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A Three Step Rational Methods for Integration of Differential Equations with Singularities

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Abstract: This study describes a three-step method for the numerical solution of ordinary differential equations with singularities. The scheme is based on rational functions approximation technique and its development and analysis is based on power series expansions (Taylor and Binomial) and Dalhquist stability test method. The scheme is convergent and A-stable. Numerical results show that the scheme is accurate, effective and efficient.

Key words: Three step raticmal method, differential equations, singularities

INTRODUCTION

The most popular methods for numerical solution of differential equations are Runge-Kutta methods developed by Runge and improved by Kutta and the linear multistep methods as discussed in Lambert[1], Gear[2], Ademiluyi[3]. These methods are based on the polynomial functions, which are normally smooth and with sufficient continuous derivatives. When such methods are use to solve differential equations with singularities they are found to be inefficient and inaccurate. To solve differential equations with singularities, an alternative method that takes into account the peculiar nature of the equation becomes imperative, Gear and Qsterby[4], Luke *et al.*[5], Fatunla[6,7] and Otunta and Ikhile[8].

This peculiarity is taking into account by the method to be developed. The method is based on inverse polynomial functions. The point of singularities of the function is chosen to coincide with singularity of the solution, so that the resultant method will step over the singular points and behave nicely in the neighborhood of singularities.

Therefore, in this study, we assumed that the numerical solution method of an Initial Value Problem (IVP).

$$y' = f(x,y);\ y(o) = y_0 \tag{1.1}$$

in which there are low order discontinuities is chosen to be of the general form

$$y_{n+k} = \frac{y_n}{1+\sum_{j=1}^{k} b_j x^j_n};\ K=2 \tag{1.2}$$

whose the parameters bj's are real valued constants and y_n is the numerical approximations to the solution at the point $x = x_n$ and k, is the step number and order of the methods.

The objectives of the study are to determine the numerical values of the parameters, bj's for k=1,2, computerized and implemented it on a microcomputer for numerical solution of some typical differential equations with singularities.

Determination of the coefficients of the methods: Setting k = 1 in Eq. 1.2, we obtained a general one step formulae in the form

$$y_{n+1} = \frac{y_n}{1+b_1 x_n} \tag{2.1}$$

Assuming that $|x|<1$, then by the adoption of binomial expansion theorem on the right hand side and ignoring terms of order higher than one, we have

$$y_{n+1} = y_n\left|1-b_1 x_n\right| + 0(x^2) \tag{2.2}$$

The first order Taylor expansion of y_{n+1} about point (x_n, y_n) gives

$$y_{n+1} = y_n + hy'_n + 0\ (h^2) \tag{2.3}$$

Substituting for y_{n+1} in (2.2), we obtained

$$y_n + hy'_n + 0\ (h^2) = y_n - b_1 y_n x_n + 0(x^2) \tag{2.4}$$

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The coefficient b_1 is evaluated by imposing the condition that Eq. 2.4 agrees term by term. Hence, we have

$$-b_1 y_n x_n = hy'_n \qquad (2.5)$$

leading to

$$b_1 = \frac{-hy'_n}{y_n x_n} \qquad (2.6)$$

Substituting this into Eq. 2.1, we obtained a one step integrator of order 1 in the form

$$y_{n+1} = \frac{y_n^{\,2}}{y_n - hy'_n} \qquad (2.7)$$

This incidentally coincides with inverse Euler method, Fatunla[9]. The formula had been used for numerical approximation of the solution of stiff and non-stiff ordinary differential equations and the performance has been found to be satisfactory.

Similarly, by setting k = 2, in (2.1), we obtained a 2nd-stage method as

$$y_{n+2} = \frac{y_n}{1 + b_1 x_n + b_2 x_n^{\,2}} \qquad (2.8)$$

Adopting binomial and Taylor's series expansion method on Eq. 2.8, then we obtained

$$\begin{aligned} y_n + hy'_n + \frac{h^2}{2} + 0(h^3) &= y_n - y_n \\ b_{1xn} - y_{nb2xn}^{\;\;2} + y_{nb1}^{\;\;2}{}_{xn}^{\;2} &+ 0(x^3) \end{aligned} \qquad (2.9)$$

using term-by-term equality principle in Eq. 2.9. We obtained the set of non-linear algebraic equations

$$\left.\begin{aligned} -b_1\, y_n x_n &= hy'_n, \\ -x^2_n\,(y_n b_2 - y_n b_1^{\,2}) &= \frac{h^2 y''_n}{2} \end{aligned}\right\} \qquad (210)$$

Solving these set of algebraic equations, we obtained

$$\begin{aligned} b_1 &= \frac{-hy'_n}{y_n x_n} \text{ and} \\ b_2 &= \frac{h^2}{2{y_n}^2 {x_n}^2}\left[2(y'_n)^2 - y_n y''_n\right] \end{aligned} \qquad (211)$$

Substituting for b_1 and b_2 in Eq. 2.8, we obtained a two-stage method of order 2 as

$$y_{n+2} = \frac{2y_n^3}{2y_n^{\,2} - 2hy_n\, y'_n + h^2\,(2(y'_n)^2 - y_n y''_n)} \qquad (212)$$

Similarly, by setting k = 3, in (1.2), we obtained a 2nd-stage method as

$$y_{n+3} = \frac{y_n}{1 + b_1 x_n + b_2 x_n^{\,2} + b_3 x_n^3} \qquad (213)$$

Following the same procedure we obtained

$$\left.\begin{aligned} b_1 &= \frac{-hy'_n}{y_n x_n} \text{ and} \\ b_2 &= \frac{h^2}{2{y_n}^2 {x_n}^2}\left[2(y'_n)^2 - y_n y''_n\right] \\ b_3 &= -\frac{h^3}{6{y_n}^3 {x_n}^3}\left[6(y'_n)^3 - 6y_n y'_n y''_n + y_n^2 y'''_n\right] \end{aligned}\right\} \qquad (214)$$

Substituting for b_1, b_2 and b_3 in Eq. 2.13, we obtained a third order as

$$y_{n+3} = \frac{6y_n^4}{6y_n^{\,3} - 6hy_n^2\, y'_n + 3h^2 y_n\,(2(y'_n)^2 - y_n y''_n) - h^3\left[6(y'_n)^3 - 6y_n y'_n y''_n + y_n^2 y'''_n\right]} \qquad (215)$$

The next stage considers the consistency and stability properties of the schemes

Properties of the formula

Consistency property

One-stage scheme: Subtracting y_n from both sides of 2.7, we obtained

$$y_{n+1} - y_n = \frac{hy_n\, y'_n}{y_n - hy'_n} \qquad (3.1)$$

which on dividing both sides by h, leads to

$$\frac{y_{n+1} - y_n}{h} = \frac{y_n y'_n}{y_n - hy'_n} \qquad (3.2)$$

Taking limit, as h tends to zero, on both sides of 3.2, we have

$$\left(\frac{y_{n+1} - y_n}{h}\right) \to y'_n = f(x_n, y_n) \qquad (3.3)$$

Suggesting that the one-stage integrator 2.7 is consistent.

Two-stage scheme: Similarly, Subtracting y_n from both sides of 2.12 and dividing by 2h, we obtained

$$\frac{y_{n+2}-y_n}{2h}=\frac{y_n^2y_n' - \frac{h}{2}y_n'(2(y_n')^2-y_ny_n'')}{y_n^2-hy_ny_n'+\frac{h^2}{2}(2(y_n')^2-y_ny_n'')} \quad (3.4)$$

Taking limit on both sides of 3.5 as h → o, we have

$$\lim_{h\to 0}\left(\frac{y_{n+2}-y_n}{2h}\right)\to y_n' \quad (3.5)$$

Implying that the two-stage scheme 2.12 is consistent.

Stability property: Applying the numerical integrator 2.12 to the Dalquist stability test equation.

$$y'=\lambda y;\quad y(x_o)=y_o \text{ and } \mathrm{Re}(\lambda)<0 \quad a\le x\le b \quad (3.6)$$

We obtained a finite difference equation

$$y_{n+1}=\left[\frac{1}{1-\lambda h}\right]y_n \quad (3.7)$$

with $\frac{1}{1-\lambda h}$ as the stability function. For the convergence of the scheme

$|M(z)| <1$, where $z = \lambda h$; Dalhquist (3.8)

Analysis of 3.8 gives $z < 0$ showing that the corresponding interval of absolute stability of the scheme is $(-\infty, 0)$.

Similarly, application of formular 2.12 to problem 3.6 yields

$$y_{n+2}=\left[\frac{1}{1-\lambda h+\frac{\lambda^2h^2}{2}}\right]y_n \quad (3.9)$$

which leads to convergent solution, whenever the stability function $\mu(\lambda h)$.

$$\mu(\lambda h)=\frac{1}{1-\lambda h+\frac{\lambda^2h^2}{2}} \quad (3.10)$$

satisfies the inequality

$$|\mu(\lambda h)|<1$$

Setting $z = \lambda h$, then the absolute stability interval of the scheme is seen to lie in the Region defined by set

$$S=\{z|\mu(z)|<1\}$$

which comes to, $S = \{z: -\infty < z < 2\}$

Showing that the two-stage scheme is stable and suitable also for stiff equations. Since the scheme is consistent and weakly stable then it is convergent, Lambert[1].

Whenever $\lambda h \in S$

Three-stage scheme: Subtracting y_n from both sides of 2.15, we obtained

$$\frac{y_{n+3}-y_n}{3h}=\frac{y_n^3y_n' - \frac{h}{2}y_n(2(y_n')^2-y_ny_n'') + \frac{h^2}{6}\left[6(y_n')^3-6y_ny_n'y_n''+y_n^2y_n'''\right]}{6y_n^3-6hy_n^2y_n'+3h^2y_n(2(y_n')^2-y_ny_n'') - h^3\left[6(y_n')^3-6y_ny_n'y_n''+y_n^2y_n'''\right]} \quad (3.11)$$

Taking limit on both sides of 3.11 as h → o, we have

$$\lim_{h\to 0}\left(\frac{y_{n+3}-y_n}{3h}\right)\to y_n'$$

Implying that the scheme 2.15 is consistent.

Stability property: Applying the numerical integrator 2.15 to the Dalquist stability test equation.

$$y'=\lambda y;\ y(x_o)=y_o \text{ and } \mathrm{Re}(\lambda)<0 \quad a\le x\le b \quad (3.12)$$

$$y_{n+3}=\left[\frac{1}{1-\lambda h+\frac{\lambda^2h^2}{2}-\frac{\lambda^3h^3}{6}}\right]y_n \quad (3.13)$$

which leads to convergent solution, whenever the stability function $\mu(\lambda h)$.

$$\mu(\lambda h)=\frac{1}{1-\lambda h+\frac{\lambda^2h^2}{2}-\frac{\lambda^3h^3}{6}} \quad (3.14)$$

satisfies the inequality

$$|\mu(\lambda h)|<1$$

Setting z = λh, then the absolute stability interval of the scheme is seen to lie in the Region defined by set

$$S=\{z|\mu(z)|<1\}$$

which comes to, $S = \{z : -\infty < z < \infty$

Showing that the scheme is stable and suitable for stiff equations. Since the scheme is consistent and stable then it is convergent, Lambert[1]

Whenever $\lambda h \in S$

Numerical experiments and results: In order to demonstrate the applicability and suitability of the scheme 2.15, we computerised the numerical formula using Fortran programming language and demonstrated it with a well known low order discontinuous differential equation and compared with known existing method as shown in problems 1 and 2

Problem 1: Consider initial value problem

$$y'=\frac{y+5x^2\exp\frac{y}{5x}}{x};\ y(1)=0$$

the theoretical solution is y(x) = -5xlog(2-x), the result is shown in Table 1

Problem 2: Consider the non-singular equation

$$y'=\frac{y\log y}{1-x};\ y(0) = \exp(0.2)$$

The theoretical solution is

$$y(x) = \exp \lambda\ (1-x)$$

the results is as shown in Table 2

RESULTS AND DISCUSSION

The problems were solved numerically with the formular 2.15 and the results are shown in Table 1 and 2. In Table 1, it can be seen that the discretisation errors e_n obtained from the solution of the low order discontinuous equation are sufficiently small compared with the known classical Runge Kutta method, because of it strong stability property, it converges as the step length approaches singularity point, x = π/4. Table 2, shows that the dicretisation error obtained from the solution are sufficiently small, this shows that the scheme is very accurate, stable and convergent in solving singular equations.

Table 1: Thenonrtical solution is y(x) = -Sx10 g (2-x)

X	Y(x)	Error classical R-K method	Proposed scheme
0.01	1.2238727362	2.717×10^{-5}	1.996×10^{-7}
0.2	1.2263982671	5.556×10^{-5}	5.365×10^{-7}
0.3	1.2289812448	8.531×10^{-5}	1.021×10^{-6}
0.4	1.2316236459	1.116×10^{-4}	1.665×10^{-6}
0.5	1.2343275382	1.149×10^{-4}	2.480×10^{-6}

Table 2: Thenonrtical solution is t(x) = exp (1-x)

X	Y(x)	Error classical R-K method	Proposed scheme
0.01	1.0101010099	1.121×10^{-5}	0.0000×10^{0}
0.1	1.1111110946	1.500×10^{-5}	1.331×10^{-8}
0.2	1.2500000279	2.311×10^{-5}	3.374×10^{-8}
0.3	1.4285713921	2.876×10^{-4}	2.323×10^{-8}
0.4	1.6666663521	3.049×10^{-4}	2.892×10^{-7}
0.5	1.9607835483	3.730×10^{-4}	7.206×10^{-7}

CONCLUSION

In this study, we have presented a three step rational method. Theoretical analysis of the method showed that the formula is consistent and stable. Computational experiments carried out confirmed the suitability of the method in solving the singular ordinary differential equations.

The method may therefore be found useful in the solution of problems arising from electrical networks, economy affected by inflations and chemical reaction. If the Richardson error control is adopted, the results obtained would be more accurate with minimum error.

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