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Fractional Calculus and Non-Differentiable Functions

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Abstract: In this study, we present investigations of certain new clarified and simple proof of the generalized Taylor series, in terms of fractional order derivatives, which is very helpful in approximating non-differentiable functions.

Key words: Fractional Taylor Series (FTS), Rieman-Liouville differentiability, non-differentiable

INTRODUCTION

Several natural phenomena lead to irregular (fractal) models. For example, typical paths of quantum mechanical particles are continuous but non-differentiable. Despite the non-differentiable structures in nature, we have few mathematical tools to deal with.

An idea is to generalize the notation of derivative in order to take into account non-differentiable functions. Many attempts already exists, in particular, the so-called fractional derivative of Riemann-Liouville, Liouville (Adda, 1997; Samko *et al*., 1993). They are all, more or less, based on a generalization of the Cauchy formula. Hence, there is no geometric idea supporting these generalizations, explaining the difficulties of using it in order to obtain information about the structure of non differentiable objects. Moreover, fractional derivative are all non local on the contrary of the classical derivative. For example, the Rieman-Liouville derivative depends on a free parameter, which relies on global information on the function. The study of non-differentiable functions via these operators is then difficult.

In Adda and Cresson (2000) a notation of (right or left) local fractional derivative, was introduce to solve this problem, by generalizing the classical Taylor series to non-differentiable cases, but unfortunately, the proof of the generalized Taylor expansion theorem is not true in general Adda (2001). Therefore, in this study, we present a clarified and simple proof to the generalized Taylor series.

RIEMAN-LIOUVILLE DIFFERENTIABILITY

Let, f be a continuous function on (a, b). For all x ϵ [a, b], we define the left (respectively right) Rieman-Liouville integral at the point x by:

$$I^{\alpha}_{a,-}(f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt$$

$$I^{\alpha}_{b,+}(f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt$$

The left (respectively right) Rieman-Liouville derivative at x is given by

$$D^{\alpha}_{a,-}(f)(x)=\frac{d I_a^{1-\alpha}(f)(x)}{dx}$$

$$D^{\alpha}_{b,+}(f)(x)=\frac{d I_b^{1-\alpha}(f)(x)}{dx}$$

Definition: We say that the function f admits a derivative of order $0<\alpha<1$ (Riemann-Liouville) at x ϵ [a, b] by below (respectively above) of D^{α}_{a} (f) (x) exists (resp. if D^{α}_{b+} (f) (x) exists).

Of course, different values of the Rieman-Liouville derivative for different values of the parameter a (resp. b) are obtained. Moreover, the derivative of a constant C $\in\Re$ is not equal to zero. Indeed, we have

$$D^{\alpha}_{a,-}(c)(x)=\frac{C}{\Gamma(1-\alpha)}\frac{1}{(x-a)^{\alpha}}$$

These give rise to great difficulties in the geometric interpretation of the Riemann-Liouville derivative (Adda, 1997). In particular; there is no relationship between the local geometry of the graph of f and its derivative. We refer to Podlubny (1999). We have

$$\frac{d^n}{dt^n}(D^p_{a,-}(f)(t))=D^{n+p}_{a,-}(f)(t) \qquad (1)$$

On the contrary, we have

$$D^p_{a,-}\left(\frac{d^n f(t)}{dt^n}\right)(t)=D^{p+n}_{a,-}(f)(t)-\sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^{j-p-n}}{\Gamma(1+j-p-n)} \qquad (2)$$

The Riemann-Liouville derivative commutes with the usual derivative if and only if $f^{(k)}(a) = 0$ for $k = 0 \ldots n-1$.

We have also the following composition formula: let $m-1 \leq p \leq m$ and $n-1 \leq q \leq n$; then

$$D^{p}_{a,-}(D^{q}_{a,-}f)(t))(t) = D^{p+q}_{a,-}(f)(t)$$
$$-\sum_{j=1}^{n}[D^{q-j}_{a,} - (f)(t)]_{t=a}\frac{(t-a)^{-p-j}}{\Gamma(1-p-j)}$$
$$D^{q}_{a,} - (D^{P}_{a,-}(f)(t))(t) = D^{p+q}_{a,t}(f)(t)$$
$$-\sum_{j=1}^{m}([D^{p=j}_{a,t}f(t)]_{t=a})(t)\frac{(t-a)^{-q-j}}{\Gamma(1-q-j)}$$

In general, we have no commutation between Riemann-Liouville derivatives. Commutation holds if and only if $f^{(j)}(a) = 0$, $j = 0,\ldots r-1$ with $r = \max(n, m)$ and similarly for the right fractional derivative.

TAYLOR SERIES EXPANSION

It is never intended for publication, Riemann scratched down a generalized Taylor-formula based on fractional derivatives, Riemann never gave any proof of this identity, he discussed the series as an asymptotic expansion for the special cases of expanding certain classes of elementary functions involving the exponential function and the binomial function, we give a proof of pointwise convergence of the general Taylor-Riemann series, the series is a form of fractional power series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^{\infty+n}$$

The via fractional calculus generalized Taylor series has down some attentions among mathematicians over the years, some alternate forms have appeared but our proof here seams to be unique in generality and gives a clarified and simplified proof.

Theorem: The (right or left) local fractional derivatives of f, $d^{\alpha}_{\sigma} f(x)$ is equal to

$$\frac{d^{\alpha}_{\sigma}f(x)}{dx^{\alpha}} = \Gamma(1+\alpha)\lim_{y\to x^{\sigma}}\frac{\sigma(f(y)-f(x))}{|y-x|^{\alpha}} \quad (3)$$

Proof: Adda and Cresson (1999).

Theorem: Let $0<\alpha<1$ and f be a continuous function such that $d^{\alpha}_{\sigma} f(y)$ exist $\sigma = \mp$ then we have

$$f(x) = f(y) + \sigma\frac{1}{\Gamma(1+\alpha)}.\frac{d^{\alpha}_{\sigma}f(y)}{dx^{\alpha}}[\sigma(x-y)]^{\alpha} + R_{\omega}(x,y) \quad (4)$$

where

$$\lim_{x\to y^{\alpha}}\frac{R_{\sigma}(x,y)}{(\sigma(x-y))} = 0$$

The proof of this theorem use the fact that the composition of the Rieman-Liouville fractional derivatives of $\Delta_{\alpha} f(x) = f(x)-f(a)$, where a is the parameter equal to $\Delta_{\alpha} f(x)$, which is not true in general. Therefore, we give a clarified and simplified proof of the following theorem:

Theorem: Let $f(x) = (z-b)^{\delta} h(z)$, where $\delta > -1$, h (z) is a function, which is analytic in some open set containing the disk $\{|z-a| \leq r\}$ for some $r>0$ and b belongs to the interior of this disk, i.e $|b-a|<r$. We then have for every $\alpha \in R$ and every Z on the circler $|z-a| = |b-a|$, $z \neq b$.

$$f(z) = \sum_{n=-\infty}^{\infty}\frac{D^{\alpha+n}_{z-b}f(a)}{\Gamma(\alpha+n+1)}(z-a)^{\alpha+n} \quad (5)$$

Here in the right hand side, for $(z-a)^{\alpha-n}$, we choose the branch, which is given by $\arg(b-a) < \arg(z-a) < \arg(b-a) + 2\pi$, where, we have fixed $-\pi \leq \arg(b-a) \leq \pi$.

Proof: For convenience, we write $\phi = \arg(b-a)$, where we require $-\pi \leq \phi \leq \pi$. We express Z on the circle $|z-a| = |b-a|$, $z \neq b$ as $z = a + |b-a|e^{i\theta}$. $\phi \in [\phi, \phi + 2\pi]$ and expand $f(z)/(z-a)^{\alpha}$ in a Fourier's series.

$$\frac{f(z)}{(z-a)^{\alpha}} = \sum_{n=-\infty}^{\infty} a_n e^{i\theta n} \quad (6)$$

By Fourier's theorem Manager (1983) and our conventions about the branches, the coefficients a_n are given by

$$a_n = \frac{1}{2\pi}\int_{\phi}^{\phi+2\pi}\frac{f(a+/b-a/e^{i\phi}}{/b-a/^{\alpha}e^{i\alpha\phi}}.e^{-in\phi}.d\phi \quad (7)$$

Note that the function $\phi \to f(a+|b-a|e^{i\phi})/|b-a|^{\alpha}.e^{i\phi}$. Has limited total fluctuation in a neighborhood of $\phi = \arg(z-a)$, since $z \neq b$; hence by Fourier theorem the right hand side in (z) is indeed convergent and the equality holds.

On other hand, we consider the Cauchy-type functional integral (Olver, 1954) of order $\alpha + n$ of f (z) at the point α:

$$D^{\alpha+n}_{z-b}f(a) = \frac{\Gamma(\alpha+n+1)}{2\pi}\int_{\theta}^{\theta+2\pi}\frac{f(a+|b-a|e^{i\theta})}{|b-a|^{\alpha+n+1}e^{i(n+1)\theta}e^{i\alpha\theta}}|b-a|ie^{i\theta}d\theta$$
$$= \frac{\Gamma(\alpha+n+1)}{2\pi}\int_{\theta}^{\theta+2\pi}\frac{f(a+|b-a|e^{i\theta})}{|b-a|^{\alpha+n}e^{i(\alpha+n)}}d\theta$$

Comparing this with Eq. 7, we see that

$$a_n = \frac{D_{z-b}^{\alpha+n} f(a)}{\Gamma(\alpha+n+1)} |b-a|^n \qquad (8)$$

So by inserting this in Eq. 6 and noticing $(z\text{-}a)^{\alpha}$ $|b\text{-}a|^n$ $e^{i\theta n} = (z\text{-}a)^{\alpha+n}$, we obtain

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{\alpha+n} f(a)}{\Gamma(\alpha+n+1)} (z-a)^{\alpha+n} \qquad (9)$$

On the circle $|z\text{-}a| = |b\text{-}a|$, $b \neq z$, which proves the theorem.

In Adda (2001), they obtained a simplified equivalent definition of local fractional derivatives. Once of the most important tools of numerical analysis is (Taylor's theorem) and the associated Taylor series. It gives a relatively simple method for approximating and computing functions f (x) by polynomials.

RESULTS AND DISCUSSION

Our main result base on theorem (3), in which, we discovered a simple method for approximating and computing f (x) by Taylor series polynomial. From few years ago many interesting applications of the so-call Fractional Taylor Series (FTS) have been published under the name (FTS) are known several different definitions of the generalization of the ordinary Taylor series expansion. this fractional series expansion have been widely applied mainly in optics and signal processing theory.

CONCLUSION

- A clarified and simple proof to the generalized Taylor series, which present in the theorem (3), use the fact that the composition of the Rieman-Liouville fractional derivatives of Δ_α f (x) = f (x) - f (a), where a is the parameter equal to Δ_α f (x)
- A simple method for a computing f (x) by Taylor series

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