Research Journal of Applied Sciences 2 (3): 289-294, 2007

Generalized Convolution Properties for Certain Classes of Analytic Functions

[1]A.T. Oladipo, [2]K.O. Babalola and [2]T.O. Opoola
[1]Department of Pure and Applied Mathematics, Ladoke Akintola
University of Technology, P.M.B. 4000, Ogbomoso, Oyo State, Nigeria
[2]Department of Mathematics, University of Ilorin
P.M.B. 1515, Nigeria

Abstract: In this study, we investigate some generalized convolution properties of certain subclasses of the class $T_n^\alpha(\beta)$, introduced by Opoola using a technique based on the application of Cauchy-Schwarz and Holder's inequalities.

Key words: Generalized convolution, analytic and univalent functions

INTRODUCTION

Let C be the complex plane. Denote by the class of normalized functions

$$g(z) = z + a_2 z^2 + \qquad (1)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. Let $\alpha > 0$ be real. Then, using binomial expansion, we can write $g(z)^\alpha$ in the form

$$g(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1} \qquad (2)$$

In (1994), Opoola introduced and studied the $T_n^\alpha(\beta)$ class consisting of functions satisfying

$$\operatorname{Re}\frac{D^n g(z)^\alpha}{\alpha^n z^\alpha} > \beta, \qquad (0 \le \beta < 1) \qquad (3)$$

where $D^n\left(n \in N_0 = \{0,1,2,...\}\right)$ is the Salagean derivative operator defined as

$$D^n g(z) = D\left[D^{n-1} g(z)\right] = z\left[D^{n-1} g(z)\right]' \qquad (4)$$
$$\text{with } D^0 g(z) = g(z)$$

Note here that the geometric condition (3) slightly modifies the given originally in (Opoola, 1994) (Badalola and Opoola).

The class $T_n^\alpha(\beta)$ is a very large family of analytic and univalent functions, which has as special cases, many other classes of functions which attracted the attention of many authors. For instance, several results concerning the cases.

$$T_1^0(0) \equiv S_0,$$
$$T_0^1(\beta) \equiv S_0(\beta),$$
$$T_1^1(0) \equiv R,$$
$$T_1^1(\beta) \equiv R(\beta),$$
$$T_1^\alpha(0) \equiv B_1(\alpha),$$
$$T_n^\alpha(0) \equiv B_n(\alpha)$$

can be found in the literatures (Abdulhalim, 1992; Macgregor, 1962; Opoola, 1994; Singh, 1973; Yamaguchi, 1966).

Let $g_j(z) \in A\left(j = 1,2,...,m\right)$ be given by

$$g_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (5)$$

then, the convolution (or Hadamard product) of is defined by

$$\prod_{j=1}^{m}{}^{*} g_j(z) \equiv \left(g_1 * ... * g_m\right)(z) = z + \sum_{k=2}^{\infty}\left(\prod_{j=1}^{m}(a_{k,j}) z^k\right) \qquad (6)$$

while the generalized convolution of is $g_j(z)$ given by

$$\prod_{j=1}^{m}{}^{\bullet} g_j(z) \equiv \left(g_1 \bullet ... \bullet g_m\right)(z) = z + \sum_{k=2}^{\infty}\left(\prod_{j=1}^{m}(a_{k,j})^{\frac{1}{p_j}}\right) z^k \qquad (7)$$

Corresponding Author: A.T. Oladipo, Department of Pure and Applied Mathematics, Ladoke Akintola University of Technology, P.M.B. 4000, Ogbomoso, Oyo State, Nigeria

where

$$\sum_{j=1}^{m}\frac{1}{p_j}=1,\quad p_j>1,\ j=1,2,...,m \tag{8}$$

See (Owa and Srivastava, 2002)

As far as the present authors are aware, no results have appeared in prints regarding the convolution of functions in the class $T_n^{\alpha}(\beta)$. Thus our concern in this research is to investigate the products $\prod_{j=1}^{m}{}^{*}g_j(z)$ and $\prod_{j=1}^{m}{}^{\bullet}g_j(z)$ in some subclasses of the class. $T_n^{\alpha}(\beta)$. The technique we shall employ is the application of the well known inequalities of Cauchy-Schwarz and Holder. Our approach depends on the binomial coefficient in (2) $a_k(\alpha)$.

First, we prove the following:

Lemma: Let $g\in A$ satisfy

$$\sum_{k=2}^{\infty}\alpha_0^n\,|a_k(\alpha)|\le 1-\beta, \tag{9}$$

where $\alpha_0=\dfrac{\alpha+k-1}{\alpha},\ k=2,3,...\ 0\le\beta<1$

then $g(z)\in T_n^{\alpha}(\beta)$.

Proof: Let $g\in A$. Then

$$g(z)^{\alpha}=z^{\alpha}+\sum_{k=2}^{\infty}a_k(\alpha)z^{\alpha+k-1}$$

so that

$$\left|\frac{D^n g(z)^{\alpha}}{\alpha^n z^{\alpha}}-1\right|=\left|\frac{\alpha^n z^{\alpha}+\sum_{k=2}^{\infty}(\alpha+k-1)^n a_k(\alpha)z^{\alpha+k-1}}{\alpha^n z^{\alpha}}-1\right|$$

$$=\left|\sum_{k=2}^{\infty}\alpha_0^n a_k(\alpha)z^{k-1}\right|\le\sum_{k=2}^{\infty}\alpha_0^n\,|a_k(\alpha)|\le 1-\beta$$

which implies that the geometric quantity $\dfrac{D^n g(z)^{\alpha}}{\alpha^n z^{\alpha}}$ lies in a circle centred at $w=1$ and of radius $1-\beta$. Hence we have

$$\mathrm{Re}\frac{D^n g(z)^{\alpha}}{\alpha^n z^{\alpha}}>\beta$$

that is,. $g\in T_n^{\alpha}(\beta)$

However, the function

$$g(z)=z\frac{1+z}{1-z}=z+2\sum_{k=2}^{\infty}z^k$$

which belongs to the class $T_0^1(0)\equiv S_0$ shows that the converse of Lemma 1.1 cannot be true in general. Thus in view of Lemma 1.1 and the binomial expansion (2) we select subclasses $T_n^{\alpha}(\beta)$ of $T_n^{\alpha}(\beta)$ as follows:

Definition : Let $A(\alpha)$ (is a positive integer) be the class of analytic functions in E of the form

$$f(z)=z^{\alpha}+\sum_{k=2}^{\infty}a_k(\alpha)z^{\alpha+k-1}$$

Then $f\in A(\alpha)$ is said to belong to $\tilde{T}_n^{\alpha}(\beta)$ if

$$\mathrm{Re}\frac{D^n g(z)^{\alpha}}{\alpha^n z^{\alpha}}>\beta\quad(0\le\beta<1) \tag{1}$$

and

$$\sum_{k=2}^{\infty}\alpha_0^n\,|a_k(\alpha)|\le 1-\beta\qquad \alpha_0=\frac{\alpha+k-1}{\alpha} \tag{ii}$$

Observe that the condition(i) above is equivalent to (3), where $f(z)=g(z)^{\alpha},\ \alpha\ge1$ is integer. Hence the class represents subclasses of $T_n^{\alpha}(\beta)$ satisfying the ine $\tilde{T}_n^{\alpha}(\beta)$ quality (ii) for some positive exponential α. The class $\tilde{T}_n^{\alpha}(\beta)$ is not empty as the following example shows.

$$f(z)=z^{\alpha}+\lambda(1-\beta)\frac{\alpha^n}{(\alpha+1)^n}z^{\alpha+1},\qquad |\lambda|=1 \tag{10}$$

Infact, functions of the form (10) will serve as extremal functions for our main results in later sections. The method of proof in the main results is basically the principle of mathematical induction.

CONVOLUTION IN $\tilde{T}_n^{\alpha}(\beta)$

Theorem 1: If $f_j(z)\in\tilde{T}_{n_j}^{\alpha}(\beta_j)\quad(j=1,2,...,m)$, then

$$\prod_{j=1}^{m}{}^{*}f_j(z)\in\tilde{T}_n^{\alpha}(\gamma_m)$$

where

$$\gamma_m=1-\left(\frac{\alpha}{\alpha+1}\right)^{\sum_{j=1}^{m}n_j-n}\prod_{j=1}^{m}(1-\beta_j) \tag{11}$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z) = z^{\alpha} + \lambda_j(1-\beta_j)\left(\frac{\alpha}{\alpha+1}\right)^{n_j} z^{\alpha+1} \quad |\lambda_j| = 1,\ j = 1,2,\ldots,m \tag{12}$$

Proof: We need to find the largest γ_m such that

$$\sum_{k=2}^{\infty} \alpha_0^{n} \prod_{j=1}^{m} |a_{k,j}(\alpha)| \le 1-\gamma_m \tag{13}$$

Note that if $m = 1$, then, $\gamma_1 = \beta_1$. Now suppose. $m = 2$. Then for functions $f_1(z) \in \tilde{T}_{n_1}^{\alpha}(\beta_1)$ and $f_2(z) \in \tilde{T}_{n_2}^{\alpha}(\beta_2)$, we have

$$\sum_{k=2}^{\infty} \alpha_0^{n_1} |a_{k,1}(\alpha)| \le 1-\beta_1$$

and

$$\sum_{k=2}^{\infty} \alpha_0^{n_2} |a_{k,2}(\alpha)| \le 1-\beta_2$$

so that

$$\sum_{k=2}^{\infty} \frac{\alpha_0^{n_1}}{1-\beta_1} |a_{k,1}(\alpha)| \le 1$$

and

$$\sum_{k=2}^{\infty} \frac{\alpha_0^{n_2}}{1-\beta_2} |a_{k,2}(\alpha)| \le 1$$

Hence, by Cauchy-Schwarz inequality, we have

$$\sum_{k=2}^{\infty} \sqrt{\frac{\alpha_0^{n_1+n_2} |a_{k,1}(\alpha)||a_{k,2}(\alpha)|}{(1-\beta_1)(1-\beta_2)}} \le 1 \tag{14}$$

In order to prove that $(f_1 * f_2)(z) \in \tilde{T}_n^{\alpha}(\gamma_2)$ it is sufficient to show that

$$\sqrt{|a_{k,2}(\alpha)||a_{k,2}(\alpha)|} \le \sqrt{\frac{\alpha_0^{n_1+n_2}}{(1-\beta_2)(1-\beta_2)}}\left(\frac{1-\gamma_2}{\alpha_0^{n}}\right) \tag{15}$$

since

$$\sum_{k=2}^{\infty} \alpha_0^{n} |a_{k,1}(\alpha)||a_{k,2}(\alpha)| \le (1-\gamma_2)$$

$$\sum_{k=2}^{\infty} \sqrt{\frac{\alpha_0^{n_1+n_2}}{(1-\beta_1)(1-\beta_2)}} |a_{k,1}(\alpha)||a_{k,2}(\alpha)|$$

$$\le (1-\gamma_2)\left\{\left(\sum_{k=2}^{\infty} \frac{\alpha_0^{n_1}}{(1-\beta_1)} |a_{k,1}(\alpha)|\right)\left(\sum_{k=2}^{\infty} \frac{\alpha_0^{n_2}}{(1-\beta_2)} |a_{k,2}(\alpha)|\right)\right\}^{\frac{1}{2}}$$

$$\le 1-\gamma_2$$

But, from (14) we have for all $k = 2,3,\ldots$

$$\sqrt{|a_{k,1}(\alpha)||a_{k,2}(\alpha)|} \le \sqrt{\frac{(1-\beta_1)(1-\beta_2)}{\alpha_0^{n_1+n_2}}}$$

Hence it is sufficient to find the largest such that

$$\sqrt{\frac{(1-\beta_1)(1-\beta_2)}{\alpha_0^{n_1+n_2}}} \le \sqrt{\frac{\alpha_0^{n_1+n_2}}{(1-\beta_2)(1-\beta_2)}}\left(\frac{1-\gamma_2}{\alpha_0^{n}}\right)$$

That is,

$$\gamma_2 \le 1-\alpha_0^{n-(n_1+n_2)}(1-\beta_1)(1-\beta_2) \tag{16}$$

$$= 1-\left(\frac{\alpha}{\alpha+k-1}\right)^{n_2+n_2-n}(1-\beta_1)(1-\beta_2)$$

It is readily seen that the right hand side of (16) is an increasing function of. Hence the largest value of γ_2 is given by

$$\gamma_2 = 1-\left(\frac{\alpha}{\alpha+1}\right)^{n_2+n_2-n}(1-\beta_1)(1-\beta_2) \tag{17}$$

Next we suppose that

$$\prod_{j=1}^{m}{}^{*} f_j(z) \in \tilde{T}_n^{\alpha}(\gamma_m)$$

where

$$\gamma_m = 1-\left(\frac{\alpha}{\alpha+1}\right)^{\sum_{j=1}^{m} n_j - n} \prod_{j=1}^{m}(1-\beta_j) \tag{18}$$

Then, by repeating the processes above we find that

$$\prod_{j=1}^{m+1}{}^{*} f_j(z) \in \tilde{T}_n^{\alpha}(\gamma_{m+1})$$

where

$$\gamma_{m+1} = 1-\left(\frac{\alpha}{\alpha+1}\right)^{m+1}(1-\beta_{M+1})(1-\gamma_m) \tag{19}$$

However, from (18) we have

$$(1-\gamma_m)=\left(\frac{\alpha}{\alpha+1}\right)^{\sum_{j=1}^{m} n_j - n}\prod_{j=1}^{m}(1-\beta_j)$$

so that (18) gives

$$\gamma_{m+1}=1-\left(\frac{\alpha}{\alpha+1}\right)^{\sum_{j=1}^{m+1} n_j - n}\prod_{J=1}^{m+1}(1-\beta_j) \quad (20)$$

Hence the conclusion follows by induction.
Consider the functions $f_j(z)$ $(j=1,2,...,m)$ given by (12). Then

$$\prod_{j=1}^{m}{}^{*} f_j(z)=z^{\alpha}+\left(\frac{\alpha}{\alpha+1}\right)^{\sum_{j=1}^{m} n_j}\prod_{j=1}^{m}\lambda_j(1-\beta_j)z^{\alpha+1} \quad (21)$$

from (21) it follows that

$$\sum_{k=2}^{\infty}\frac{\alpha_0^n}{1-\gamma_m}|a_{k,j}(\alpha)|=1$$

which shows that the result is sharp and this completes the proof.

Corollary 2: If $f_j(z)\in \tilde{T}_n^{\alpha}(\beta)$, then

$$\prod_{j=1}^{m}{}^{*} f_j(z)\in \tilde{T}_n^{\alpha}(\gamma_m)$$

where

$$\gamma_m=1-\left(\frac{\alpha}{\alpha+1}\right)^{(m-1)n}(1-\beta)^m \quad \beta=\beta_j(,j=1,2,...,m)$$

The result is sharp for the function $f_j(z)$ given by

$$f_j(z)=z^{\alpha}+\lambda(1-\beta)\left(\frac{\alpha}{\alpha+1}\right)^{n}z^{\alpha+1},\ |\lambda|=1,\quad j=1,2,...,m \quad (22)$$

Corollary 3: If $f_j(z)\in \tilde{T}_n^{1}(\beta)$, then

$$\prod_{j=i}^{m}{}^{*} f_j(z)\in \tilde{T}_n^{1}(\gamma_m)$$

where

$$\gamma_m=1-2^{(1-m)n}(1-\beta)^m \quad (23)$$

The result is sharp for the function $f_j(z)$ given by

$$f_j(z)=z+\frac{\lambda(1-\beta)}{2}z^2,\ |\lambda|=1,\ j=1,2,...,m \quad (24)$$

Corollary 4: $f_j(z)\in \tilde{T}_n^{1}(0)$, then

$$\prod_{j=i}^{m}{}^{*} f_j(z)\in \tilde{T}_n^{1}(\gamma_m)$$

where

$$\gamma_m=1-2^{(1-m)n} \quad (25)$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z)=z+\frac{\lambda}{2}z^2 \qquad |\lambda|=1 \qquad j=1,2,...,m \quad (26)$$

Corollary 5: If, $f_j(z)\in \tilde{T}_0^{1}(0)$ then

$$\prod_{j=i}^{m}{}^{*} f_j(z)\in \tilde{T}_1^{1}(0)$$

The result is sharp for the function $f_j(z)$ given by

$$f_j(z)=z+\frac{\lambda}{2}z^2 \qquad |\lambda|=1 \qquad j=1,2,...,m \quad (27)$$

GENERALIZED CONVOLUTION IN $\tilde{T}_n^{\alpha}(\beta)$

Theorem 6: If $f_j(z)\in \tilde{T}_{n_j}^{\alpha}(\beta_j)$ $(j=1,2,...,m)$, then

$$\prod_{j=1}^{m}{}^{\bullet} f_j(z)\in \tilde{T}_n^{\alpha}(\gamma_m)$$

where

$$\gamma_m=1-\left(\frac{\alpha}{\alpha+1}\right)^{\sum_{j=1}^{m}\frac{n_j}{p_j}-n}\prod_{j=1}^{m}(1-\beta_j)^{\frac{1}{p_j}} \quad (28)$$

provided that. $\sum_{j=1}^{m}\frac{n_j}{p_j}\ge n$. The result is sharp for the functions $f_j(z)$ given by (12).

Proof: We shall find the largest γ_m such that

$$\sum_{k=2}^{\infty}\alpha_0^n\prod_{j=1}^{m}|a_{k,j}(\alpha)|^{\frac{1}{p_j}}\le 1-\gamma_m \quad (29)$$

For $m=1$, we have $\gamma_1=\beta_1$. Suppose $m=2$, then for $f_1(z)\in \tilde{T}^{\alpha}_{n_1}(\beta_1)$ and $f_2(z)\in \tilde{T}^{\alpha}_{n_2}(\beta_2)$ we have

$$\sum_{j=1}^{\infty}\frac{\alpha_0^{n_1}}{1-\beta_1}\left|a_{k,1}^{(\alpha)}\right| \le 1$$

and

$$\sum_{j=1}^{\infty}\frac{\alpha_0^{n_2}}{1-\beta_2}\,|\,a_{k,1}(\alpha)\,| \le 1$$

so that

$$\prod_{j=1}^{2}\left\{\sum_{k=2}^{\infty}\left(\left(\frac{\alpha_0^{n_j}}{1-\beta_j}\right)^{\frac{1}{p_j}}|\,a_{k,j}(\alpha)\,|^{\frac{1}{p_j}}\right)^{p_j}\right\}^{\frac{1}{p_j}} \le 1$$

Using Holder's inequalities, we have

$$\sum_{k=2}^{\infty}\left(\prod_{j=1}^{2}\left(\frac{\alpha_0^{n_j}}{1-\beta_j}\right)^{\frac{1}{p_j}}|\,a_{k,j}(\alpha)\,|^{\frac{1}{p_j}}\right) \le 1 \qquad (30)$$

which implies that

$$\prod_{j=1}^{2}|\,a_{k,j}(\alpha)\,|^{\frac{1}{p_j}} \le \prod_{j=1}^{2}\left(\frac{1-\beta_j}{\alpha_0^{n_j}}\right)^{\frac{1}{p_j}}$$

From (29), we need to find the largest γ_2 such that

$$\sum_{k=2}^{\infty}\alpha_0^{n}\prod_{j=1}^{2}|\,a_{k,j}(\alpha)\,|^{\frac{1}{p_j}} \le 1-\gamma_2$$

But

$$\sum_{k=2}^{\infty}\frac{\alpha_0^{n}}{1-\gamma_2}\prod_{j=1}^{2}|\,a_{k,j}(\alpha)\,|^{\frac{1}{p_j}} \le \sum_{k=2}^{\infty}\frac{\alpha_0^{n}}{1-\gamma_2}\prod_{j=1}^{2}\left(\frac{1-\beta_j}{\alpha_0^{n_j}}\right)^{\frac{1}{p_j}} \qquad (31)$$

Hence it is sufficient to prove that

$$\frac{\alpha_0^{n}}{1-\gamma_2} \le \prod_{j=1}^{2}\left(\frac{\alpha_0^{n_j}}{1-\beta_j}\right)^{\frac{1}{p_j}}$$

that is,

$$\gamma_2 \le 1-\frac{\prod_{j=1}^{2}(1-\beta_j)^{\frac{1}{p_j}}}{\alpha_0^{\sum_{j=1}^{2}\frac{n_j}{p_j}-n}} = 1-\left(\frac{\alpha}{\alpha+k-1}\right)^{\sum_{j=1}^{2}\frac{n_j}{p_j}-n}\prod_{j=1}^{2}(1-\beta_j)^{\frac{1}{p_j}} \qquad (32)$$

We observe that the right hand side of (32) is an increasing function of k, provided. $\sum_{j=1}^{2}\frac{n_j}{p_j}-n\ge 0$. Hence the largest value of γ_2 is given by

$$\gamma_2 = 1-\left(\frac{\alpha}{\alpha+1}\right)^{\sum_{j=1}^{2}\frac{n_j}{p_j}-n}\prod_{j=1}^{2}(1-\beta_j)^{\frac{1}{p_j}}.$$

This implies that the assertion is true for $m=2$. Next we assume that it is true for m, that is,

$$\prod_{j=1}^{m}{}^{\bullet} f_j(z)\in \tilde{T}^{\alpha}_{n}(\gamma_m)$$

where

$$\gamma_m = 1-\left(\frac{\alpha}{\alpha+1}\right)^{\sum_{j=1}^{m}\frac{n_j}{p_j}-n}\prod_{j=1}^{m}(1-\beta_j)^{\frac{1}{p_j}} \qquad (33)$$

Following the same process as above we can show that

$$\prod_{j=1}^{m+1}{}^{\bullet} f_j(z)\in \tilde{T}^{\alpha}_{n}(\gamma_{m+1})$$

where

$$\gamma_{m+1} = 1-\left(\frac{\alpha}{\alpha+1}\right)^{\frac{n_{m+1}}{p_{m+1}}}(1-\beta_{m+1})^{\frac{1}{p_{m+1}}}(1-\gamma_m)$$

Therefore, by (33), we have

$$\gamma_{m+1} = 1-\left(\frac{\alpha}{\alpha+1}\right)^{\sum_{j=1}^{m+1}\frac{n_j}{p_j}-n}\prod_{j=1}^{m+1}(1-\beta_j)^{\frac{1}{p_j}}$$

Thus the result follows by induction.

If we consider the function (2), we find that

$$\sum_{k=2}^{\infty}\frac{\alpha_0^{n}}{1-\gamma_m}\,|\,a_{k,j}(\alpha)\,|^{\frac{1}{p_j}} = 1$$

which shows that the result is sharp.

Corollary 7: If $f_j(z)\in \tilde{T}^{\alpha}_{n}(\beta_j)$, then

$$\prod_{j=1}^{m}{}^{\bullet} f_j(z)\in \tilde{T}^{\alpha}_{n}(\gamma_m)$$

where

$$\gamma_m = 1-(1-\beta)^m$$

The result is sharp for the function (12).

Proof: The corollary follows from Theorem 6 and the condition (7) when $n_j = n$, $\beta_j = \beta$ for all. $j = 1,2,...,m$

Corollary 10: If $f_j(z) \in \tilde{T}_n^1(\beta_j)$, then

$$\prod_{j=1}^{m} {}^{\bullet} f_j(z) \in \tilde{T}_n^1(\gamma_m)$$

where

$$\gamma_m = 1-(1-\beta)^m$$

The result is sharp for the functions (23)

Corollary 9: I $f_j(z) \in \tilde{T}_n^1(0)$ f, then

$$\prod_{j=1}^{m} {}^{\bullet} f_j(z) \in \tilde{T}_n^1(0)$$

The result is sharp for the functions (24).

Corollary 10: If $f_j(z) \in \tilde{T}_0^1(0)$, then

$$\prod_{j=1}^{m} {}^{\bullet} f_j(z) \in \tilde{T}_0^1(0)$$

The result is sharp for the functions (25).

REFERENCES

Abduhalim, S., 1992. On a class of Analytic Functions involving the Salagean Differential Operator, Tamkang Journal Math.

Babalola, K.O. and T.O. Opoola, Iterated integral transforms of Caratheodory functions and their applications to analytic and univalent functions Tamkang Journal Math.

Macgregor, T.H., 1962. Functions whose derivatives have positive real part. Trans. Am. Math. Soc. MR, MR, 25-797, 104: 532-537.

Opoola, T.O. 1994. On a new subclass of univalent functions. Matematika tome, 36: 195-200.

Owa, S. and H.M. Srivastava, 2002. Some generalized convolution Properties associated with certain subclasses of analytic functions, J. Inequalities in Pure and Applied Math, 42: 1-13.

Singh, R., 1973. On Bazilevic functions. Proc. Am. Soc. 38 MR 47 # 449, 38: 261-271.

Yamaguchi, K., 1966. On functions satisfying $\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > 0$ Proc. Amer. Math. Soc. 17: 588-591, MR 33 # 268.