Journal of Modern and Mathematics Statistics

Research Article
doi: 10.59218\makjmms.2022.1.7

OPEN ACCESS

Key Words
Repairable systems reliability, bathtub failure intensity, imperfect maintenance, estimation, likelihood

Corresponding Author
Makram Krit
Higher Institute of Companies Administration, University of Gafsa, Tunisia

Received: 12 December 2021
Accepted: 2 January 2022
Published: 28 January 2022

Citation: Makram Krit, 2022. Asymptotic Properties of Estimating Parameters of Intensity Function and Maintenance Effect. J. Mod. Math. Stat., 16: 1-7, doi: 10.59218\makjmms.2022.1.7

Asymptotic Properties of Estimating Parameters of Intensity Function and Maintenance Effect

Makram Krit
Higher Institute of Companies Administration, University of Gafsa, Tunisia

ABSTRACT
The aim of this study is to study the asymptotic behavior of the Arithmetic Reduction of Intensity (ARI) and Arithmetic Reduction of Age (ARA) models as two imperfect maintenance models. These models have been proposed by Doyen and Gaudoin, the failure process is simply Non Homogeneous Poisson Process (NHPP). The maintenance effect is characterized by the change induced on the failure intensity before and after failure during degradation period. To simplify study, the asymptotic properties of failure process are derived. Then, the asymptotic normality of several maintenance efficiency estimators can be proved in the case where the failure process without maintenance is known. Practically, the coverage rate of the asymptotic confidence intervals issued from those estimators is studied.

INTRODUCTION

Throughout their operational life, the industrial systems are subjected to actions of preventive and corrective maintenance. The essential assumptions on maintenance efficiency are known as minimal maintenance and perfect maintenance. Further realistic ideas of maintenance are somewhat intermediary between these two extremes. Various models have already been proposed for imperfect maintenance effects, for example[1]. Single a minority imperfect maintenance models have been statistically studied, particularly regarding the estimation of maintenance efficiency. For virtual age models, some empirical studies on maximum likelihood estimators have been published[2]. The entire of these articles are based on simulation results. Recent articles Peña *et al.*[3] and Doyen[4] deal with simulation and theoretical statistical results in a general class of repair models that include the Kijima[5] models and the Brown-Proschan model as the work of Makram *et al.*[6]. But in these articles, authors consider multiple independent and identical systems over a finite time interval.

Thus, the system behavior without maintenance is known and the failure intensity is then supposed to be as a function of the single efficiency parameter. For this fact, we try to proceed in the same way as Doyen, by introducing in the first place the properties of maximum likelihood estimator and in the second place by interesting in exposing an explicit estimator. Several works was carried out on the parametric statistical inference in imperfect repair models. We refer for example to the Shin *et al*.[7] study in which authors developed a preventive maintenance policy, as the work for Sheu *et al.*[8]. For the case of the Arithmetic Reduction of Intensity (ARI) and the Arithmetic Reduction of Age (ARA) models models, we evoke the Doyen and Gaudoin[9] and Doyen[10] works. The numerical results for our study were at the estimate base by the maximization likelihood method and its properties.

The study is organized as follows: Section 2 discusses the properties of the failure process. Section 3 analyses the failure intensity first order asymptotic extension. Section 4 derives the cumulative failure intensity second order asymptotic extension. Section 5 introduces the maintenance efficiency estimation. A simulation phase is done in section 6. Finally, section 7 gives conclusions.

Properties of the failure process: The basic outline of the ARI model is to ensure that the maintenance effect shoulders on the failure intensity itself. The fundamental idea is at the origin of the Chan and Shaw[11] work, for whom the maintenance effect is to reduce the failure intensity with quantity proportional to its value right before maintenance. This model is characterized by the following failure intensity:

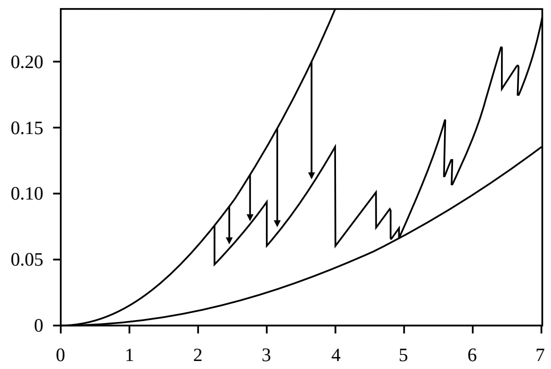

Fig. 1: Failure intensity for the model

$$\lambda_t = \lambda(t) - \rho \sum_{j=0}^{\mathcal{N}_t} (1-\rho)^j \lambda(\mathcal{T}_{\mathcal{N}_t - j}) \qquad (1)$$

Doyen and Gaudoin extended the of Chan-Shaw model. The construction principle of their model appeared in two stages. The first stage is to formulate a model similar to that of Chan-Shaw, for which the maintenance effect is to not reduce the failure intensity but its increase since last maintenance. In a second stage, authors have defines the ARI models with memory m (ARI_m) using intensity written in the following way:

$$\lambda_t = \lambda(t) - \rho \sum_{j=0}^{\min(m-1, \mathcal{N}-1)} (1-\rho)^j \lambda(\mathcal{T}_{\mathcal{N}_t - j}) \qquad (2)$$

Under these conditions, the model defined by the intensity (1) can thus be called ARI model with infinite memory (ARI_∞). Figure 1 represents the generally appearance of the failure intensity of the ARI_m model. The first fine line represents the initial intensity and the second, the minimal degradation intensity, $\lambda_{min}(t)$. This function is defined as maximum lower limit for the failure intensity. For the ARI_m model, minimal degradation intensity is:

$$\forall t \geq 0, \lambda_{min}(t) = (1-\rho)^m \lambda(t) \qquad (3)$$

Practically, that wants to say that the examined system is degraded faster than a system whose failure intensity is $\lambda_{min}(t)$ and slower that a system whose failure intensity is the initial intensity $\lambda(t)$. Near the ARI_m models, the failure intensity is vertically parallel to the initial intensity, as the arrows indicate it on Fig. 1. The maintenance efficiency is allotted by the estimatedvalue of the parameter ρ, called improvement factor.

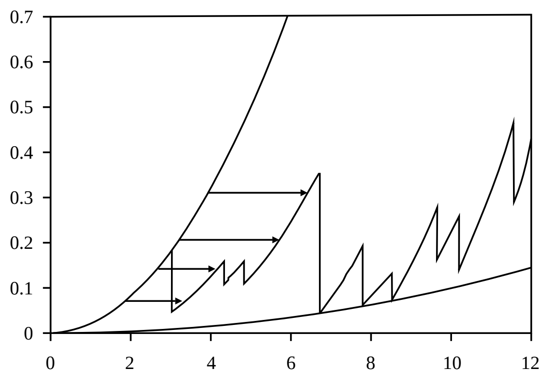

Fig. 2: Failure intensity for the model

The principle of the ARA models is considered in the manner that the maintenance causes to renovate the system. This is with the intuition that its failure intensity at the instant t is equal to the initial intensity at one instant considered as the virtual age of the system, in perception where we consider that the real age of the system put under operation at instant 0 is t. The failure intensity of the ARA model is defined by:

$$\lambda_t = \lambda\left(t - \rho\sum_{j=0}^{\min(m-1,\mathcal{N}_t)}(1-\rho)^j \mathcal{T}_{\mathcal{N}_t - j}\right) \tag{4}$$

Figure 2 represents the failure intensity behavior of the model ARA_m. We see that at any instant, the intensity is parallel to the initial intensity. Nevertheless, this time the correspondence is horizontal (direction of the arrows). The minimal degradation intensity for the ARA_m model is:

$$\forall t \geq 0, \lambda_{\min}(t) = \lambda\left((1-\rho)^m t\right) \tag{5}$$

The failure intensity, as well for the ARA_m models that for the ARI_m models, expressed using the m last failure instants. Thus, the memory imitate a Markovian property, remaining the maximum number of the failures which can influence the failure intensity.

Failure intensity first order asymptotic extension: In this section, the idea is to show that the failure intensity and the asymptotic intensity have an identical behavior. Thus, we recall the property presented in, that if exists a function λ_{min}, not decreasing and verifies for our model $\forall t>\gamma_1$: $\lambda_{min}(t)$ hence for all:

$$k \geq 0 : t - T_{Nt-k} = o(t)$$

We develop the asymptotic intensity with the same principle which followed by Doyen[4], like the minimal degradation intensity for the effective maintenance and the maximal degradation intensity for the harmful maintenance. Consequently, we admit for our generalization of the ARA_m model, like asymptotic failure intensity, the function: $\lambda_\infty = \lambda((1-\rho)^m(t))$ And the function defined by $\lambda_\infty = (1-\rho)m\ \lambda(t)$ is considered as an asymptotic failure intensity of the ARI_m generalized model.

In continuation, the initial intensity, $\lambda(t)$, is supposed to be as a deterministic function, which is not identically null and which is increasing during the period of the system degradation (defined in Makram *et al*.[6] without maintenance process). These conditions necessarily imply:

$$\lim_{t\to+\infty} \Lambda(t) = +\infty$$

The function Λ is the cumulative failure intensity. If moreover λ is a regular variation function, then for $t>\gamma_1$: $\lambda(t)-\lambda(t+0(1)) = o(\lambda(t))$.

Thereafter, the whole of asymptotic results of this study are rested on a rewriting of the failure intensity, considered exclusively by finished memory models. By means of:

$$\rho\sum_{k=0}^{m-1}(1-\rho)^k = 1-(1-\rho)^m$$

This new form of failure intensity is defined, for the ARI_m generalized model, $\forall t \geq T_m \geq T\gamma_1$ as:

$$\lambda_t = \lambda_\infty(t) - \frac{\rho}{(1-\rho)^m}\sum_{k=0}^{m-1}(1-\rho)^k\left[\lambda_\infty(t) - \lambda_\infty\left(t + \left(t - \mathcal{T}_{\mathcal{N}_t - k}\right)\right)\right] \tag{6}$$

By means of the foregoing property, this formula is written as follows:

$$\lambda_t = \lambda_\infty(t) - \frac{\rho}{(1-\rho)^m}\sum_{k=0}^{m-1}(1-\rho)^k\left[\lambda_\infty(t) - \lambda_\infty(t + o(1))\right]$$
$$= \lambda_\infty(t) + o(\lambda_\infty(t))$$

In same way, for the ARA_m generalized models:

$$\lambda_t = \lambda_\infty(t) - \left[\lambda_\infty(t) - \lambda_\infty\left(t + \frac{\rho}{(1-\rho)^m}\sum_{k=0}^{m-1}(1-\rho)^k\left(t - \mathcal{T}_{\mathcal{N}_t - k}\right)\right)\right] \tag{7}$$

And that:

$$\lambda_t = \lambda_\infty(t) - \left[\lambda_\infty(t) - \lambda_\infty(t + o(1))\right] = \lambda_\infty(t) + o(\lambda_\infty(t))$$

Consequently, for our generalizations of the ARI_m and ARA_m models, the failure intensity, for all $t>\gamma_1$, verify: $\lambda t = \lambda\infty(t)+o(\lambda\infty(t))$. Under the same conditions, the cumulative failure intensity proves: Λ_t $\Lambda_\infty(t)+o(\Lambda_t)(t)$. This first order of asymptotic expansion of the failure intensity, make possible to verifies that the increasing phase of the failure intensity and the asymptotic intensity of the ARI_m and ARA_m generalized models of finished memory have a same asymptotic behavior.

Cumulative failure intensity second order asymptotic extension: Using the second order of asymptotic expansion of the cumulative failure intensity, Shin *et al*.[7] goes more and expresses the difference between failure and asymptotic intensities. The author proved that the cumulative failure intensity for the and models with the power failure intensity. In consequence, for our generalization with bath-tub failure intensity, the cumulative failure intensity of the model can be written, for all as:

$$\Lambda_t = \Lambda_\infty(t) + \frac{\rho}{(1-\rho)^m}\sum_{k=0}^{m-1}(1-\rho)^k \int_{\gamma_1}^{t} \lambda_\infty(s) - \lambda_\infty\left(\mathcal{T}_{\mathcal{N}_s - k}\right) ds \quad (8)$$

Thereafter, let's suppose that through the asymptotic intensity, or in an equivalent way, during the degradation phase of the system that the initial intensity is divergent. That's to say then the proposal that the cumulative failure intensity of the ARI_m generalized models ensures:

$$\Lambda_t = \Lambda_\infty(t) + \frac{1-(1+m\rho)(1-\rho)^m}{\rho(1-\rho)^m}\ln\lambda(t) + o(\ln\lambda(t)) \quad (9)$$

By analogy with the ARI_m generalized models is defined by:

$$\lambda_{]\gamma_1,+\infty}(t) = \frac{1}{\eta_0} + \frac{\beta_2}{\eta_2}\left(\frac{t-\gamma_1}{\eta_2}\right)^{\beta_2 - 1}$$

Then the cumulative failure intensity of the ARA_m generalized models verify:

$$\Lambda_t = \Lambda_\infty(t) + (\beta_2 - 1)\frac{1-(1+m\rho)(1-\rho)^m}{\rho(1-\rho)^m}\ln(t) + o(\ln(t)) \quad (10)$$

The two relations 9 and 10 indicate the asymptotic behavior of the failure process of the ARI_m and ARA_m models. In fact, this behavior with finished memory is overall the same one as that NHPP with intensity λ_∞. Obviously, for the models with finished memory, if $\lambda(t)$ is concave (respectively convex), for the same parameter ρ such as $0 \le \rho \le 1$, the asymptotic degradation speed of the ARA_m model is larger (respectively smaller) than that of the model. Consequently, ARI_m ARA_m and models with asymptotic intensities having different asymptotic behaviors are similar to NHPP with different failure intensities. This way, ARI_m and ARA_m models with same parameters are not comparable because they have very different degradation speeds. Nevertheless, it can happen there that if the initial intensity is a function power, the values of the maintenance efficiency parameters such as the models are comparable. And each of the two models has its own maintenance efficiency.

Maintenance efficiency estimation: The object now is to study some estimators of maintenance efficiency since initial intensity is known. In that case the failure intensity is supposed to depend on a simple parameter ρ. The true value of this parameter will be noted ρ_0. The maximum likelihood estimators MLE of maintenance efficiency, denoted $\hat{\rho}_t^{ML}$. For the ARI_m generalized model the MLE of maintenance efficiency parameter checks:

$$\sqrt{\frac{\Lambda(t)}{(1-\rho_0)^m}}(1-\rho_0)^m - (1-\hat{\rho}_t^{ML})^m \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \quad (11)$$

The MLE of maintenance efficiency parameter of the ARA_m generalized model, for only one observation of the failure process proves:

$$\sqrt{\frac{(t-\gamma_1)^{\beta_2}}{\eta_2(1-\rho_0)^{m(\beta_2-1)}}}(1-\rho_0)^{m(\beta_2-1)} - (1-\hat{\rho}_t^{ML})^{m(\beta_2-1)} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \quad (12)$$

As it was seen according to preceding assumptions, we do not know how to prove that the MLE is convergent when the maximization of likelihood is made on]-∞, 1]. So the MLE must be required in compact of]-∞, 1] containing the true value ρ_0 of maintenance efficiency. The explicit estimators (EE), which are not present in this problem type, can exist. These EE verify the same asymptotic properties as the MLE. For the ARI_m generalized model for only one observation of the failure process over $]\gamma_1, t]$, the EE of maintenance efficiency parameter is given by:

$$\hat{\rho}_t^E = 1 - \left[\frac{\mathcal{N}_t}{\Lambda(t)}\right]^{1/m} \quad (13)$$

Similarly with the ARI_m generalized model, an EE can be defined for the ARA_m generalized model using the initial intensity in bath-tub form. Thus, we define near last generalization, for only one observation of the failure process the EE of maintenance efficiency parameter. This estimator is expressed by:

$$\hat{\rho}_t^E = 1 - \left[\frac{\eta_2 \mathcal{N}_t}{t^{\beta_2}}\right]^{1/[m(\beta_2-1)]} \quad (14)$$

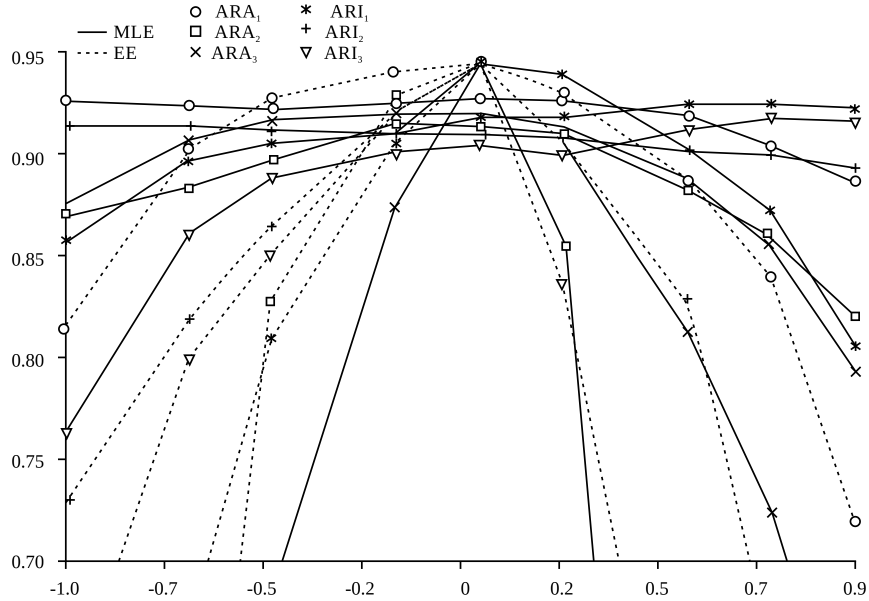

Fig. 3: CR(ρ); $\beta_1 = 0.75$, $\beta_2 = 3$, n = 60

In consideration to asymptotic normality of the estimators introduced in front, we maintain to define the Asymptotic Confidence Intervals (ACI). It is clear that for a same model, MLE and EE verify the same properties, then they describe the identical ACI. Thus, we can assimilate to the model ARI_m generalized model with finished memory, simultaneously for two estimators the ACI for $(1-\rho_0)^m$ at level δ, given by:

$$ACI(\rho) = (1-\hat{\rho})^m + \frac{u_\delta^2 \pm \sqrt{u_\delta^2\left[4\Lambda(t)(1-\hat{\rho})^m + u_\delta^2\right]}}{2\Lambda(t)} \quad (15)$$

where, u_δ indicate it $1-\frac{\delta}{2}$ quantile of the reduced-centered normal law, $\hat{\rho}$ indicate the MLE or EE and:

$$\Lambda_{]\gamma_1,+\infty}(t) = \frac{1}{\eta_0}t + \left(\frac{t-\gamma_1}{\eta_2}\right)^{\beta_2}$$

In a similar way, we can define an ACI for the ARA_m generalized model. The ACI for $(1-\rho_n)^m$ of level δ is defined as follows:

$$ACI(\rho) = (1-\hat{\rho})^{m(\beta_2-1)} + \frac{\eta_2\left(u_\delta^2 \pm \sqrt{u_\delta^2\left[\frac{4}{\eta_2}(t-\lambda_1)^{\beta_2}(1-\hat{\rho})^{m(\beta_2-1)} + u_\delta^2\right]}\right)}{2(t-\lambda_1)^{\beta_2}} \quad (16)$$

Simulation phase: Using simulations groping of the ARI_m and ARA_m models one next to one and for a given ACI, we estimate the coverage Rate (CR). This rate is expressed as the simulations proportion for which the true value of the parameter is in the confidence interval. Obviously, the CR converges to 1-δ when the number of observed failures n increases, where, δ represents the ACI threshold. Practically, the CR is a function only of the estimator quality used to build the ACI. Thus, we have estimated over 10000 simulations, the CR of the ACI at level 95 for m = 1, 2 or 3, $\beta_1 = 0.75$, $\beta_2 = 3$, ρ = -1, -0.7, -0.5, -0.2, 0, 0.2, 05, 0.7 or 0.9 and n = 60. The following notations are used in the Fig. 1:

Figure 3 represents the CR evolution according to the value of maintenance efficiency parameter ρ. So the CR of EE depends closely of the ρ value. The EE provides the most correct ACI for maintenance efficiency close to the minimal case (ρ near to), even for a low values of the number of observed failures. On the other hand when maintenance efficiency is too different to the minimal assumption, the converges less quickly especially in the EE case. This result is a consequence due to the EE, which is founded on an equivalence property between cumulative failure and asymptotic intensities. This equivalence relation is made with a near remainder which is asymptotically equal to:

$$r(t) = \frac{(\beta_2-1)}{(1-\rho)^m}\frac{1-(1+m\rho)(1-\rho)^m}{\rho}\ln(t-\gamma_1) \quad (17)$$

At a certain instant $t>\gamma_1$ and for ρ = 0, this quantity is null by hypothesis. In fact in this case $\Lambda_t = \Lambda_\infty(t)$. Whereas, when ρ tends to 1 by lower values, the above difference diverges. finally, since the maintenance efficiency is degraded and becomes more and more harmful, the cumulative asymptotic intensity increase and the difference r(t) tends to a constant limit equal to m(β_2-1).

The MLE are characterized by CR, which are less sensitive to the value of ρ, but it is always under the assumption of minimal maintenance efficiency that the estimators are most correct. This CR behavior whether through of the MLE or EE, can be owed to the operation of the system in the improvement and service life periods and is maintained by minimal maintenance actions. It appears clearly, on the one hand for low numbers of failures ACI and on the other hand for the models with high enough memories. Thus, for a great number of failures the are the good approximations for the practical value of maintenance efficiency.

CONCLUSION

In this study we generalized two classes of imperfect maintenance models using failure intensity in bath-tub shape. We gave new results on our generalizations of arithmetic reduction of age or intensity with memory m. We have shown that ARIm and ARAm models with finite memories are adapted to reparable systems. In fact, they are asymptotically equivalent to a non homogeneous Poisson process with no decreasing failure intensity. Their failures process is characterized by equivalence between cumulative failure intensity and cumulative asymptotic intensity. In the application, it is proposed that if the initial intensity is unidentified, then an estimate could be used. Except, this also guides to a different property of the estimator of maintenance efficiency. Further study can be to extend the statistical properties to this case. For the ARI and ARA generalized models with finished memory, we proposed the explicit estimators of maintenance efficiency parameter. Then, we presented theoretical statistical results for the estimate of maintenance efficiency. The convergence properties relative to maximum likelihood and explicit estimators were derived. Thus, we could assume that the asymptotic confidence intervals are issued from those estimators.

REFERENCES

1. Krit, M. and A. Rebai, 2012. An estimate of maintenance efficiency in brown-proschan imperfect repair model with bathtub failure intensity. J. Ind. Eng. Manage., 5: 88-101.
2. Kvam, P.H. and E.A. Peña, 2005. Estimating load-sharing properties in a dynamic reliability system. J. Am. Stat. Assoc., 100: 262-272.
3. Peña, E.A., E.H. Slate and J.R. González, 2007. Semiparametric inference for a general class of models for recurrent events. J. Stat. Plann. Inference, 137: 1727-1747.
4. Doyen, L., 2010. Asymptotic properties of imperfect repair models and estimation of repair efficiency. Naval Res. Logistics (NRL), 57: 296-307.
5. Kijima, M., 1989. Some results for repairable systems with general repair. J. Applied Probab., 26: 89-102.
6. Makram, K., R. Abdelwaheb and B. Habib, 2011. General parametric reliability model for reparable system. J. Comp. Commun., 8: 509-516.
7. Shin, I., T.J. Lim and C.H. Lie, 1996. Estimating parameters of intensity function and maintenance effect for repairable unit. Reliab. Eng. Syst. Saf., 54: 1-10.
8. Sheu, S.H., Y.B. Lin and G.L. Liao, 2005. Optimal policies with decreasing probability of imperfect maintenance. IEEE Trans. Reliab., 54: 347-357.
9. Doyen, L. and O. Gaudoin, 2004. Classes of imperfect repair models based on reduction of failure intensity or virtual age. Reliab. Eng. Syst. Saf., 84: 45-56.
10. Doyen, L., 2005. Repair efficiency estimation in the ARI_1 imperfect repair model. In: Series on Quality, Reliability and Engineering Statistics, Wilson, A., N. Limnios, S. Keller-McNulty and Y. Armijo, (Eds.)., World Scientific,, Singapore, ISBN-17: 978-981-4480-25-3, pp: 153-168.
11. Chan, J.K. and L. Shaw, 1993. Modeling repairable systems with failure rates that depend on age and maintenance. IEEE Trans. Reliab., 42: 566-571.