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Asymptotic Analysis of the Boundary Layerby Matching the WKB Solutions of the Inner and Outer Layers of a Neo-Hookean Cylindrical Shell

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Abstract: We analyzed and compared the asymptotic outer, inner and the matching solutions with the numerical counterpart results of the eigen-value problem of a neo-Hookean elastic cylindrical shell of arbitrary thicknesses subjected to an external hydrostatic pressure. In order to study thin-walled shells (i.e., a thin layer between the two regions A1-1 = O(1) and A1-1 = O(1/n), where A1 and a1 are the inner radii of the shell before and after deformation respectively on 0<A1<1) and for the purpose of matching the two regions, it is necessary to reconsider the asymptotic solutions obtained previously and offer the summarized relations of the relevant eigenvalues, i.e., $\mu = a_1/A1$. For analyzing thin-walled shells, the theory of boundary layer and also Van Dyke's matching rule has been employed.

Key words: WKB method, boundary layer theory, Van Dyke's matching rule, finite elasticity, thin-walled shells

INTRODUCTION

WKB theory is a powerful tool for obtaining a global approximation to the solution of a linear differential equation whose highest derivative is multiplied by a small parameter like, it contains boundary layer theory as an special case. The boundary layer theory shown how to construct an approximate solution to a differential equation containing a small parameter. This construction requires one to match slowly varying outer solutions to rapidly varying inner solutions. We will consider the method of matched asymptotic expansions applied to the fourth-order differential equation related to an incompressible isotropic homogeneous elastic shell. The purpose of this study is to introduce the notion of matched asymptotic expansions. Asymptotic matching is an important perturbation method which is used often in both boundary layer and WKB theories to determine analytically the approximate global properties of the solution to a differential equation. Asymptotic matching is usually used to determine a uniform approximation to the solution of a differential equation and to find other global properties of differential equations such as the eigen values. The procedure of finding the leading-order boundary layer approximation applied to the solution of a differential equation of an elastic shell may be formulated and one can obtain and asymptotically match the layers in an overlap region. The self-consistency of the boundary layer theory depends on the success of the asymptotic matching. Ordinarily, if the inner and outer solutions match to all thicknesses, then boundary layer theory gives an asymptotic approximation to the exact solution of the differential equation. The possibility of matching the outer solution with the inner solution to form a composite solution is discussed. It should be noticed that since the outer solution is independent of A1, the composite solution is simply the inner solution according to Van Dyke's matching rule (Bush, 1992). Therefore, if Van Dyke's matching rule works for this problem, the inner solution should also be valid in the outer region where A1-1 = O(1). It is verified here that the inner and outer solutions can be matched at leading order but not at the second order. The failure of Van Dyke's matching rule at second order is discussed in details on (Fu, 1998). We can show and indeed provide a uniformly valid curve for μ_1 against A1 for A1-1 = O (1) and A1-1 = O(1/n) by solving numerically Eq. 19 with the use of compound matrix method (Fig. 1) (Sanjaranipour, 2010). Since, the numerical solution corresponding to Eq. 19 fails for A1-1<O(1/n) and in order to obtain a proper thickness, we have to continue our asymptotic analysis for the case of two term inner solution. Fu and Lin (2002) performed successfully matching the outer and inner solutions concerning the analysis of the buckling of an spherical shell made of an elastic Neo-Hookean material which perfectly

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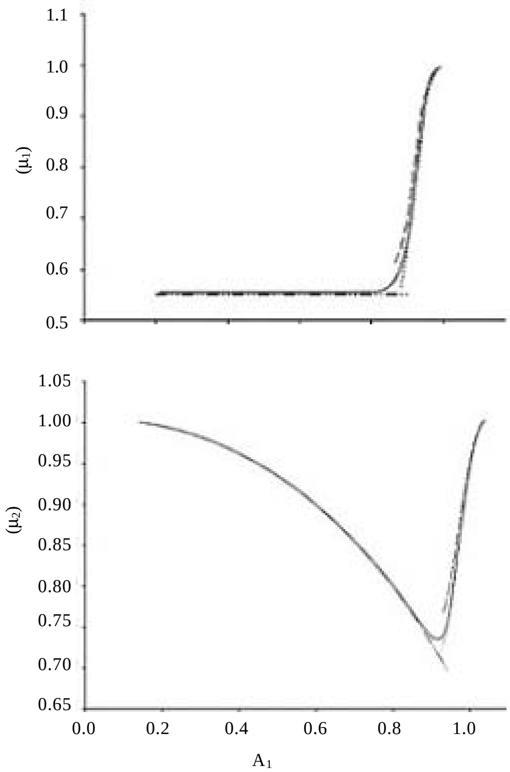

Fig. 1: Neutral stability curves for n = 20. Solid lines: numerical result; dashed lines: one-term asymptotic result; dotted lines: two-term asymptotic results

conforms the results obtained of thin shells with the numerical results but this conformation has not been done in cylindrical tube which is going to be study here.

Definition of the problem: We consider an incompressible isotropic homogeneous elastic shell andassume that the outer surface of the shell is subjected to a hydrostatic pressure and that the tube is in a state of plain strain. The co-ordinates of a representative material particle in the un-deformed and deformed states are (R, Θ, Z) and (r, θ, z), respectively. Thus, the hollow cylinder in its un-deformed configuration is defined by 0<A1 = R = A2, 0 = Θ = 2π, 0 = Z = L, the plane-strain deformation is denoted by:

$$r = r(R) \tag{1}$$

Where:

- r = The radial coordinate of a material point which is at a distance
- R = The radial coordinate) from the center of the shell before the pressure is applied
- A1 and A2 = are the inner and outer radii, respectively and the deformed state is:

$$0<a1 \le r \le a2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le l, \tag{2}$$

where, a1 and a2 are the deformed inner and outer radii, respectively. We assume that:

$$a1 = \mu 1 A1 \text{ and } a2 = \mu 2 A2 \tag{3}$$

where μ1 and μ2 are controlling parameters which for the compression problem satisfy 0<μ1, μ2<1. Incompressibility implies no volume change andso μ1 and μ2 are related by:

$$\begin{aligned} &(\mu 1^2-1)A_1^{\,2} = (\mu_2^2-1)A_1^2 \\ &\mu_2^2 = 1-(1-\mu_1^2)\frac{A_1^2}{A_2^2},\ 0<A_1<1 \end{aligned} \tag{4}$$

Either μ1 or μ2 can be taken as the controlling parameters and the aim ofa linear stability analysis is to find the value of μ1 or μ2 at which the shellbuckles. We shall assume that all variables and parameters which have adimension of length have been scaled by A2. Thus, we have:

$$\begin{aligned} &a_1 = \mu_1 A_1,\ a_2 = \mu_2, \\ &\mu_2^2 = 1-(1-\mu_1^2)A_1^2,\ 0<A_1<1 \end{aligned} \tag{5}$$

The eigen value problem resulting from a linear stability analysis can be found in Haughton and Ogden (1979) and also in refs (Sanjaranipour, 2010) and consists of solving the following fourth-order differential equation for a neo-Hookean material:

$$\begin{aligned} &V''''+\left(\frac{6}{r}+\frac{2k}{r^3}\right)+QV'''+\left[\frac{-k^2\left(3+n^3\right)}{r^6}+\frac{2k\left(1-n^2\right)}{r^4}+\frac{\left(5-2n^2\right)}{r^2}\right] \\ &Q^2V''+\left[\frac{k^3\left(3+n^3\right)}{r^9}+\frac{k^2\left(5+n^2\right)}{r^7}\right]+ \\ &\left[\frac{k\left(1-4n^2\right)}{r^5}+\frac{\left(1+2n^3\right)}{r^3}\right] \\ &Q^3V''+\left(n^2-1\right)\left[\frac{3k^2}{r^3}+\frac{2k}{r^6}+\frac{\left(n^2-1\right)}{r^4}\right]Q^2V=0 \end{aligned} \tag{6}$$

Subj ect to the boundary conditions:

$$V''+\frac{1}{r}V'+\frac{n^2-1}{r^2}V=0 \tag{7}$$

$$V'''Q^{-1}+\left[\frac{2k}{r^3}+\frac{4}{r}\right]V''-\left[Q\left(\frac{k^2}{r^6}+\frac{n^2-1}{r^2}\right)+\frac{2n^2}{r^2}Q^{-1}\right] V'+\left[\frac{n^2-1}{r^3}-\frac{k\left(n^2-1\right)}{r^6}\right]V=0 \quad (8)$$

onr = a1, a2, where V = V (r), a prime denotes differentiation with respect to r, n is the longitudinal mode number and:

$$Q=\frac{r^2}{k+r^2}, k=A_1^2\left(1-\mu_1\right)^2 \quad (9)$$

It is necessary to mention that this problem bas been solved by the use of the asymptotic WKB method and the results are related to the outer i.e., A1-1 = O(1) and inner, i.e., A1-1 = O(1/n) solutions separately and are compared with the counter part numerical results obtained throughout the whole region, i.e., 0<A1<1. Our main aim on this study is to solve this logarithmic layer which appears in between the two mentioned layers with the help of the Van Dyke matching rule and also the boundary layer theory. In order to be able to analysis this thin-walled shells, first a brief description about the previously studied inner and outer solutions (Sanjaranipour, 2010) should be written down here.

MATERIALS AND METHODS

Asymptotic analysis for thick shells in order to find the Boundary layer. The linear eigen-value problem is to solve Eq. 3 subject to Eq. 8-9 onr = a1, a2. In order to find the eigenvalue μ_1 and in large n limit, we look for a WKB solution e.g., Bender and Orszag (1978) and (Sanjaranipour, 2010) of the following form:

$$V=\exp\left(n\int_{a1}^{r}s\left(u\right)du\right) \quad (10)$$

$$S=S_0+\frac{1}{n}S_1+\frac{1}{n^2}S_2+...=n\sum_{m=0}^{\infty}\frac{1}{n^m}S_m \quad (11)$$

By substituting relations Eq. 10-11 in Eq. 6-8 and by equatingthe coefficients of the like powers of n with the aid of Mathematica, we obtain anumber of differential equations for S_0, S_1,…. There are four independent solutions for S. We denote the ith solution by S(i) and write:

$$S^{(i)}=nS_0^{(i)}+S_1^{(i)}+\frac{1}{n^2}S_2^{(i)}+... \quad (12)$$

Where the details of $S_0^{(i)}$, $S_1^{(i)}$, $S_3^{(i)}$, ..., (i = 1, 2, 3, 4) are given on (Sanjaranipour, 2010). All the above functions are analytic in the region of interest, i.e., 0<a1 = r = a2 = 1. Now, we focus on the general solution which for V is given by:

$$V=\sum_{m=1}^{4}k_i\exp\left(n\int_{a1}^{r}s^{(i)}\left(u\right)du\right) \quad (13)$$

where, Ki are constants and by substituting Eq. 13 and the relevant derivativesinto the equations of the boundary conditions Eq. 7 and 8 we obtain a matrix equation of the form:

$$\sum_{j=1}^{4}C_{ij}K_j=0,\left(i=1,2,3,4\right) \quad (14)$$

where, (C_{ij}) is given by:

$$C_{ij}=\begin{bmatrix} F_1(a_1)F_2(a_1)F_3(a_1)F_4(a_1) \\ G_1(a_1)G_2(a_1)G_3(a_1)G_4(a_1) \\ E_1F_1(a_1)E_2F_2(a_1)E_3F_3(a_1)E_4F_4(a_1) \\ E_1G_1(a_1)E_2G_2(a_1)E_3G_3(a_1)E_4G_4(a_1) \end{bmatrix} \quad (15)$$

The constants E_j and function $F_j(r)$ and $G_j(r)$ in the above matrix are defined by:

$$E_j=\exp\left(n\int_{a_1}^{a_2}s^{(i)}\left(u\right)du\right) \quad (16)$$

$$F_j\left(r\right)=\left(S^{(i)}\right)^2+\frac{1}{n}\left(S^{(j)}\right)^1+\frac{1}{nr}S^{(j)}+\frac{n^2-1}{n^2r^2} \quad (17)$$

$$G_j\left(r\right)=Q^{-1}\left[\left(S^{(j)}\right)^3+\frac{3}{n}\left(S^{(j)}\right)'+\frac{1}{n^2}S^{(j)''}\right]+\left[\frac{2k}{r^3}+\frac{4}{r}\right]\left(\frac{1}{n^2}S^{(j)}\right)'+\frac{1}{n}\left(S^{(j)}\right)^2-\left[Q\left(\frac{k^2}{r^6}+\frac{n^2-1}{r^2}\right)+\frac{2n^2}{r^2}Q\right]\frac{1}{n^2}S^{(j)}+\frac{1}{n^3}\left[\frac{n^2-1}{r^3}+\frac{k\left(n^2-1\right)}{r^5}\right] \quad (18)$$

Where:

$S_{(j)}$ = Denotes differentiation with respect to r
a2-a1 = O(1) = The constants
E_1 and E_3 = Are exponentially large whereas
E_2 and E_4 = Are exponentially small. A none trivial solution for (K_i) requires:

$$\det\left(Cij\right)=0 \quad (19)$$

Which is the required bifurcation condition in the large n limit. In order to show the dependence of mu1 on A1, this algebraic bifurcation condition should be solved numerically. Because of the appearance of E_1 and E_3 in C_{ij} which are exponentially large, we first find more explicit results and then in order to be able to compare this asymptotic results with the counter part numerical ones, the numerical determinant method (Haughton and Orr, 1997) is used and presented.

Asymptotic results for A1-1 = O(1): For A1-1 = O(1), we have a2 -a1 = O(1). It means that a2>>a1 and clearly the term proportional to E_1E_3 is exponentially dominant and hencewe may write Eq. 19 as:

$$\frac{\det C_{ij}}{E_1E_3} = \begin{bmatrix} F_2(a_1) & F_4(a_1) \\ G_2(a_1) & G_4(a_1) \end{bmatrix} \begin{bmatrix} F_1(a_1) & F_3(a_1) \\ G_1(a_1) & G_3(a_1) \end{bmatrix} + E.S.T = 0 \quad (20)$$

where, E.S.T. stands for exponentially small terms. By solving the matrix equation Eq. 14 and finally by expanding $\mu_1$in termsof 1/n, for the case of A1-1 = O(1), we obtain:

$$\mu_1^{out} = 0.5436 + \frac{0.3522}{n} - \frac{3.7132}{n^2} + \ldots \quad (21)$$

as remarked earlier, this expression is independent of A1. This fact ic confirmedby our numerical results shown in Fig. 1.

Asymptotic results for A_1-1 = O(1/n): It is clear from the Fig.1 that, by increasing A1, μ1 is constant on A1-1 = O(1). Since for A_1-1 = O(1/n), a_2-$a_1$is small, the exponentials E_1 and E_3 are no longer exponentially large and Eq. 20 fails to approximate Eq. 19. It can be deduced that E_1 and E_3 become O(1) and hence all the C_{ij}'sbecome O(1) when A_1-1 = O(1/n). In the latter regime we write:

$$A_1 = 1 + \frac{1}{n}\xi \quad (22)$$

Where ξ is an O(1) constant and look for an asymptotic solution for $\mu_1$ofthe form:

$$\mu_1 = {}_1 + \frac{1}{n}{}_2 + \ldots \quad (23)$$

where ${}_1$and ${}_2$ are to be determines. On substituting Eq. 22 and 23 in Eq. 5 and 9 we have:

$$a_1 = 1 + \frac{1}{n}({}_2 + {}_2\xi) + \frac{1}{n^2}2\xi + a_2 = 1 + \frac{1}{n}\left(2 + 1\xi - \frac{\xi}{{}_1}\right) + \ldots..$$
$$k = 1 + {}_1^{\,2} + \frac{2}{n}\left(\xi - \xi_1^{\,2} + {}_{12}\right) \quad (24)$$

By substituting the new a_1, a_2 and k from Eq. 24 into Eq. 15 and by collecting the coefficients of the leading order terms of det(C_{ij}) = 0 with the aid ofMathematica, we obtain the following simplified expression:

$$16z(1+z^2)^2 - (1+4z+2z^2+z^4)^2\cosh\left(\xi - \frac{\xi}{z}\right) + (1-4z+2z^2+z^4)^2\cosh\left(\xi + \frac{\xi}{z}\right) = 0 \quad (25)$$

where we used z = ${}^2{}_1$ and multiplied the second and fourth rows by 1^{-1} We may extract 1 by expanding:

$$\cosh\left(\xi - \frac{\xi}{z}\right)$$

and:

$$\cosh\left(\xi - \frac{\xi}{z}\right)$$

Eq. 25 and finally get:

$${}_1 = 1 - \frac{\xi^2}{18} - \frac{\xi^4}{1080} + \ldots. \quad (26)$$

Equation 26 satisfied by 2 which is linear in 2 is obtained by collecting the coefficients of the next order terms in det (C_{ij}) = 0 with the aid of Mathematica and is given by:

$${}_2\xi_1 + \xi_2 = 0 \quad (27)$$

Since, the relations of $\xi_1$and $\xi_2$are too long to be written out here, we referthe reader to Eq. 27 in (Sanjaranipour, 2010). For any given mode number nand any value of A_1 close to unity, the corresponding value of ξ is determined by Eq. 22. Now by substituting ξ from Eq. 26 into Eq. 27 and by expanding wemay find 2 as follows:

$${}_2 = -\frac{5\xi^2}{18} + \frac{5\xi^3}{36} + \frac{\xi^4}{135} + \ldots \quad (28)$$

The value of $\mu_1$is then calculated by using 1 and 2 from Eq. 26 and 28 respectively in Eq. 23. In Fig. 1, we have shown the comparison between the numerical results and the 1-term (corresponding to μ1 = 1) and 2-term (correspondingto μ_1 = 1+ 2/n) asymptotic results for n = 20. We see that the 2-term asymptotic results approximate the numerical results extremely well over the region of their validity.

In Fig. 1 we find that for n»1, $\mu_1$is constant over almost the entire region of 0 <A_1<1 and decreases sharply from this constant value to unity as A_1 tends to unity (the thin shell limit). Comparison between WKB and the numerical results shows that agreement is not good over a thin region between A_1-1 = O(1) and A_1-1 = O(1/n) where this region has aboundary layer structure.

Solving boundary layer: We now arrive to the case of forming a composite solution by matching the outer solution Eq. 21 with the inner solution Eq. 23. As we mentioned earlier the outer solution Eq. 21 is independent of A_1 and because of that the compositesolution is simply the inner solution according to Van Dyke's matching rule (Bush, 1992). It is evident from the study has been done by Fu in the caseof spherical shell (Van Dyke, 1975) that Van Dyke rule works for this problem also, i.e., theinner solution also valid in the outer region where A1-1 = O(1). By focusing on the Fig. 1ab we see that, this is indeed the case for the leading orderinner solution but not for the case of two-term inner solution. As mentioned on Eq. 3, Van Dyke (1975) matching rule fails at second order. According to the results obtained by Fu for the spherical shell on Fu, 1998, it is seen that Eq. 19 does indeed provide a uniformly valid solution for A1-1 = O(1) and A1-1 = O(1/n). However, the numerical solution corresponding to Eq. 19 fails for A1-1<O(1/n) which will be studied in the following section.

Asymptotic analysis for thin-walled shells: The logarithmic layer is a thin layer between the two inner and outerregions which a rapid change occurs in the value of the dependent variable. For solving this logarithmic layer it is necessary to reconsider the layer on a thin regions like A1-1 = O(ϵ^p), 0»p = 2 and for matching the twosolutions, we can use the theory of boundary layer and also Van Dyke's matching rule. If the boundary layer is located in the region, the appropriate stretching transformation is x = x-x_0/∂(ϵ) the choice of the power p, indeed more generally the choice of function ∂(ϵ) to use in the stretching transformation x = x-x_0/∂(ϵ) is determined by the need to represent the regionof rapid change correctly. We must ensure that the boundary layer solution contains rapidly varying functions. In order to choose the boundary layer thickness, it is necessary to seek a stretching transformation which retains the largest number of terms in the dominant equation governing μ_1. This is referred to as the principle of least degeneracy by Van Dyke (1975). It can be deduced from Eq. 25 and the expression for η_2 that, as ξ→0, we have:

$$\mu_1^i = 1-\frac{\xi^2}{18}-\frac{\xi^4}{1080}+\ldots+ \frac{1}{n}\left(-\frac{5\xi^2}{18}+\frac{5\xi^3}{36}+\frac{\xi^4}{135}+\ldots\right)+\ldots \quad (29)$$

The inner expansion Eq. 29 is not uniformly valid for all n, because this asymptotic expansion becomes disordered and hence invalid when ξ_4 and ξ_2/n are ofthe same order, i.e., when A_1-1 = O(1/3/n2). For matching outer and inner solutions and in order to find a uniform solution, we must find a thickness which includes the region of rapid changes. This is the regime we considerin the next part.

Asymptotic results for A_1-1 = O(1/3/n2): For A1-1 as small as O(1/3/n2), a_1 is close to $a_2$and it is convenient to define a new stretched variable x through:

$$x = \frac{r-a_1}{a_2-a_1}, 0 < a_1 \le r \le a_2 \quad (30)$$

In terms of x, the inner and outer surfaces of the shell correspond to x = 0, 1 and the governing Eq. 6 and the boundary conditions Eq. 7 and 8 becomes:

$$\begin{aligned}&\frac{d^4V}{dx^4}+Q\left[\frac{6}{r}+\frac{2k}{r^3}\right]\epsilon\delta\frac{d^3V}{dx^3}+\\&Q^2\left[\frac{-k\left(1+3\epsilon^2\right)}{r^6}+\frac{2k\left(\epsilon^2-1\right)}{r^4}+\frac{\left(5\epsilon^2-2\right)}{r^2}\right]\delta^2\\&\frac{d^2V}{dx^2}+Q^3\left[\begin{matrix}\frac{k^3\left(3\epsilon^2+1\right)}{r^9}+\frac{k^2\left(5\epsilon^2+1\right)}{r^7}+\frac{k\left(\epsilon^2-4\right)}{r^5}\\-\frac{\left(\epsilon^2+2\right)}{r^3}\end{matrix}\right]\\&\epsilon\delta^3\frac{dV}{dx}+Q^2\left[\left(1-\epsilon^2\right)\left(\frac{3k^2\epsilon^2}{r^8}+\frac{2k\epsilon^2}{r^6}+\frac{\left(1-\epsilon^2\right)}{r^4}\right)\right]\delta^4V=0\end{aligned} \quad (31)$$

$$\begin{aligned}&\frac{d^2V}{dx^2}+\frac{\epsilon\delta}{r}\frac{dV}{dx}+\frac{\left(1-\epsilon^2\right)}{r^2}V=0, \frac{d^3V}{dx^3}+Q\left[\frac{4}{r}+\frac{2k}{r^3}\right]\\&\epsilon\delta\frac{d^2V}{dx^2}-\left[Q^2\frac{\left(k^2m^2\right)}{r^6}+\frac{\left(1-\epsilon^2\right)}{r^2}\right]\delta\frac{dV}{dx}+\\&\left[\frac{\left(m+m^3\right)}{r^3}-\frac{k\left(m+m^3\right)}{r^5}\right]\delta^3V=0\end{aligned} \quad (32)$$

These two boundary conditions hold on x = 0, 1. For convenience we have introduced $\epsilon = 1/n$, $\delta = (a_2 - a_1)$ n. Now we write:

$$A_1 = 1 + \epsilon^{\frac{3}{2}}\xi \tag{33}$$

and look for a solution for μ_1 of the form:

$$\mu_1 = 1 + \epsilon_1^{(1)} + \epsilon^{\frac{3}{2}}{}_2^{(1)} + \epsilon^{2(1)}_{3} + ... \tag{34}$$

For V we assume:

$$V = V_0^{(1)} + \epsilon V_1^{(1)} + \epsilon^{\frac{3}{2}} V_2^{(1)} + \epsilon^2 V_3^{(1)} + ... \tag{35}$$

By substituting Eq. 34-36 into Eq. 31-33 and equating coefficients of similar powers of ϵ, we obtain a couple of differential equations and boundary conditions which can be derived and solved successively withthe aid of Mathematica. By solving these equations up-to and including $O(\epsilon)$, we find:

$$\mu_1^{ii} = 1 - \epsilon \frac{5\xi^2}{18} - \epsilon^2 \left(\frac{1}{2} + \frac{5\xi^2}{18} + \frac{37\xi^4}{1080} \right) + ... \tag{36}$$

This solution for A_1-1 = O(ϵ3/2) can be matched with the solution Eq. 23 validfor A_1- 1 = O(ϵ) by using Van Dyke (1975) matching rule, i.e., by substituting $\epsilon = 1/n$ we have $\mu_1^{ii} = \mu_1^{i}$ which indicates that on the leading order μ_1^{ii} is matched with μ_1^{i} . But it cannot be matched for two-term, therefore, the mentioned thickness does not include all the rapid changes anticipated and hence, we should continue our study further to reach to the proper thickness. In terms of the original variables A1 and n, (Eq. 37) becomes:

$$\begin{aligned} \mu_1^{ii} = 1 - \frac{1}{18} n^2 (A_1 - 1)^2 - \frac{1}{2n^2} - \frac{5}{18} \\ n(A_1 - 1)^2 - \frac{37}{1080} n^4 (A_1 - 1)^4 + O(\frac{1}{n^3}).... \end{aligned} \tag{38}$$

To determine the critical mode number at which μ_1^{ii} attains its minimum, we consider:

$$\begin{aligned} \frac{\partial \mu_1^{ii}}{\partial n} = -\frac{1}{9} n (A_1 - 1)^2 - \frac{1}{n^3} - \frac{5}{18} (A_1 - 1)^2 \\ -\frac{37}{270} n^3 (A_1 - 1)^3 + O\left(\frac{1}{n^4} \right) \end{aligned} \tag{39}$$

For A_1-1 = O(1/3/n2), the first term on the right-handed side of Eq. 39 is the only leading order term and is negative. Thus, μ_1^{ii} is a decreasing function of n. It is clear that the critical mode number cannot be of order (A_1-1)3/2. It should be noticed that $\partial\mu_1^{ii}/\partial\mu_1^{i}$ may become zero if the first termcan be balanced by the higher-order terms. This occurs when A_1-1 = O ($1/n^2$) which is the regime we consider in the next subsection.

RESULTS AND DISCUSSION

Asymptotic results for A_1- 1 = O($1/n^2$): An inspection of Eq. 37 shows that this asymptotic expansion becomes invalid when $\epsilon\xi$2 = 0(ϵ^2). In terms of A1 this is A1= O(ϵ^2). Inthe new regime we write:

$$A_1 = 1 + \epsilon^2 \tag{40}$$

where is an O(1) constant and we look for the following form of solutionsfor μ1 and V:

$$\mu_1 = 1 + \epsilon^{2(2)}_{1} + \epsilon^{3(2)}_{2} + \epsilon^{4(2)}_{3} + ... \tag{41}$$

$$V = V_0^{(2)} + \epsilon^2 V_1^{(2)} + \epsilon^3 V_2^{(2)} + \epsilon^3 V_3^{(2)} + ... \tag{42}$$

Where the expansions for $\mu_1$and V are deduced from Eq. 36 and 37 by replacing ξ by $\xi^{3/2}\Phi$. By substituting Eq. 40-42 into Eq. 31-33 and equating coefficients of similar powers of ϵ, we obtain an infinite number of differential equations and boundary conditions. As in the pervious subsection, these equations and boundary conditions can again be derived andsolved successively with the aid of Mathematica. By solving these equationsupto and including $O(\epsilon)^8$ we find:

$$\begin{aligned} \mu_1^{iii} = 1 - \epsilon^2 \left(\frac{1}{2} + \frac{5}{18} F^2 \right) + \epsilon^3 \left(\frac{1}{2} - \frac{5}{18} F^2 \right) \\ + \epsilon^4 \left(-\frac{1}{135} F^4 + \frac{5}{36} F^3 - \frac{1}{18} F^2 + \frac{1}{9} F - \frac{1}{8} \right) + ... \end{aligned} \tag{43}$$

We note that with Φ replaced by ζ/ϵ $\zeta/1/\epsilon_2$, $\mu_1$iii and reduces to equations $\mu^{i/1}$ and $\mu^{ii/1}$, respectively. Thus equation $\mu^{iii/1}$ include all the regions of rapid change and can be matched by equationμ_1^{out} (i.e., (21s) directly. With the use of Van Dyke's matching rule by following form:

$$\lim_{x \to \infty} \lim_{x \to \infty} \mu_1^{iii} = \lim_{r \to 1} \lim_{r \to 1} \mu_1^{out} \tag{44}$$

where X and (x-x_0)/$\delta(\in)$ = O(1), when x→∞$\in^2$, δ→0, we obtain μ_1^{matc} h by replacing $\Phi = \zeta/\epsilon$ into expansion Eq. 43 as follows $\mu_1^{match} = {}^{\epsilon 2-0\lim}\mu_1{}^{iii}$:

$$1-\frac{5\xi^2}{18}+\ldots+\epsilon\left(-\frac{5\xi^2}{18}+\frac{5\xi^3}{36}+\ldots\right) \quad (45)$$

where, μ_1^{match} is the expansion of either the inner or outer approximationsin the matching region. We find the following composite solution which isuniformly valid for A_1-1 = O (ϵ^p), 0 ≤m≤ = 2 :

$$\mu_1^{comp}=\mu_1^{out}-\frac{\in^{2\,\mu_1}}{2}s^{comp}+\frac{\in^3}{2}+ \\ \in^{4\mu_1^{outs}}\left(-\frac{1}{8}+^{\mu_1\,iii}+\frac{1}{9}\Phi+\frac{1}{18}\Phi^2\right)+\ldots \quad (46)$$

A_1-1 = O(1) and A_1-1 = O(1/n^2) by using μ_1^{match}. We have shown the composite solution is the cause of the complete matching of the two regions A_1-1 = O(1) and A_1-1 = O(1/n^2). We compared neutral stability curves of asymptotic matching solution (uniformly valid curve for $\mu_1$and μ_2 against A_1 for A_1-1 = O(1) and A_1-1 =O(1/n^2) by solving numerically Eq. 19 with the use of Compound matrix method. Matching two regions A_1-1 = O (1) and A_1-1 = O(1/n^2) (Fig. 2 and 3) solid line: Numerical result. The numerical solution, $\mu_1$and μ_{12} for n = 20 are smaller than those forn = 15. We find a critical neutral curve of Eq. 43 which is the envelope of the neutral curves corresponding to different mode numbers. In terms of the original variables A1 and n, Eq. 43 takes the form:

$$\mu_1^{iii}=1-\frac{1}{2n^2}-\frac{5}{18}n^2(A_1-1)^2+ \\ \frac{1}{2n^3}-\frac{5}{18}n(A_1-1)^2+O\left(\frac{1}{n^4}\right) \\ \frac{\partial\mu_1^{iii}}{\partial n}=-\frac{5}{9}n\left(A_1-1\right)^2-\frac{5}{18} \\ \left(A_1-1\right)^2+\frac{1}{n^3}-\frac{3}{4n^4}+O\left(\frac{1}{n^5}\right) \quad (47)$$

By solving $\partial\mu_1^{iii}/\partial nss$ = 0s we obtain the critical mode number:

$$n=\frac{1}{2}+\frac{5}{9\sqrt{14}}\left(A_1-1\right)^2 \quad (48)$$

On substituting Eq. 48 into Eq. 47, we obtain the following expression for the critical neutral curve which is the envelope of the neutral curves corresponding to different mode numbers:

$$\mu_1=1+\frac{1}{9}\left(A_1-1\right)-\frac{13}{14}\left(A_1-1\right)^2+\ldots$$

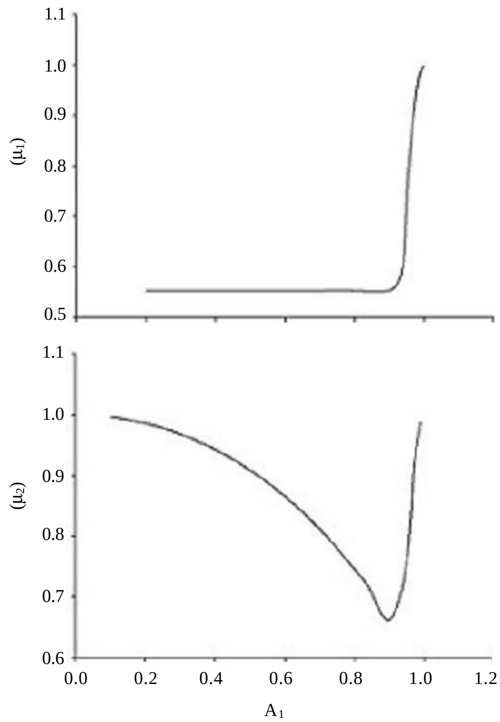

Fig. 2: Neutral stability curve for n = 20 of the asymptotic result obtained form matching two regions

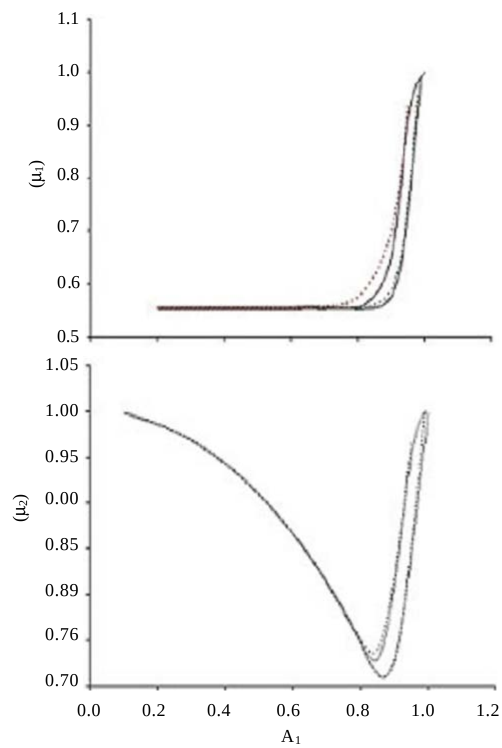

Fig. 3: Neutral stability curves for n = 15, 20. Dotted line: the asymptotic result obtained form

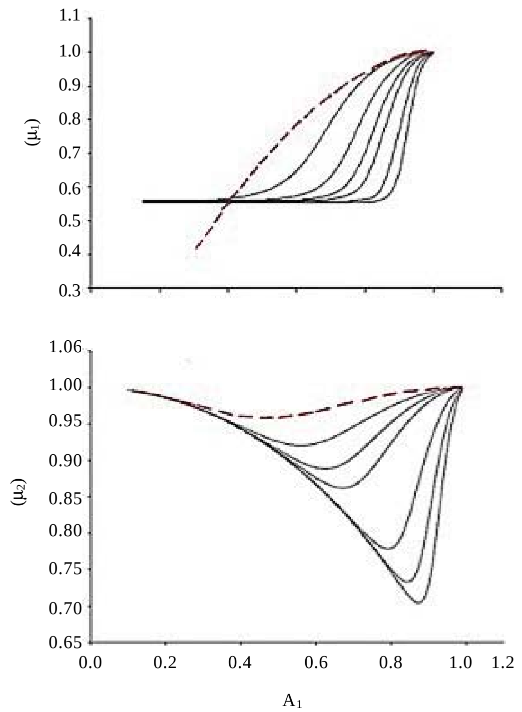

Fig. 4: The solid lines represent the numerical results for n = 5, 8, 10, 15, 20, 40. The dotted lines are the asymptotic critical neutral curves corresponding to Eq. 48

We have shown the asymptotic results Eq. 48 together with the results from a fully numerical integration. The validity of Eq. 48 is obvious (Fig. 4).

CONCLUSION

On solving the fourth order eigenvalue problem of the neo-Hookean cylindrical shell by WBK method which has been done in two layers (i.e., outerA_1-1 = O(1) and inner A_1-1= O(1/n) on $0<A_1<1$), we observe that athin layer exists between the two solutions. In this study, our main concern isto find and solve the asymptotic solution of this logarithmic layer and finally compare this solution with the counterpart numerical one. In order to be able to study this layer, we are writing down a brief description of the asymptoticinner and outer solutions previously done by M. Sanjaranipour (2010). In order to obtain the proper thickness, the boundary layer theory and Van Dyke's matching rule were employed. For A_1-1, we checked different thickness (i.e., O (1/n). O ($1/3n^2$) and O ($1/n^2$). Finally by obtaining a composite solution which is uniformly valid for A_1-1 = O ($\in^m$) $0\leq m\leq 2$, we could match theouter region (A_1-1 = O (1)) with the inner region A_1-1 = O (1/n). The asymptotic matching solution is coincident with the numerical solution. A_1 critical neutral curve which enveloped the neutral curves corresponding to different mode numbers is also verified.

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