In order to use the Adomain decomposition method we need to represent $g_i(t,r)=(\underline{g}_i(t,r),\overline{g}_i(t,r))$ in a series form:

$$\underline{g}_i(t,r)=\sum_{m=0}^{\infty}\underline{g}_{im}(t,r)$$
$$\overline{g}_i(t,r)=\sum_{m=0}^{\infty}\overline{g}_{im}(t,r) \quad (20)$$

and letting:

$$N_i(\underline{g}_1,\ldots,\underline{g}_n)(t,r)=\sum_{m=0}^{\infty}\underline{A}_{im}$$
$$N_i(\overline{g}_1,\ldots,\overline{g}_n)(t,r)=\sum_{m=0}^{\infty}\overline{A}_{im},\quad m=0,1,\ldots \quad (21)$$

where, $A_{im}=(\underline{A}_{im},\overline{A}_{im})$ are Adomain polynomials. Upon, using Eq. 21 and 22 can be written as:

$$\sum_{m=0}^{\infty}\underline{g}_{im}=\underline{f}_i+\sum_{m=0}^{\infty}\underline{A}_{im}(\underline{g}_{10},\ldots,\underline{g}_{1m},\ldots,\underline{g}_{n0},\ldots,\underline{g}_{nm})$$
$$\sum_{m=0}^{\infty}\overline{g}_{im}=\overline{f}_i+\sum_{m=0}^{\infty}\overline{A}_{im}(\overline{g}_{10},\ldots,\overline{g}_{1m},\ldots,\overline{g}_{n0},\ldots,\overline{g}_{nm}) \quad (22)$$

To obtain the Adomain's polynomial we introduce a parameter for convenience, so we have:

$$\underline{g}_{i\lambda}(t,r)=\sum_{m=0}^{\infty}\underline{g}_{im}(t,r)\lambda^m$$
$$\overline{g}_{i\lambda}(t,r)=\sum_{m=0}^{\infty}\overline{g}_{im}(t,r)\lambda^m \quad (23)$$

and:

$$N_{i\lambda}(\underline{g}_{1?},\ldots,\underline{g}_{n\lambda})=\sum_{m=0}^{\infty}\underline{A}_{im}\lambda^m$$
$$N_{i\lambda}(\overline{g}_{1\lambda},\ldots,\overline{g}_{n\lambda})=\sum_{m=0}^{\infty}\overline{A}_{im}\lambda^m,\quad m=0,1,\ldots \quad (24)$$

then, we get $A_{im}=(\underline{A}_{im},\overline{A}_{im})$ as:

$$\overline{A}_{im}=\frac{1}{m!}\left[\frac{d^m}{d\lambda^m}N_{i\lambda}(\underline{g}_{1\lambda},\ldots,\underline{g}_{n\lambda})\right]_{\lambda=0}$$
$$\overline{A}_{im}=\frac{1}{m!}\left[\frac{d^m}{d\lambda^m}N_{i\lambda}(\overline{g}_{1\lambda},\ldots,\overline{g}_{n\lambda})\right]_{\lambda=0} \quad (25)$$

From Eq. 22 and 24, we obtain:

$$\overline{A}_{im}\left(\underline{g}_{10},\ldots,\underline{g}_{1m},\ldots,\underline{g}_{n0},\ldots,\underline{g}_{nm}\right) =$$
$$\int_a^b\sum_{j=1}^{n}K_{ij}(s,t)\frac{1}{m!}\frac{d^m}{d\lambda^m}\sum_{l=0}^{\infty}\underline{g}_{jm}\lambda^l\Bigg|_{\lambda=0} ds = \int_a^b\sum_{j=1}^{n}K_{ij}(s,t)\underline{g}_{jm}(s,r)ds$$
$$\overline{A}_{im}\left(\overline{g}_{10},\ldots,\overline{g}_{1m},\ldots,\overline{g}_{n0},\ldots,\overline{g}_{nm}\right) =$$
$$\int_a^b\sum_{j=1}^{n}K_{ij}(s,t)\frac{1}{m!}\frac{d^m}{d\lambda^m}\sum_{l=0}^{\infty}\overline{g}_{jm}\lambda^l\Bigg|_{?=0} ds = \int_a^b\sum_{j=1}^{n}K_{ij}(s,t)\overline{g}_{jm}(s,r)ds \tag{26}$$

From Eq. 22 the solution of Eq. 6 will be as follows:

$$\begin{aligned}\underline{g}_{i0} &=_i \underline{f}\\ \underline{g}_{i,m+1} &= \int_a^b\sum_{j=1}^{n}K_{ij}(s,t)\underline{g}_{jm}(s,r)ds\end{aligned} \tag{27}$$

and:

$$\begin{aligned}\underline{g}_{i0} &= \overline{f}_i\\ \overline{g}_{i,m+1} &= \int_a^b\sum_{j=1}^{n}K_{ij}(s,t)\underline{g}_{jm}(s,r)ds\end{aligned} \tag{28}$$

We usually approximate $g(t,r)=(\underline{g}(t,r),\overline{g}(t,r))$ by (Babolian *et al*., 2005):

$$\begin{aligned}\underline{\phi}_{in} &= \sum_{m=0}^{n-1}\underline{g}_{im}(t,r)\\ \overline{\phi}_{in} &= \sum_{m=0}^{n-1}\overline{g}_{im}(t,r)\end{aligned} \tag{29}$$

Where:

$$\lim_{n\to\infty}\underline{\phi}_{in} = \underline{g}_i(t,r)$$
$$\lim_{n\to\infty}\overline{\phi}_{in} = \overline{g}_i(t,r)$$

RESULTS AND DISCUSSION

Examples and results: In order to test the effeciency of our analytical methods, we consider the following test example. We compare approximate solutions with exact solutions using metric of Definition (5.1) (Table 1 and 2).

Further comparison between the approximate solutions and the exact solutions can be seen in Fig. 1 and 2 for fixed t = 1.

Definition (5.1) (Ziari *et al*., 2012): For arbitrary fuzzy u, v∈E numbers the quantity

$$D(u,v)=\sup_{0\le r\le 1}\left\{\max\max\left|\overline{u}(r)-\overline{v}(r)\right|,\max\left|\overline{u}(r)-\overline{v}(r)\right|\right\}$$

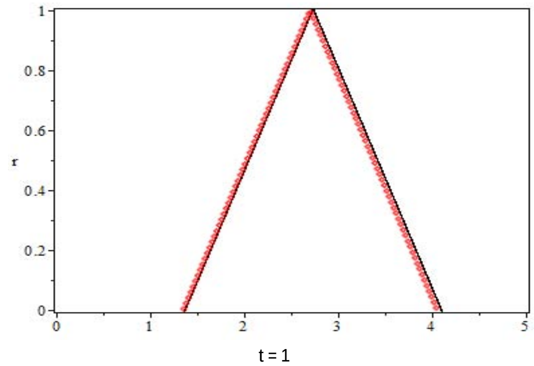

Fig. 1: Exact solution and approximate solution for t = 1

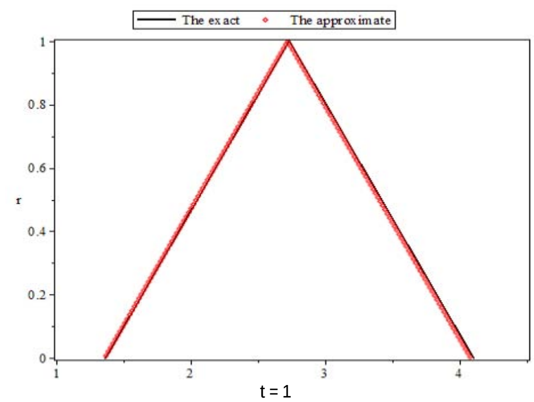

Fig. 2: Exact solution and approximate solution for t = 1

defines the distance between u and v.

Example (40) (Homotopy analysis method): Consider the following fuzzy Fredholm integral equations:

$$\underline{g}(t,r)=(r+1).(e^{-t}+t-\sin t)+\int_0^1\frac{1}{2}.e^s.\sin t.\underline{g}(s,r)ds$$
$$\overline{g}(t,r)=(3-r).(e^{-t}+t-\sin t)+\int_0^1\frac{1}{2}.e^s.\sin t.\overline{g}(s,r)ds \tag{40}$$

have the exact solutions:

$$\begin{aligned}\underline{g}(t,r)&=(r+1).(e^{-t}+t)\\ \overline{g}(t,r)&=(3-r).(e^{-t}+t)\end{aligned} \tag{41}$$

On the interval [0, 1]. The first terms of homotopy series are:

$$\underline{u}_0(t,r)=\overline{0}$$
$$\underline{u}_1(t,r)=-h\underline{f}(t,r)=-h(r+1).\left(e^{-t}+t-\sin(t)\right)$$

Table 1: The error resulted for h = -1 at t = 1

r	$\overline{g}$ exact	$\underline{g}$ exact	$\underline{g}$ approximate	$\overline{g}$ approximate	Error = D (g_{exact}, g_{approx})
0.0	1.367879441	4.103638323	1.351530171	4.054590513	4.904781003×10^{-2}
0.1	1.504667385	3.966850379	1.488322192	3.919437496	4.741288300×10^{-2}
0.2	1.641455329	3.830062435	1.625114212	3.784284479	4.577795597×10^{-2}
0.3	1.778243274	3.693274491	1.761906233	3.649131462	4.414302899×10^{-2}
0.4	1.915031218	3.556486547	2.035490274	3.513978445	4.250810198×10^{-2}
0.5	2.051819162	3.419698603	2.051880310	3.378825428	4.087317401×10^{-2}
0.6	2.188607106	3.282910659	2.172282295	3.243672410	3.923824799×10^{-2}
0.7	2.325395050	3.146122715	2.309074316	3.108519393	3.760332097×10^{-2}
0.8	2.462182994	3.009334771	2.445866337	2.973366376	3.596839402×10^{-2}
0.9	2.598970938	2.872546826	2.582658357	2.838213359	3.433346699×10^{-2}
1.0	2.735758882	2.735758882	2.719450378	2.703060342	3.269854000×10^{-2}

Table 2: The error resulted at t = 1

r	$\underline{g}$ exact	$\overline{g}$ exact	$\underline{g}$ approximate	$\overline{g}$ approximate	Error = D (g_{exact}, g_{approx})
0.0	1.367879441	4.103638323	1.360445994	4.081337982	2.230034101×10^{-2}
0.1	1.504667385	3.966850379	1.496490593	3.945293383	2.155699600×10^{-2}
0.2	1.641455329	3.830062435	1.632535193	3.809248783	2.081365199×10^{-2}
0.3	1.778243274	3.693274491	1.768579792	3.673204184	2.007030701×10^{-2}
0.4	1.915031218	3.556486547	1.904624392	3.537159584	1.932696300×10^{-2}
0.5	2.051819162	3.419698603	2.040668991	3.401114985	1.858361701×10^{-2}
0.6	2.188607106	3.282910659	2.176713590	3.265070386	1.784027198×10^{-2}
0.7	2.325395050	3.146122715	2.312758190	3.129025786	1.709692800×10^{-2}
0.8	2.462182994	3.009334771	2.448802789	2.992981187	1.635358300×10^{-2}
0.9	2.598970938	2.872546826	2.584847389	2.856936587	1.561023900×10^{-2}
1.0	2.735758882	2.735758882	2.720891988	2.720891988	1.486689400×10^{-2}

$$\underline{u}_2(t,r)=(1+h)\underline{u}_1(t,r)-h\int_0^1\frac{1}{2}e^s\sin(t)\underline{u}_1(s,r)ds$$
$$=-((1+h))h(r+1)(e^{-t}+t-\sin(t))+$$
$$0.5453346632h^2\sin(t)(r+1)$$
$$\underline{u}_3(t,r)=(1+h)\underline{u}_2(t,r)-h\int_0^1\frac{1}{2}e^s\sin(t)\underline{u}_2(s,r)ds$$
$$=-((((r+1)h^2+(2r+2)h+r+1)e^{-t}+$$
$$((-1.842724558-1.842724558r)h^2+$$
$$(-3.090669326r-3.090669326)h-$$
$$r-1)\sin(t)+(rt+t)h^2+(2t+2rt)h+rt+1t))h$$
$$\underline{u}_4(t,r)=(1+h)\underline{u}_3(t,r)-h\int_0^1\frac{1}{2}e^s\sin(t)\underline{u}_3(s,r)ds=$$
$$-((((r+1)h^3+(3r+3)h^2+(3r+3)h+r+1)e^{-t}+$$
$$((-2.004901576r-2.004901576)h^3+$$
$$(-5.528173674r-5.528173674)h^2+$$
$$(-4.636003989-4.636003989r)h-r-1)\sin(t)+$$
$$(rt+t)h^3+(3t+3rt)h^2+(3t+3rt)h+rt+t))h$$
$$\overline{u}_5(t,r)=(1+h)\overline{u}_4(t,r)-h\int_0^1\frac{1}{2}e^s\sin(t)\overline{u}_4(s,r)ds=$$
$$-h(((r+1)h^4+(4r+4)h^3+(6+6r)h^2+$$
$$(4r+4)h+r+1)e^{-t}+((-2.093342326-$$
$$2.093342326r)h^4+(-8.019606304-8.019606304r)h^3+$$
$$(-11.05634734r-11.05634734)h^2+$$
$$(-6.181338652r-6.181338652)h-r-1)\sin(t)+$$
$$(rt+t)h^4+(4t+4rt)h^3+(6t+6rt)h^2+(4t+4rt)h+rt+t)$$
$$\underline{u}_6(t,r)=(1+h)\underline{u}_5(t,r)-h\int_0^1\frac{1}{2}e^s\sin(t)\underline{u}_5(s,r)ds$$
$$=-(1(((r+1)h^5+(5r+5)h^4+(10r+10.)h^3+$$
$$(10r+10)h^2+(r+5)h+r+1)e^{-t}+$$
$$((-2.141572132r-2.141572132)h^5+$$
$$(-10.46671163-10.46671163r)h^4+$$
$$(-20.04901575r-20.04901575)h^3+$$
$$(-18.42724557r-18.42724557)h^2+$$
$$(-7.726673315-7.726673315r)h-$$
$$r-1)\sin(t)+(rt+t)h^5+(5t+5rt)h^4+$$
$$(10t+10rt)h^3+(10t+10rt)h^2+$$
$$(5t+5rt)h+rt+t))h$$

and:

$$\overline{u}_0(t,r)=\overline{0}$$
$$\overline{u}_1(t,r)=-h\overline{f}(t,r)=-h(r+1).(e^{-t}+t-\sin(t))$$
$$(t,r)=(1+h)\underline{u}_1(t,r)-h\int_0^1\frac{1}{2}e^s\sin(t)\underline{u}_1(s,r)ds=$$
$$-((1+h))h(r+1)(e^{-t}+t-\sin(t))+$$
$$0.5453346632h^2\sin(t)(r+1)\overline{u}_3(t,r)$$
$$=(1+h)\overline{u}_2(t,r)-h\int_0^1\frac{1}{2}e^s\sin(t)$$
$$\overline{u}_2(s,r)ds=-((((r+1)h^2+$$
$$\underline{u}_2(2r+2)h+r+1)e^{-t}+$$
$$((-1.842724558-1.842724558r)h^2+$$

$$(-3.090669326r-3.090669326)h-r-1)$$
$$\sin(t)+(rt+t)h^2+(2t+2rt)h+rt+1t))h$$

$$\bar{u}_4(t,r)=(1+h)\bar{u}_3(t,r)-h\int_0^1\frac{1}{2}e^s$$
$$\sin\sin(t)\bar{u}_3(s,r)ds$$
$$=-((((r+1)h^3+(3r+3)h^2+$$
$$(3r+3)h+r+1)e^{-t}+((-2.004901576r-$$
$$2.004901576)h^3+(-5.528173674r-$$
$$5.528173674)h^2+(-4.636003989-$$
$$4.636003989)rh-r-1)\sin(t)+(rt+t)h^3+$$
$$(3t+3rt)h^2+(3t+3rt)h+rt+t))h$$

$$\bar{u}_5(t,r)=(1+h)\bar{u}_4(t,r)-h\int_0^1\frac{1}{2}e^s\sin(t)\bar{u}_4(s,r)ds$$
$$=-h(((r+1)h^4+(4r+4)h^3+(6+6r)h^2+$$
$$(4r+4)h+r+1)e^{-t}+((-2.093342326-$$
$$2.093342326r)h^4+$$
$$(-8.019606304-8.019606304r)h^3+$$
$$(-11.05634734r-11.05634734)h^2+$$
$$(-6.181338652r-6.181338652)h-$$
$$r-1)\sin(t)+(rt+t)h^4+(4t+4rt)h^3+$$
$$(6t+6rt)h^2+(4t+4rt)h+rt+t)$$

$$\bar{u}_6(t,r)=(1+h)\bar{u}_5(t,r)-h\int_0^1\frac{1}{2}e^s\sin(t)\bar{u}_5(s,r)ds=$$
$$-(1(((r+1)h^5+(5r+5)h^4+(10r+10.)h^3+$$
$$(10r+10)h^2+(r+5)h+r+1)e^{-t}+$$
$$((-2.141572132r-2.141572132)h^5+$$
$$(-10.46671163-10.46671163r)h^4+$$
$$(-20.04901575r-20.04901575)h^3+$$
$$(-18.42724557r-18.42724557)h^2+$$
$$(-7.726673315-7.726673315r)h-$$
$$r-1)\sin(t)+(rt+t)h^5+(5t+5rt)h^4+$$
$$(10t+10rt)h^3+(10t+10rt)h^2+$$
$$(5t+5rt)h+rt+t))h$$

then, we approximate $\underline{g}(t,r)$ with:

$$\underline{g}_6(t,r)=\sum_{n=1}^{6}\underline{u}_n(t,r)=-(((((r+1)h^5+(6r+6)h^4+(15r+15)h^3+$$
$$(20r+20)h^2+(15r+15)h+5r+5)e^{-t}+(r+1)e^{-t}+$$
$$((-2.141572132r-2.141572132)h^5+$$
$$(-12.56005396r-12.56005395)h^4+$$
$$(-30.07352363r-30.07352363)h^3+$$
$$(-36.85449115r-36.85449115)h^2+$$
$$(-23.18001994r-23.18001994)h-6r-6)\sin(t)$$
$$+(rt+t)h^5+(6rt+6t)h^4+(15t+15rt)h^3+$$
$$(20t+20rt)h^2+(15t+15rt)h+6rt+6t))h$$

and $\bar{g}(t,r)$ with:

$$\bar{g}_6(t,r)=\sum_{n=1}^{6}\bar{u}_n(t,r)=h(((-3+r)h^5+(-18+6r)h^4+(15r-45)h^3+$$
$$(20r-60)h^2+(15r-45)h+5r-15)e^{-t}+(-3+r)e^{-t}+$$
$$((-2.141572132r+6.424716396)h^5+$$
$$(-12.56005396r+37.68016186)h^4+$$
$$(-30.07352363r+90.22057091)h^3+$$
$$(-36.85449113r+110.5634735)h^2+$$
$$(-23.18001994r+69.54005985)h-6r+18)\sin(t)+$$
$$(rt-3t)h^5+(-18t+6rt)h^4+(-45t+15rt)h^3+$$
$$(-60t+20rt)h^2+(-45t+15rt)h+6rt-18t)$$

Example (5.2) (Adomain Decomposition Method): The fuzzy Fredholm integral Eq. 40 have the exact solution Eq. 41, on the interval [0, 1]. Some of first terms of Adomian decomposition series are

$$\bar{u}_0(t,r)=(r+1).(e^{-t}+t-\sin(t))$$
$$\bar{u}_1(t,r)=\int_0^1\frac{1}{2}e^s\sin(t).\bar{u}_0(s,r)ds=0.5453346632(r+1).\sin(t)$$
$$\bar{u}_2(t,r)=\int_0^1\frac{1}{2}e^s\sin(t).\bar{u}_1(s,r)ds=0.2479447683(r+1).\sin(t)$$
$$\bar{u}_3(t,r)=\int_0^1\frac{1}{2}e^s\sin(t).\bar{u}_2(s,r)ds=0.1127318916(r+1).\sin(t)$$
$$\bar{u}_4(t,r)=\int_0^1\frac{1}{2}e^s\sin(t).\bar{u}_3(s,r)ds=0.05125528346(r+1).\sin(t)$$
$$\bar{u}_5(t,r)=\int_0^1\frac{1}{2}e^s\sin(t).\bar{u}_4(s,r)ds=0.02330400072(r+1).\sin(t)$$
$$\bar{u}_6(t,r)=\int_0^1\frac{1}{2}e^s\sin(t).\bar{u}_5(s,r)ds=0.01059552134(r+1).\sin(t)$$

and:

$$\bar{u}_0(t,r) = (3-r).\left(e^{-t} + t - \sin(t)\right)$$

$$\bar{u}_1(t,r) = \int_0^1 \frac{1}{2} e^s \sin(t).\bar{u}_0(s,r)\,ds = 0.5453346632(3-r).\sin(t)$$

$$\bar{u}_2(t,r) = \int_0^1 \frac{1}{2} e^s \sin(t).\bar{u}_1(s,r)\,ds = 0.2479447683(3-r).\sin(t)$$

$$\bar{u}_3(t,r) = \int_0^1 \frac{1}{2} e^s \sin(t).\bar{u}_2(s,r)\,ds = 0.1127318916(3-r).\sin(t)$$

$$\bar{u}_4(t,r) = \int_0^1 \frac{1}{2} e^s \sin(t).\bar{u}_3(s,r)\,ds = 0.05125528346(3-r).\sin(t)$$

$$\bar{u}_5(t,r) = \int_0^1 \frac{1}{2} e^s \sin(t).\bar{u}_4(s,r)\,ds = 0.02330400072(3-r).\sin(t)$$

$$\bar{u}_6(t,r) = \int_0^1 \frac{1}{2} e^s \sin(t).\bar{u}_5(s,r)\,ds = 0.01059552134(3-r).\sin(t)$$

then we approximate $g(t, r)$ with

$$\underline{\phi}_7(t,r) = (r+1).\left(e^{-t} + t - 0.0088338714\sin(t)\right)$$

and:

$$\bar{\phi}_7(t,r) = (3-r).\left(e^{-t} + t - 0.0088338714\sin(t)\right)$$

Figure 2 compares the exact solution and the approximate solution for a fixed t = 1.

CONCLUSION

In this study, some analytical methods, namely; the homotopy analysis method and the Adomain decomposition method have been investigated and implemented to obtain approximate solutions of the fuzzy Fredholm integral equation of the second kind. In fact these methods have shown credibility and great potential to solve linear fuzzy Fredholm integral equations. A comparison between these methods shows the Adomain decomposition method is more efficient than the homotopy analysis method. Moreover, these results have been justified in (Amawi, 2014).

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