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# Numerical Scheme for Wave Propagation Problem

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**Abstract:** Turkel studied a hyperbolic equation using composite method. Motivated by Turkel, we study a corresponding problem (wave propagation problem) using a numerical scheme, namely, central difference scheme. The stability and convergence of the scheme are investigated. From the analysis the order of dissipativity for the central difference scheme is four compared to that of the composite scheme, which is six; hence, the central difference scheme is less dissipative than the composite scheme. Another feature of the central difference scheme is that it allows more refinement of the grid size than the composite scheme and this refinement helps to improve the accuracy of the central difference scheme. A numerical example based on a wave propagation problem (initial boundary value problem) is solved using analytic and computer methods and the results are compared with that of the central difference scheme.

**Key words:** Numerical method, wave propagation, central difference scheme, IBVP, hyperbolic equation, Nigeria

## INTRODUCTION

There are several procedures that come under the numerical methods. Nonlinear problems in fluid dynamics, elasticity and potential theory involving two and three dimensions are being solved today using numerical techniques that were not even considered a few years ago. Turkel (1977) studied a hyperbolic problem using composite scheme. Other contributors to solutions of problems using finite difference scheme include, Notably (Bilbao and Smith, 2003; Courant *et al*., 1928; Forsythe and Wasow, 1960; Furihata, 2001; Gustafsson, 1975; Khalifa and Raslan, 1999; Luo *et al*., 2001; Oliveira, 2003; Potter and Okoniewski, 2001; Shao and Liu, 2001; Zingg, 2000).

In this research, we solve a wave propagation problem using central difference scheme. The result is compared with that of the composite scheme (Turkel, 1977). We then proceed to solve the Initial Boundary Value Problem (IBV P) using the analytic and computer methods. Comparison of the results with that of the central difference scheme shows that the central difference scheme is sufficiently accurate.

## RECURRENCE SCHEME FOR WAVE PROPAGATION EQUATION

A uniform x-t grid with mesh size $\Delta x = h$ in the x-direction and $\Delta t = k$ in the t-direction is introduced. Thus, the difference formula for approximating the wave Eq. 1:

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) \;\; 0 < x < a, \;\; 0 < t < b \tag{1}$$

using central difference method is as:

$$u_{tt}(x,t) = \frac{u(x,t+k) - 2u(x,t) + u(x,t-k)}{k^2} + O(k^2) \tag{2}$$

$$c^2 u_{xx}(x,t) = c^2 \left\{ \frac{\begin{matrix} u(x+h,t) - 2u(x,t) \\ + u(x-h,t) \end{matrix}}{h^2} \right\} + O(h^2) \tag{3}$$

Dropping the terms $O(k^2)$ and $O(h^2)$ and using the approximation $U_{i,j}$ for $u(x_i, t_j)$ in (Eq. 2 and 3), which in turn are substituted into the PDE Eq. 1, we obtain:

$$U_{i,j+1} - 2U_{i,j} + U_{i,j-1} = \frac{c^2k^2}{h^2}\left(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right) \tag{4}$$

For convenience, we introduce the substitution $r = ck/h$ (where, r is called the Courant number) in (Eq. 4) and we obtain the relation:

$$U_{i,j+1} - 2U_{i,j} + U_{i,j-1} = r^2\left[U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right] \tag{5}$$

Solving (Eq. 5) for $U_{i,j+1}$ we obtain a recurrence scheme:

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$$U_{i,j+1} = (2-2r^2)U_{i,j} + r^2\left[U_{i+1,j} + U_{i-1,j} - U_{i,j-1}\right], \quad i = 1,2,\ldots,n-2 \tag{6}$$

where, n is the number of grid points in the x-direction. Accordingly, the number of grid points in the t-direction w ill be designated by m.

## STABILITY AND CONVERGENCE ANALYSIS

Using Von Neumann's method or Fourier series method, namely,

$$U_{i,j} = e^{\sqrt{-1}\beta ih} e^{\alpha jk}$$

in (Eq. 6) (where, β and α are any real numbers), the amplification factor $\xi = e^{\alpha k}$ is easily shown to be:

$$\begin{aligned}\xi &= \left(1-r^2\right) + \frac{r^2}{2}\left(e^{i\beta h} + e^{-i\beta h}\right) - r/_2\left(e^{e\beta h} - e^{-i\beta h}\right)\\ &= 1 - r^2 + r^2\cos\beta h - ir\sin\beta h\\ &= 1 - r^2 + r^2\left[\cos^2\frac{\beta h}{2} - \sin^2\frac{\beta h}{2}\right] - ir\sin\beta h\\ &= \left[1 - 2r^2\sin^2\frac{\beta h}{2}\right] - ir\sin\beta h\end{aligned}$$

where, r = ck/h. This gives:

$$\begin{aligned}|\xi| &= \left|\left[1 - 2r^2\sin^2\frac{\beta h}{2}\right] - ir\sin\beta h\right|\\ &= \left\{\left[1 - 2r^2\sin^2\frac{\beta h}{2}\right] + \left(-r\sin\beta h\right)^2\right\}^{1/2}\\ &= \left[1 - 4r^2\sin^2\frac{\beta h}{2_-} + 4r^4\sin^4\frac{\beta h}{2} + 4r^2\sin^2\beta h\cos^2\frac{\beta h}{2}\right]^{1/2}\\ &= \left[1 - 4r^2\left(1-r^2\right)\sin^4\frac{\beta h}{2}\right]^{1/2}\end{aligned} \tag{7}$$

from which it follows that $|\xi| \le 1$ if $0 \le r \le 1$. Hence, the (Eq. 6) is stable if $0 \le r \le 1$.

The convergence of the scheme follows the Courant *et al*. (1928) (C.F.L.) condition for convergence, which applies to explicit t difference replacement of hyperbolic equations. It requires the (Eq. 6) to be convergent when $0 \le r \le 1$. Thus, the stability condition coincides with the C.F.L. condition.

## NUMERICAL ILLUSTRATION: WAV E PROPAGATION PROBLEM

$$\text{PDE} = u_{tt}(x,t) = c^2 u_{xx}(x,t),\ 0 < x < a,\ 0 < t < b \tag{8}$$

Table 1: Finite difference result of the IBVP

| | x | | | | | |
|---|---|---|---|---|---|---|
| t | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| 0 | 0 | 0.58778 | 0.95105 | 0.95105 | 0.58778 | 0 |
| 0.1 | 0 | 0.47552 | 0.76941 | 0.76941 | 0.47552 | 0 |
| 0.2 | 0 | 0.18163 | 0.29388 | 0.29388 | 0.18163 | 0 |
| 0.3 | 0 | -0.18164 | -0.29390 | -0.29390 | -0.18164 | 0 |
| 0.4 | 0 | -0.47553 | -0.76942 | -0.76942 | -0.47553 | 0 |
| 0.5 | 0 | -0.58778 | -0.95105 | -0.95105 | -0.58778 | 0 |

$$\text{B.Cs} = \left.\begin{aligned}u(0,t) &= 0\\ u(a,t) &= 0\end{aligned}\right\}\ 0 < t < b \tag{9}$$

$$\text{I.Cs} = \left.\begin{aligned}u(x,0) &= \sin(\pi x)\\ u_t(x,0) &= 0\end{aligned}\right\}\ 0 < x < a \tag{10}$$

Here, we give the numerical solution of the IBVP for x = 0 (0.2) 1, t = 0 (0.1) 0.5, c = 2. Here, a = 1, b = 0.5. For r = 1 (Eq. 6) becomes:

$$U_{i,j+1} = \left(U_{i-1,j} + U_{i+1,j}\right) - U_{i,j-1,}\ i = 1(1)4 \tag{11}$$

The boundary conditions in finite difference form are:

$$\left.\begin{aligned}U_{0,j} &= o, \quad j = 0(1)5\\ U_{5,j} &= o, \quad j = 0(1)5\end{aligned}\right\} \tag{12}$$

while, the initial conditions are:

$$\left.\begin{aligned}U_{i,0} &= \sin\pi x, \quad i = 0(1)5\\ \left(U_{i,1} - U_{i,-1}\right)/2k &= 0, i = 0(1)5\end{aligned}\right\} \tag{13}$$

Applying the Eq. 11-13, successively to generate rows and computing the pivotal value in each row produce the finite difference solution to the IBVP corrected to five decimal places shown in Table 1.

## ANALYTICAL SOLUTION OF THE IBVP

Using d' Alembert's method the required analytical solution for the IBVP (Eq. 8)-(10) is:

$$u(x,t) = \sin\pi x\cos 2\pi t \tag{14}$$

The analytical values computed for x = 0 (0.21) 1 and t = 0 (0.1) 0.5 and corrected to five places of decimal are shown in Table 2.

Table 2: Analytical result of the IBVP

| | x | | | | | |
|---|---|---|---|---|---|---|
| t | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| 0 | 0 | 0.58778 | 0.95106 | 0.95106 | 0.58778 | 0 |
| 0.1 | 0 | 0.47553 | 0.76942 | 0.76942 | 0.47553 | 0 |
| 0.2 | 0 | 0.18163 | 0.29389 | 0.29389 | 0.18163 | 0 |
| 0.3 | 0 | -0.18163 | -0.29389 | -0.29389 | -0.18163 | 0 |
| 0.4 | 0 | -0.47553 | -0.76942 | -0.76942 | -0.47553 | 0 |
| 0.5 | 0 | -0.58778 | -0.95106 | -0.95106 | -0.58778 | 0 |

## A BASIC PROGRAM FOR THE FINITE DIFFERENCE SOLUTION OF THE WAVE EQUATION

```
REM Finite Difference Solution for the Wave Equation
REM
DIM U(100,100)
DEF FNF (I) = SIN (H* I* 3. 142)
DEF FNG (I) = 0
IN PUT          A
IN PUT          B
INPUT           C
INPUT           M
INPUT           N
H       =       A/(N-1)
K       =       B/(M-1)
R       =       C×K/H
R2      =       R^2
R22     =       R×R/2
S1      =       1-R^2
S2      =       2-2×R^2
Check out loop number 1
FOR J   =       0 TO M-1
U (O,J) =       0
NEXT J
FOR J   =       0 TO M-1
U (N-1,J) =     0
NEXT J
Check out loop number 2
FOR I   =       1 TO N-2
U (I,0) =       FNF (I)
U (I,1) =       S1×FNF(I) + K× FNG (I)+ R22×(FNF (I+1) + FNF (I-
                1))
NEXT I
Check out loop number 3
FOR J   =       2 TO M-1
FOR I   =       1 TO N2-
U(I, J) =       S2×U(I,J-1)+R2×(U(I-1,J-1)+U(I+1, J-1))-U(I, J-2)
NEXT I
NEXT J
Check out loop number 4
CLS
L PRINT =       L PRINT
FOR J   =       0 TO M-1
FOR I   =       0 TO N-1
L PRINT U (I, J);
NEXT I
L PRINT =       L PRINT
NEXT J
End
```

The computer values obtained for the particular data A = 1, B = 0.5, C = 2, M = 6, N = 6 and rounded-off to five decimal places are shown in Table 3.

Table 3: Computer result of the IBVP

| | x | | | | | |
|---|---|---|---|---|---|---|
| t | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
| 0 | 0 | 0.58785 | 0.95111 | 0.95098 | 0.58752 | 0 |
| 0.1 | 0 | 0.47555 | 0.76942 | 0.76931 | 0.47529 | 0 |
| 0.2 | 0 | 0.18156 | 0.29376 | 0.29376 | 0.18179 | 0 |
| 0.3 | 0 | -0.18179 | -0.29413 | -0.29376 | -0.18156 | 0 |
| 0.4 | 0 | -0.47569 | -0.76931 | -0.76942 | -0.47555 | 0 |
| 0.5 | 0 | -0.58752 | -0.95098 | -0.95111 | -0.58785 | 0 |

## DISCUSSION

We observe that the recurrence relation (Eq. 6) for the wave (Eq. 1) is stable when the dimensionless parameter (Courant number) r is equal to or <1 and convergent in the sense of Courant, Friedrichs and Lewy (C.F.L.) condition for convergence; that is, the central difference scheme (Eq. 6) is convergent for $0<r\leq 1$. Thus, stability and convergence conditions for (Eq. 6) coincide. From (Eq. 7) the (Eq. 6) is evidently dissipative for long wave lengths and the order of dissipativity is four. Thus, the central difference scheme is better than the composite scheme (Turkel, 1977), in which the order of dissipativity is six. Also the central difference scheme allows more refinement of the mesh size than the composite scheme and this refinement property helps to improve the accuracy of the central difference scheme. Finally, comparison of Table 1-3 reveals that the central difference scheme based on the IBVP agrees reasonably with the analytical and computer result, thus attesting to the accuracy of the central difference scheme.

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