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An Improved Error Estimation of the Tau Method for Boundary Value Problems in Ordinary Differential Equations

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Abstract: We constructed a polynomial error approximant of the error function $e_n(x)$ of the Lanczos Tau method for ordinary differential equations, based on the error of the Lanczos economization process. In the present research, we modify this approximant for boundary value problems in ordinary differential equations by perturbing some of the homogenous condition of $e_n(x)$ and show that the new approximant, thus obtained, yields a more accurate estimate of the maximum error. Numerical results further confirm that the order of the Tau approximant is also accurately estimated.

Key words: Tau method, differential form, integrated form, recursive form, Tau approximant, error estimate, variant

INTRODUCTION

The Tau method of Lanczos (1938), Coleman (1976) and Okunuga (1984) solves the class of the m-th order linear differential equation:

$$Ly(x) = \sum_{r=0}^{m}\left(\sum_{k=0}^{N_r} p_{rk}x^k\right) y^{(r)}(x) = \sum_{r=0}^{F} f_r x^r, \quad a \le x \le b \qquad (1.1a)$$

with the associated multi-point boundary conditions

$$L^*y(x_{rk}) = \sum_{r=0}^{m-1} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, \; k = (1)m \qquad (1.1b)$$

and where, N_r, F are given non-negative integers; a_{rk}, x_{rk}, α_k, f_r, P_{rk} for r = 0(1) m, k = 1 (1) m.

The method seeks an approximant (Tau approximant):

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \quad n < +\infty \qquad (1.2)$$

of y(x) which satisfies exactly the corresponding perturbed problem:

$$Ly_n(x) := \sum_{r=0}^{F} f_r x^r + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \; T_{n-m+r+1}(x) \qquad (1.3a)$$

$$L^*y_n(x_{rk}) = \alpha_{rk}, \; k = 1(1)m \qquad (1.3b)$$

where, Tau parameters (hence, the name 'Tau method'), τ_r, r = 1(1) m+s, are to be determined from Eq.(1.3) along with a_r, r = 0(1) m. We shall denote a typical Chebyshev polynomial $T_k(x)$ by:

$$T_k(x) = \sum_{r=0}^{k} c_r^{(k)} x^k \qquad (1.4)$$

The parameter s called the number of over determination of Eq. (1.1a), is defined by (Fox, 1968):

$$s = \max\{|N_r - r| : 0 \le r \le m\} \qquad (1.5)$$

This original formulation of the Tau method shall herein be referred to as the Differential Form of the method.

Two other variants of the method are the Integrated form (Adeniyi, 2000; Fox, 1962; Fox and Parker, 1968; Ortiz, 1974) and the recursive form Lanczos (1956), Ortiz (1969, 1974) and Freilich and Oritz (1982).

If

$$\int\int \underset{\dots}{m} \int g(x)\,dx$$

denotes the m-times indefinite integration of g(x) and

$$I_L = \int\int \underset{\dots}{m} \int L(\bullet)\,dx \qquad (1.6)$$

then the Integrated Form of (1.1a) is

$$I_L(y(x)) = \int\int \underset{\dots}{m} \int f(x)\,dx + c_m(x) \qquad (1.7)$$

where, $c_m(x)$ is an arbitrary polynomial of degree (m-1) arising from the constants of integration. Thus, the integrated Tau problem corresponding to Eq.(1.3) is the perturbed problem.

$$I_L(y_n(x)) = \int\int\underset{\dots}{m}\int f(x)dx + c_m(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \quad T_{n+r+1}(x) \qquad (1.8a)$$

$$L \times y_n(x_{rk}) = \alpha_k, \quad k = 1(1)m \qquad (1.8b)$$

The problem (1.8) often gives a more accurate approximant of y(x) than Eq.(1.3) does, as it involves a higher order perturbation term (Fox, 1962; Fox *et al*., 1968).

For the recursive form we define the canonical polynomials $Q_r(x)$ $r \in N_0$-S, by:

$$LQ_r(x) = x^r \qquad (1.9)$$

where, S is small finite or empty set of indices with cardinality s (Ortiz, 1969, 1974; Crisci and Ortiz, 1981). We adopt Eq.(1.9) in Eq.(1.3a), that is, in

$$Ly_n(x) = \sum_{r=0}^{F} f_r x^r + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)} x^k$$

to have

$$\begin{aligned} Ly_n(x) &= \sum_{r=0}^{F} f_r LQ_r(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)} LQ_k(x) \\ &= L\left\{\sum_{r=0}^{F} f_r Q_r(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)} Q_k(x)\right\} \end{aligned}$$

since L is linear. If L^{-1} exists then

$$y_n(x) = \sum_{r=0}^{F} f_r Q_r(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+r+1)} Q_k(x) \qquad (1.10)$$

This is the Tau approximant of y(x) by the recursive form.

AN ERROR ESTIMATION OF THE TAU METHOD

In Adeniyi (1991), we constructed the error polynomial function:

$$(e_n(x))_{n+1} = \frac{\phi_n U_m(x) T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}}, \quad a \le x \le b \qquad (2.1)$$

as an approximant of the error function of

$$e_n(x) = y(x) - y_n(x)$$

An approximant of the error function: The Lanczos Tau approximant $y_n(x)$ of y(x) is an economized polynomial function implicitly defined by a differential equation. So, logically, based on the error of the Lanczos economization process (Gerald, 1970), it could be possible to construct an approximant of the error $e_n(x)$ in $y_n(x)$ as follows:

By the Lauczos economization process, we have that:

$$\begin{aligned} y_n(x) &= \sum_{r=0}^{n-1} b_r x^r + b_n x^n \\ &= \sum_{r=0}^{n-1} b_r x^r + b_n \left\{\frac{T_n(x)}{C_n^{(n)}} - \frac{1}{C_n^{(n)}} \sum_{r=0}^{n-1} C_r^{(n)} x^r\right\} \end{aligned} \qquad (2.2)$$

$$\begin{aligned} y_n(x) &= \sum_{r=0}^{n-1} \left\{b_r - \frac{b_n}{C_n^{(n)}} C_r^{(n)}\right\} x^r + \frac{b_n T_n(x)}{C_n^{(n)}} \\ &\equiv \sum_{r=0}^{n-1} a_r x^r + \frac{b_n T_n(x)}{C_n^{(n)}} \end{aligned} \qquad (2.3)$$

Thus,

$$y_n(x) \cong \sum_{r=0}^{n-1} a_r x^r$$

with an error

$$e_n(x) = \frac{b_n T_n(x)}{C_n^{(m)}} \qquad (2.4)$$

To make Eq.(2.4) suitably appropriate for all members of Eq.(1.1), we modify it to have the (n + 1)-th degree approximant

$$(e_n(x))_{n+1} = \frac{\varphi_n U_m(x) T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \qquad (2.5)$$

of the error

$$e_n(x) = y(x) - y_n(x)$$

in $y_n(x)$, where, the form of $U_m(x)$ depends on the boundary conditions Eq.(1.1b) and it is chosen to ensure that $(e_n(x))_{n+1}$ satisfies all the homogenous conditions of $e_n(x)$; φ_n is a parameter to be determined. Once $(e_n(x))_{n+1}$ is constructed, error estimation of the Tau method, as explained in the next section, is made possible.

Once $U_m(x)$ is determined, an error estimation of the Tau method, as explained in the next study, is also then made possible.

We remar here that error estimations for related integration schemes have been reported in the literature (Oliver, 1969; Zadunaisky, 1976).

AN IMPROVED ERROR ESTIMATE OF THE TAU METHOD

The choice of $U_m(x)$ which ensures that $(e(x))_{n+1}$ satisfies,

$$L\times\left(e_n\left(x_{rk}\right)\right)=0, \quad k=1(1)m \tag{3.1}$$

is desirable. However, it may not always guarantee a most accurate estimate of the maximum error

$$\varepsilon=\max|e_n(x)|, \quad a\le x\le b \tag{3.2}$$

in the range of consideration (a, b). Fox (1968) had earlier, confirmed this and suggested that the perturbation of some of the homogenous conditions of $e_n(x)$ for $(e_n(x))_{n+1}$ may give the desired accuracy.

This suggestion led us to consider the choice

$$U_m(x)=x^m \tag{3.3}$$

which, for boundary value problems, will not make Eq.(2.5),
that is:

$$\left(e_n(x)\right)_{n+1}=\frac{\varphi_n\ U_m(x)\ \ T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \tag{3.4}$$

satisfy all the homogeneous conditions Eq.(3.1).

These homogenous conditions of $e_n(x)_{n+1}$, which are not exactly satisfied by the new form

$$\left(e_n(x)\right)_{n+1}=\frac{\varphi_n\ x^m\ \ T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \tag{3.5}$$

may thus be considered perturbed for $(e_n(x))_{n+1}$.

The flexibility in the mode of choice of $U_m(x)$ as described in the study makes the choice Eq.(3.3) more appropriate and as will be seen later in the study, the choice Eq.(3.3) for $U_m(x)$ and hence Eq.(3.5) for $(e_n(x))_{n+1}$ leads to dramatic improvement in the accuracy of

$$\varepsilon_1^*=\max_{a\le x\le b}\left|\left(e_n(x)\right)_{n+1}\right|\cong\max_{a\le x\le b}\left|e_n(x)\right|=\varepsilon^* \tag{3.6}$$

Now having fixed the choice of $U_m(x)$ as Eq.(3.3) and consequently $(e_n(x))_{n+1}$ as Eq.(3.5), we then follow Adeniyi *et al.* (1990), Adeniyi and Onumanyi (1991) and Adeniyi and Erebhole (2007) to estimate ε^* by considering the perturbed error problem

$$L\left(e_n(x)\right)_{n+1}=-\sum_{r=0}^{m+s-1}\tau_{m+s-r}\ T_{n-m+r+1}(x)+\sum_{r=0}^{m+s-1}\bar{\tau}_{m+s-r}\ T_{n-m+r+2}(x) \tag{3.7a}$$

$$L^*\left(e_n\left(x_{rk}\right)\right)_{n+1}=0 \tag{3.7b}$$

The extra $\bar{\tau}$'s in Eq.(2.2) are fixed and are to be determined along with φ_n by solving the system of equations resulting from equating coefficients of x^{n+s+1}, x^{n+s},..., x^{n+m+1} in Eq.(2.2a). However, as only φ_n is needed in Eq.(2.1), we employ a forward elimination process for its determination. It is to be noted that the τ's are already known from the Tau approximation process of the study.

This then leads to the estimate:

$$\varepsilon_1=\max_{a\le x\le b}\left|\left(e_n(x)\right)_{n+1}\right|=\frac{|\varphi_n|}{\left|C_{n-m+1}^{(n-m+1)}\right|}\cong\max_{a\le x\le b}\left|e_n(x)\right|=\varepsilon^*$$

For the integrated formulation, the corresponding error problem is

$$I_L\left(e_n(x)\right)_{n+1}=-\int\int\underline{m}\int\left(\sum_{r=0}^{m+s-1}\tau_{m+s+r}\ \ T_{n-m+r+1}(x)\right)dx+C_m(x)+\sum_{r=0}^{m+s-1}\bar{\tau}_{m+s-r}\ \ T_{n-m+r+3}(x) \tag{3.9a}$$

$$L^*\left(e_n\left(x_{rk}\right)\right)_{n+1}=0, \quad k=1(1)m \tag{3.9b}$$

By equating coefficients of $x^{n+s+m+1}$, x^{n+s+m},......, x^{n+1} in Eq.(3.9) we get the (m + s + 1) equations for the determination of, $\bar{\tau}_1$, $\bar{\tau}_2$, ……, $\bar{\tau}_{m+s}$ and $\widetilde{\varphi_n}$, where, $\widetilde{\varphi_n}$ replaces $\widetilde{\varphi_n}$ in Eq.(3.4). Again, a forward elimination process is recommended. Subsequently, we get a second estimate,

$$\varepsilon_2=\left|\widetilde{\varphi_n}\right|/\left|C_{n-n+1}^{(n-m+1)}\right| \tag{3.10}$$

For an error estimation of the recursive form, a slight perturbation of the condition Eq.(1.1b) by ε_1 will be considered so as to obtain an estimate of ε_{m+s} in terms of canonical polynomials. Once this is achieved, a new error estimate ε_3 can then be obtained (Crisci and Ortiz, 1981; Onumanyi, 1981; Onumanyi and Ortiz, 1982; Adeniyi and Onumanyi, 1991).

For the purpose of our discussion in the study we let

$$R_v^{(k)}=\sum_{r=0}^{v}C_r^{(v)}\ Q_r^{(k)}(x), \quad k=0(1)m-1 \tag{3.11}$$

denote the derivative of order k of $R_v(x)$ with respect to x, with $R_v^{(0)}=R_v(x)$ and where, $Q_r(x)$, $r\ge0$, is the set of canonical polynomials associated with Eq.(1.1a).

NUMERICAL EXAMPLES

We consider here 2 examples for discussion on the 2 possible choices of $U_m(x)$ and hence $(e_n(x))_{n+1}$ of the preceding study and then show that the choice Eq.(3.3) and consequently Eq.(3.5) is more accurate.

Example 4.1 (a midly stiff problem): A boundary value problem in a fourth order differential equation

$$Ly(x)=\left(\frac{d^4}{dx^4}-3601\frac{d^2}{dx^2}+3600\right) \quad y(x)=-1+800x^2,\ \ 0\le x\le 1 \tag{4.1a}$$

$$y(0)=1,\ y'(0)=1,\ y(1)=1.5+\sinh(1),\ \ y'(1)=1+\cosh(1) \tag{4.1b}$$

The analytical solution is

$$y(x)=1+0.5x^2+\sinh(x)$$

This problem has been used quite of ten to test methods (Conte, 1966; Delves, 1976; Davey, 1980).

We shall first use the form

$$U_4(x)=x^2(x-1)^2 \tag{4.2}$$

which ensures the exact satisfaction of the homogenous conditions

$$e_n(0)=e_n'(0)=e_n(1)=e_n'(1)=0$$

By:

$$(e_n(x))_{n+1}=\frac{\varphi_n x^2(x-1)^2\ \ T_{n-3}(x)}{C_{n-3}^{(n-3)}}$$

$$=\theta\sum_{r=0}^{n}C_r^{(n-3)}\left(x^{r+4}-2x^{r+3}+x^{r+2}\right) \tag{4.3}$$

Where,

$$\theta=\varphi_n\left(C_{n-3}^{(n-3)}\right)^{-1}$$

and then the form

$$U_4(x)=x^4 \tag{4.4}$$

for which only the condition $e_n(0) = e'_n(0) = 0$ are exactly satisfied by

$$(e_n(x))_{n+1}=\frac{\varphi_n x^4\ \ T_{n-3}(x)}{C_{n-3}^{(n-3)}}=\theta\sum_{r=0}^{n-3}C_r^{(n-3)}x^{r+4} \tag{4.5}$$

while, the other 2 conditions $e_n(1) = e'_n(1) = 0$ are perturbed for Eq.(4.5)

Now let us insert Eq.(4.3) in the corresponding error problem

$$\begin{aligned}L(e_n(x))_{n+1}&:=\left(e_n^{(4)}(x)\right)_{n+1}-3601\left(e_n^{(2)}(x)\right)_{n+1}+3600\left(e_n(x)\right)_{n+1}\\&=-\tau_1T_n(x)-\tau_2T_{n-1}(x)-\tau_3T_{n-2}(x)-\tau_4T_{n-3}(x)\\&\quad+\bar{\tau}_1T_{n+1}(x)+\bar{\tau}_2T_n(x)+\bar{\tau}_3T_{n-1}(x)+\bar{\tau}_4T_{n-2}(x)\end{aligned} \tag{4.6}$$

to have

$$\begin{aligned}&\theta\sum_{r=0}^{n-3}C_r^{(n-3)}\left\{\begin{aligned}&3600x^{r+4}-7200x^{r+3}+\left[3600-3601(r+4)(r+3)\right]x^{r+2}\\&+7202(r+3)(r+2)(r+1)rx^{r-1}+(r+2)(r+1)r(r-1)x^{r-2}\end{aligned}\right\}\\&=\tau_1\sum_{r=0}^{n+1}C_r^{(n+1)}x^r+\left(\bar{\tau}_2-\tau_1\right)\sum_{r=0}^{n}C_r^{(n)}x^r+\left(\bar{\tau}_3-\tau_2\right)\sum_{r=0}^{n-1}C_r^{(n-1)}x^r\\&\quad+\left(\bar{\tau}_4-\tau_3\right)\sum_{r=0}^{n-2}C_r^{(n-2)}x^r-\tau_4\sum_{r=0}^{n-3}C_r^{(n-3)}x^r\end{aligned}$$

That is,

$$\begin{aligned}&\theta\{3600C_{n-3}^{(n-3)}x^{n+1}+3600\left(C_{n-4}^{(n-3)}-2C_{n-3}^{(n-3)}\right)x^n+\left[\left(3600-3601n-3601n^2\right)C_{n-3}^{(n-3)}-7200C_{n-4}^{(n-3)}+3600C_{n-5}^{(n-3)}\right]x^{n-1}\\&\left[7202n(n-1)C_{n-3}^{(n-3)}+\left(3600+3601n-3601n^2\right)C_{n-4}^{(n-3)}-7200C_{n-5}^{(n-3)}+3600C_{n-6}^{(n-3)}\right]x^{n-2}\\&+\left[\begin{aligned}&\left(n^4-2n^3-3602n^2+10{,}805n-7202\right)C_{n-3}^{(n-3)}+7202\left(n^2-3n+2\right)C_{n-4}^{(n-3)}-\left(3601n^2-10{,}803n+3602\right)C_{n-5}^{(n-3)}\\&-7200C_{n-6}^{(n-3)}+3600C_{n-7}^{(n-3)}\end{aligned}\right]x^{n-3}+\cdots\cdots\\&=\bar{\tau}_1C_{n+1}^{(n+1)}x^{n+1}+\left[\bar{\tau}_1C_n^{(n+1)}+\left(\bar{\tau}_2-\tau_1\right)C_n^{(n)}\right]x^n+\left[\bar{\tau}_1C_{n-1}^{(n+1)}+\left(\bar{\tau}_2-\tau_1\right)C_{n-1}^{(n)}+\left(\bar{\tau}_3-\tau_2\right)C_{n-1}^{(n-1)}\right]x^{n-1}\\&+\left[\bar{\tau}_1C_{n-2}^{(n+1)}+\left(\bar{\tau}_2-\tau_1\right)C_{n-2}^{(n)}+\left(\bar{\tau}_3-\tau_2\right)C_{n-2}^{(n-1)}+\left(\bar{\tau}_4-\tau_3\right)C_{n-2}^{(n-2)}\right]x^{n-2}\\&+\left[\bar{\tau}_1C_{n-3}^{(n+1)}+\left(\bar{\tau}_2-\tau_1\right)C_{n-3}^{(n)}+\left(\bar{\tau}_3-\tau_2\right)C_{n-3}^{(n-1)}+\left(\bar{\tau}_4-\tau_3\right)C_{n-3}^{(n-2)}-\tau_4C_{n-3}^{(n-3)}\right]x^{n-3}+\cdots\cdots\end{aligned}$$

We equate coefficients of x^{n+1}, x^n, x^{n-1}, x^{n-2} and x^{n-3} to obtain the system

$$\begin{aligned}&\bar{\tau}_1C_{n+1}^{(n+1)}=3600\theta C_{n-1}^{(n-1)}\\&\bar{\tau}_1C_n^{(n+1)}+\left(\bar{\tau}_2-\tau_1\right)C_n^{(n)}=3600\theta\left[C_{n-4}^{(n-3)}-2C_{n-3}^{(n-3)}\right]\\&\bar{\tau}_1C_{n-1}^{(n+1)}+\left(\bar{\tau}_2-\tau_1\right)C_{n-1}^{(n)}+\left(\bar{\tau}_3-\tau_2\right)C_{n-1}^{(n-1)}=\theta\left[\left(3600-3601n-36001n^2\right)C_{n-3}^{(n-3)}-7200C_{n-3}^{(n-3)}+3600C_{n-5}^{(n-3)}\right]\end{aligned} \tag{4.7}$$

$$\bar{\tau}_1 C_{n-2}^{(n+1)} + \left(\bar{\tau}_2 - \tau_1\right) C_{n-2}^{(n)} + \left(\bar{\tau}_3 - \tau_2\right) C_{n-2}^{(n-1)} + \left(\bar{\tau}_4 - \tau_3\right) C_{n-2}^{(n-2)}$$
$$= \theta\left[7202n(ny)C_{n-3}^{(n-3)} + \left(3600 + 3601n - 3601n^2\right)C_{n-4}^{(n-3)} - 7200C_{n-5}^{(n-3)} + 3600C_{n-6}^{(n-3)}\right]$$
$$\bar{\tau}_1 C_{n-3}^{(n+1)} + \left(\bar{\tau}_2 - \tau_1\right) C_{n-3}^{(n)} + \left(\bar{\tau}_3 - \tau_2\right) C_{n-3}^{(n-1)} + \left(\bar{\tau}_4 - \tau_3\right) C_{n-3}^{(n-2)} - \tau_4 C_{n-3}^{(n-3)}$$
$$= \theta\begin{bmatrix}\left(n^4 - 2n^3 - 3602n^2 + 10{,}805n - 7202\right)C_{n-3}^{(n-3)} + 7202\left(n^2 3n + 2\right)C_{n-4}^{(n-3)} \\ -\left(3601n^2 - 10{,}803n + 3602\right)C_{n-5}^{(n-3)} - 7200C_{n-6}^{(n-3)} + 3600C_{n-7}^{(n-3)}\end{bmatrix}$$

We solve this for φ_n by forward elimination using well-known relations $C_k^{(k)} = 2^{2k-1}$ and $C_k^{(k)} = 1/2kC_k^{(k)}$ subsequently we obtain, from (ii), our first error estimate for the differential form,

$$\varepsilon_1 = \max_{0\le x\le 1}|\varphi_n| / \left|C_{n-3}^{(n-3)}\right| = \frac{2^{4n-6}|\tau_4|}{D} \quad (4.8)$$

Where,

$$D = 2^{2n-3}225C_{n-3}^{(n+1)} + \left(16\alpha_1 - 225C_{n-1}^{(n+1)}\right)C_{n-3}^{(n-1)} - 2^{2n-5}(n-2)\left\{16\beta_1 - 450C_{n-2}^{(n+1)} + (n-1)\left(16\alpha_1 - 225C_{n-1}^{(n+1)}\right)\right\} - 2^{2n+1}\lambda_1$$

and

$$\alpha_1 = 3600C_{n-5}^{(n-3)} - \left(3601n^2 + n + 7200\right)C_{n-3}^{(n-3)}$$
$$\beta_1 = \left(3601n^3 - 7201n + 10{,}800\right)C_{n-3}^{(n-3)} - 14{,}400C_{n-5}^{(n-3)} + 7200C_{n-6}^{(n-3)}$$
$$\lambda_1 = \left(n^4 - 3603n^3 + 18{,}004n^2 - 28{,}806n + 14{,}404\right)C_{n-3}^{(n-3)}$$
$$-\left(3602n^2 - 10{,}803n + 3602\right)C_{n-5}^{(n-3)} - 7200C_{n-6}^{(n-3)} + 3600C_{n-7}^{(n-3)}$$

A similar procedures yields, for the form (iv), the estimate

$$\bar{\varepsilon}_1 = \frac{2^{4n-6}|\tau_4|}{|k_1|} \quad (4.10)$$

where,

$$k_1 = 225\left[2^{2n-3}C_{n-3}^{(n+1)} + 2^{2n}C_{n-3}^{(n)}\right] + \left[16\alpha_2 - 225C_{n-1}^{(n+1)} + 450n2^{2n}\right]C_{n-3}^{(n-1)} - (n-2)2^{2n-5}$$
$$\left\{32\beta_2 - 450C_{n-2}^{(n+1)} - 3600C_{n-2}^{(n)} + (n-1)\left[16\alpha_2 - 225C_{n-1}^{(n+1)} + 450n2^{2n}\right]\right\} - 2^{2n+1}\gamma_2 \quad (4.11)$$

and

$$\alpha_2 = 3600C_{n-5}^{(n-3)} - 3601n(n+1)C_{n-3}^{(n-3)}$$
$$\beta_2 = 3600C_{n-6}^{(n-3)} - 3601n(n-1)C_{n-4}^{(n-3)}$$
$$\gamma_2 = (n-2)(n-1)_n(n+1)C_{n-3}^{(n-3)} - 3601(n-2)(n-1)C_{n-5}^{(n-3)} + 3600C_{n-7}^{(n-3)}$$

For the integrated formulation, we have from Eq.(3.9a), that in this case,

$$I_L\left(e_n(x)\right)_{n+1} = -\int_0^x\int_0^w\int_0^\vartheta\int_0^u\left[\tau_1 T_n(t) + \tau_2 T_{n-1}(t) + \tau_3 T_{n-2}(t) + \tau_4 T_{n-3}(t)\right]dt\ du\ d\vartheta\ dw$$
$$+ \bar{\tau}_1 T_{n+5}(x) + \bar{\tau}_2 T_{n+4}(x) + \bar{\tau}_3 T_{n+3}(x) + \bar{\tau}_4 T_{n+2}(x) \quad (4.12)$$

where,

$$I_L\left(e_n(x)\right)_{n+1} = \left(e_n(x)\right)_{n+1} - 3601\int_0^x\int_0^w\left(e_n^{(2)}(\vartheta)\right)_{n+1} d\vartheta\ dw + 3600\int_0^x\int_0^w\int_0^\vartheta\int_0^u\left(e_n^{(4)}(t)\right)_{n+1} dt\ du\ d\vartheta\ dw$$

with Eq.(4.3) in Eq.(4.12) we get

$$\frac{Q_n}{C_{n-3}^{(n-3)}}\sum_{r=0}^{n}\left\{\begin{aligned}&x^{r+4} - 2x^{r+3} + x^{r+2} - \frac{3601x^{r+6}}{(r+5)(r+6)} + \frac{7202x^{r+5}}{(r+4)(r+5)} - \frac{3601x^{r+4}}{(r+3)(r+4)} + \frac{3600x^{r+8}}{(r+5)(r+6)(r+7)(r+8)}\\ &- \frac{7200x^{r+7}}{(r+4)(r+5)(r+6)(r+7)} + \frac{3600x^{r+6}}{(r+3)(r+4)(r+5)(r+6)}\end{aligned}\right\}$$

$$= \bar{\tau}_1 \sum_{r=0}^{n+5} C_r^{(n+5)} x^r + \bar{\tau} \sum_{r=0}^{n+4} C_r^{(n+4)} x^r - \tau_1 \sum_{r=0}^{n} \frac{C_r^{(n)} x^{r+4}}{(r+1)(r+2)(r+3)(r+4)}$$
$$+ \bar{\tau}_3 \sum_{r=0}^{n+3} C_r^{(n+3)} x^r - \tau_2 \sum_{r=0}^{n-1} \frac{C_r^{(n-1)} x^{r+4}}{(r+1)(r+2)(r+3)(r+4)}$$
$$+ \bar{\tau}_4 \sum_{r=0}^{n} C_r^{(n+2)} x^r - \tau_3 \sum_{r=0}^{n-2} \frac{C_r^{(n-2)} x^{r+4}}{(r+1)(r+2)(r+3)(r+4)}$$
$$- \tau_4 \sum_{r=0}^{n-3} \frac{C_r^{(n-3)} x^{r+4}}{(r+1)(r+2)(r+3)(r+4)}$$

We proceed from here with same approach which led to the system Eq. (4.7) to get

$$\bar{\tau}_1 C_{n+5}^{(n+5)} = \frac{3600\widetilde{\varphi_n}}{(n+2)(n+3)(n+4)(n+5)}$$
$$\bar{\tau}_1 C_{n+5}^{(n+5)} + \bar{\tau}_2 C_{n+4}^{(n+4)} - \frac{\tau_1 C_n^{(n)}}{(n+1)(n+2)(n+3)(n+4)} = \frac{\left[3600 C_{n-4}^{(n-3)} - 7200 C_{n-3}^{(n-3)}\right]\widetilde{\varphi_n}}{(n+1)(n+2)(n+3)(n+4)C_{n-3}^{(n-3)}}$$
$$\bar{\tau}_1 C_{n+3}^{(n+5)} + \bar{\tau}_2 C_{n+3}^{(n+4)} + \bar{\tau}_3 C_{n+3}^{(n+3)} - \frac{\left[\tau_1 C_{n-1}^{(n)} + \tau_2 C_{n-1}^{(n-1)}\right]}{n(n+1)(n+2)(n+3)} = \frac{\beta_1 \widetilde{\varphi_n}}{n(n+1)(n+2)(n+3)C_{n-3}^{(n-3)}} \quad (4.13)$$
$$\bar{\tau}_1 C_{n+2}^{(n+5)} + \bar{\tau}_2 C_{n+2}^{(n+4)} + \bar{\tau}_3 C_{n+2}^{(n+3)} + \bar{\tau}_4 C_{n+2}^{(n+2)} - \frac{\left[\tau_1 C_{n-2}^{(n)} + \tau_2 C_{n-2}^{(n-1)} + \tau_3 C_{n-2}^{(n-2)}\right]}{(n-1)n(n+1)(n+2)} = \frac{\beta_2 \widetilde{\varphi_n}}{(n-1)n(n+1)(n+2)C_{n-3}^{(n-3)}}$$
$$\bar{\tau}_1 C_{n+1}^{(n+5)} + \bar{\tau}_2 C_{n+1}^{(n+4)} + \bar{\tau}_3 C_{n+1}^{(n+3)} + \bar{\tau}_4 C_{n+1}^{(n+2)} - \frac{\left[\tau_1 C_{n-3}^{(n)} + \tau_2 C_{n-3}^{(n-1)} + \tau_3 C_{n-3}^{(n-2)} + \tau_4 C_{n-3}^{(n-3)}\right]}{(n-2)(n-1)n(n+1)} = \frac{\beta_3 \widetilde{\varphi_n}}{(n-2)(n-1)n(n+1)C_{n-3}^{(n-3)}}$$

We solve this for ϕ_n by forward elimination and then obtain the estimate

$$\varepsilon_2 = \frac{\left|\widetilde{\varphi_n}\right|}{2^{2n-7}} = \frac{2^{2n+12}(n+5)}{D_2}\left[\left|\left\{\xi_1\tau_1 + 2(n+4)\xi_2\tau_2 + 2^{2n+2}(n+2)(n+3)(n+4)\tau_4\right\}\right|\right] \quad (4.14)$$

where,

$$D_2 = 3600 2^{2n+5}(n-2)(n-1)n(n+1)C_{n+1}^{(n+5)} + (n-2)(n-1)\xi_1 C_{n+1}^{(n+3)}\delta_1$$
$$- 2^{2n+3}(n-2)(n+2)\delta_2 - 2^{2n+21}(n+2)(n+3)(n+4)(n+5)\beta_3$$

and

$$\xi_1 = 512(n+2)(n+3)(n+4)C_{n-3}^{(n)} + (n-2)(n+2)\delta_3 - 2(n-2)(n-1)nC_{n+1}^{(n+4)}$$
$$\xi_2 = 256(n+2)(n+3)C_{n-3}^{(n-1)} - (n-2)(n-1)C_{n+1}^{(n+3)}$$
$$\delta_1 = 2^{16}(n+4)(n+5)\beta_1 - 3600n(n+1)C_{n+3}^{(n+5)}$$
$$\delta_2 = 2^{17}(n+3)(n+5)\beta_2 - 7200(n-1)n(n+1)C_{n+2}^{(n+5)} + (n-1)(n+3)\delta_1$$
$$\delta_3 = 256(n+3)(n+4)C_{n-2}^{(n)} - (n-1)_n C_{n+2}^{(n+4)}$$

For the choice Eq. (4.4), we similarly obtain

$$\bar{\varepsilon}_2 = \frac{2^{2n+12}(n+5)}{|k_2|}\left|\left[\eta_1\tau_1 + 2(n+4)\eta_2\tau_2 + 2^{2n+2}(n+2)(n+3)(n+4)\tau_4\right]\right| \quad (4.16)$$

where,

$$
\begin{aligned}
k_2 &= (n-2)\mu_1 - 2^{2n+21}(n+2)(n+3)(n+4)(n+5)\mu_2 \\
\mu_1 &= 512(n+2)(n+3)(n+4)C_{n-3}^{(n)} + 256(n-2)(n+2)(n+3)(n+4)C_{n-2}^{(n)} \\
&\quad + (n-2)(n-1)_n (n+2)C_{n+2}^{(n+4)} - 2(n-2)(n-1)_n C_{n+1}^{(n+4)} \\
\mu_2 &= 256(n+2)(n+3)C_{n-3}^{(n-1)} - (n-2)(n-1)C_{n+1}^{(n+3)} \\
\mu_1 &= 14400(n-1)_n (n+1)2^{2n+3}C_{n+1}^{(n+5)} + 115200(n-1)_n (n+5)2^{2n+3}C_{n+1}^{(n+4)} \\
&\quad - (n+2)v_1 + (n-1)C_{n+1}^{(n+3)}v_2 \\
v_1 &= 2^{17}(n+3)(n+4)(n+5)\rho - 7200(n-1)_n (n+1)C_{n+2}^{(n+5)} - 57600(n-1)_n (n+5) \\
&\quad + (n-1)(n+3)v_2 \\
v_2 &= 2^{16}.3600(n+4)(n+5)C_{n-5}^{(n-3)} - 2^{2n+9}(3601n-1)_n (n+4)(n+4) - 3600n(n+1)C_{n+3}^{(n+5)} \\
\rho &= 360C_{n-6}^{(n-3)} + 3601(n-3)(n-1)_n 2^{2n-8} \\
\mu_2 &= 3600C_{n-7}^{(n-3)} - 3601(n-2)(n-1)C_{n-5}^{(n-3)} + 2^{2n-7}(n-2)(n-1)_n (n+1)
\end{aligned}
\tag{4.17}
$$

Finally, for the recursive form, we have from Eq. (1.10) and Eq.(3.11), that

$$
y_n(x) = -Q_0(x) + 1800Q_2(x) + \tau_1 R_n(x) + \tau_2 R_{n-1}(x) + \tau_3 R_{n-2}(x) + \tau_4 R_{n-3}(x)
$$

where, the canonical polynomials $Q_r(x)$, r = 0, are defined recursively by

$$
Q_r(x) = \left[x^r + 3601(r-1)r\, Q_{r-2}(x) - (r-3)(r-2)(r-1)r\, Q_{r-4}(x)\right]/3600
$$

By applying the boundary condition Eq. (4.1b) in Eq.(4.18), we have

$$
\begin{aligned}
&-Q_0(0) + 1800Q_2(0) + \tau_1 R_n(0) + \tau_2 R_{n-1}(0) + \tau_3 R_{n-2}(0) + \tau_4 R_{n-3}(0) = 1 \\
&-Q_0(0) + 1800Q_2'(0) + \tau_1 R_n'(0) + \tau_2 R_{n-1}'(0) + \tau_3 R_{n-2}'(0) + \tau_4 R_{n-3}'(0) = 1 \\
&-Q_0(1) + 1800Q_2(1) + \tau_1 R_n(1) + \tau_2 R_{n-1}(1) + \tau_3 R_{n-2}(1) + \tau_4 R_{n-3}(1) = 1.5 + \sinh(1) \\
&\qquad = 1 + \cosh(1)
\end{aligned}
$$

We eliminate τ_1, τ_2 and τ_3 from this system to get

$$
\begin{aligned}
&\tau_4\left[\sigma_3\left\{v_3 R_{n-3}(0) - v_1 R_n(0) - v_2 R_{n-1}(0)\right\} + \sigma_2 v_3 R_{n-2}(0)\right] \\
&\quad = \sigma_3\left[\eta_1 v_3 - \gamma_1 R_n(0) - v_2 R_{n-1}(0)\right] - \sigma_1 v_3 R_{n-2}(0)
\end{aligned}
\tag{4.18}
$$

The parameter v_1, v_2, v_3, σ_1, σ_2 and σ_3 in Eq. (4.18) are defined below
hence,

$$
\begin{aligned}
&|\tau_4|\left\|\left[\left\{v_3 R_{n-3}(0) - v_1 R_n(0) - v_3 R_{n-1}(0)\right\} + \sigma_2 v_3 R_{n-2}(0)\right]\right\| \\
&\quad \equiv \left\|\left[\sigma_3\left\{\eta_1 v_3 - \gamma_1 R_n(0) - v_2 R_{n-1}(0)\right\} - \sigma_1 v_3 R_{n-2}(0)\right]\right\| + \varepsilon_1
\end{aligned}
$$

where, $\varepsilon_1 = 2^{4n-6}\,|\tau_4|\,|D_1|^{-1}$. This leads to

$$
\left|\bar{\tau}_4\right| \le \frac{\left\|\left[\sigma_{13}\left\{\eta_1 v_3 - \gamma_1 R_n(0) - v_2 R_{n-1}(0)\right\} - \sigma_1 v_3 R_{n-2}(0)\right]|D_1|\right\|}{|D_1|\left\|\left[\sigma_3\left\{v_3 R_{n-3}(0) - v_1 R_n(0) - v_2 R_{n-1}(0)\right\} + \sigma_2 v_3 R_{n-2}(0)\right]\right\| - 2^{4n-6}} \cong \left|\bar{\tau}_4\right|
\tag{4.20}
$$

as an estimate of $|\tau_4|$. Thus, again from Eq.(4.8) we get a new error estimate

$$\varepsilon_3 = \frac{\left\|\left[\sigma_3\left\{\gamma_1 v_3 - \gamma_1 R_n(0) - v_2 R_{n-1}(0)\right\} - \sigma_1 v_3 R_{n-2}(0)\right]\right\|}{2^{6-4n}\left|D_1\right|\left\|\left[\sigma_3\left\{v_3 R_{n-3}(0) - v_1 R_n(0) - v_2 R_{n-1}(0)\right\} + \sigma_2 v_3 R_{n-2}(0)\right]\right\| - 1} \tag{4.21}$$

where,

$$\sigma_1 = v_3 - \gamma_1 R_n'(0) - \gamma_2 R_{n-1}'(0)$$
$$\sigma_2 = v_1 R_n'(0) + v_2 R_{n-1}'(0) - v_3 R_{n-3}'(0)$$
$$\sigma_3 = v_3 R_{n-2}'(0)$$
$$v_1 = \rho_1\rho_5 + \rho_2\rho_6$$
$$v_2 = \rho_3\rho_5 + \rho_4\rho_6$$
$$v_3 = \rho_2\rho_3 - \rho_1\rho_4$$
$$\gamma_1 = (\rho_1\eta_2 + \eta_3\rho_4)R_{n-1}(0) - \eta_4\left(\rho_1 R_{n-2}'(0) + \rho_2 R_{n-2}'(0)\right)$$
$$\gamma_2 = (\rho_3\eta_2 + \eta_3\rho_4)R_{n-2}(1) - \eta_4\left(\rho_3 R_{n-2}'(1) + \rho_4 R_{n-2}'(0)\right)$$
$$\eta_1 = 1 + Q_0(0) - 1800 Q_2(0)$$
$$\eta_2 = 1 + \cosh(1) + Q_0'(1) + 1800 Q_2'(1)$$
$$\eta_3 = 1 + Q_0'(0) + 1800 Q_2'(0)$$
$$\eta_4 = 1.5\sinh(1) + Q_0(1) + 1800 Q_2(1)$$
$$\rho_1 = R_{n-1}(1)R_{n-2}'(0) - R_{n-1}'(0)R_{n-2}(1)$$
$$\rho_2 = R_{n-2}(1)R_{n-1}'(1) - R_{n-2}'(1)R_{n-1}(1)$$
$$\rho_3 = R_{n-2}(1)R_n'(0) - R_{n-2}'(0)R_n(1)$$
$$\rho_4 = R_n(1)R_{n-2}'(1) - R_n'(1)R_{n-2}(1)$$
$$\rho_5 = R_{n-2}(1)R_{n-3}'(1) - R_{n-2}'(1)R_{n-3}(1)$$
$$\rho_6 = R_{n-2}(1)R_{n-3}'(0) - R_{n-2}'(0)R_{n-3}(1)$$

For the choice Eq. (4.4), the same result holds except that D_1 is replaced by k_1 given by Eq. (4.10); the estimate, in this case, is denoted by $\bar{\varepsilon}_3$.

Numerical results for the example are presented in Table 1.

Example 4.2 (The Runge's function):

$$Ly(x) = \left((1+x^2)\frac{d^2}{dx^2} + 4x\frac{d}{dx} + 2\right)y(x) = 0, \quad 0 \le x \le 1$$
$$y(0) = 1, \qquad y(1) = \frac{1}{2}$$

The theoretical solution is the Runge's function:

$$y(x) = \frac{1}{1+x^2}$$

For this example, we have the choices of $(e_n(x))_{n+1}$ as:

$$(e_n(x))_{n+1} = \frac{x(x-1)\varphi_n T_{n-1}(x)}{C_{n-1}^{(n-1)}} \tag{4.22}$$

which satisfies the condition $(e_n(x))_{n+1}$ and

Table 1: Error and error estimate for example 4.1

Error $V_4(x)$ n	Error estimate $x^2(x-1)^2$	x^4	Error $V_4(x)$ exact error
7	4.44×10^{-8}	1.53×10^{-9}	3.04×10^{-9}
8	1.42×10^{-8}	6.15×10^{-10}	4.71×10^{-10}
9	4.07×10^{-10}	2.08×10^{-11}	1.62×10^{-11}
10	9.23×10^{-12}	5.33×10^{-13}	4.16×10^{-13}

Table 2: Error estimate for example 4.2

Error $V_4(x)$ n	Error estimate x (x-1)	x^2	Error $V_4(x)$ exact error
6	2.71×10^{-4}	6.11×10^{-5}	6.35×10^{-5}
7	3.01×10^{-4}	6.68×10^{-5}	4.42×10^{-5}
8	6.92×10^{-5}	1.52×10^{-5}	0.94×10^{-5}
9	1.92×10^{-6}	4.17×10^{-7}	5.18×10^{-7}
10	3.71×10^{-6}	7.99×10^{-7}	4.53×10^{-7}

$$(e_n(x))_{n+1} = \frac{x^2 Q_n T_{n-1}(x)}{C_{n-1}^{(n-1)}}$$

Which satisfies $e_n(0) = 0$ but perturbed for $e_n(1) = 0$.

The error estimate obtained using Eq.(4.22) for the differential form, integrated form and the recursive form are, respectively.

$$\varepsilon_1 = \frac{2^{2n+1}|\tau_2|}{|\alpha_1|},\ \varepsilon_2 = \frac{2^{2n+5}|\tau_2|}{|\beta_1|},\ \varepsilon_3 = \frac{|\lambda||2R_n(1) - R_n(0)|}{\left[2^{-2n}|\alpha_1||\xi| - 1\right]}$$

Where,

$$\alpha_1 = (n+2)(n+3)C_{n-1}^{(n+1)} - 16n(n+1)$$
$$\left[C_{n-3}^{(n-1)} - C_{n-2}^{(n-1)}\right] - 2^{2n}n(n+1)(n+2)$$
$$\beta_1 = C_{n+1}^{(n+3)} - 2^{2n+4}(n+2) - 256\left[C_{n-1}^{(n-1)} - C_{n-2}^{(n-1)} + C_{n-3}^{(n-1)}\right]$$
$$\xi = R_n(1)R_{n-1}(0) - R_n(0)R_{n-1}(1)$$

For the 2nd choice Eq.(4.16), we have the corresponding error estimate as:

$$\bar{\varepsilon}_1 = \frac{2^{2n+1}|\tau_2|}{|\alpha_2|},\ \bar{\varepsilon}_2 = \frac{2^{2n+5}|\tau_2|}{|\beta_2|},\ \bar{\varepsilon}_3 = \frac{|2R_n(1) - R_n(0)|}{2^{-2n}|\alpha_1||\xi_1| - 1}$$

Where,

$$\alpha_2 = (n+2)(n+3)C_{n-1}^{(n+1)} - 16n(n+1)C_{n-3}^{(n-1)} - 2^{2n+1}n(n+1)(n+3)$$
$$\beta_2 = n(n+1)\left[C_{n-1}^{(n+3)} - 2^{2n+5}(n+2) - 256\left\{C_{n-1}^{(n-1)} + C_{n-3}^{(n-1)}\right\}\right]$$

and ξ retains its definition above. Numerical results for this example are presented in Table 2.

CONCLUSION

A modification of the polynomial error approximant $(e_n(x))_{n+1}$ of the error function $e_n(x)$ of the Tau approximant $y_n(x)$ of $y(x)$ has been presented and shown to yield a more accurate estimate of the maximum error in $y_n(x)$ in the range of definition when compared to that earlier proposed in Adeniyi and Onumanyi (1991). The estimate obtained also confirm the order of the Tau approximation as it accurately estimates the order of accuracy of the Tau approximant.

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