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A New Mixture of Gamma Shape Distributions: Properties Applications

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Abstract: Over the last few decades, mixture distributions are used in creating population from two or more distributions. Mixture distributions are a good application in the applications of Medical Science, Biology, Engineering, Finance and Economics. Gaussian mixture models have broad utility including their usage for model-based clustering framework. Recently, there are indications to use of non-Gaussian mixture distributions to skewed and asymmetric data. We propose a mixture model of Inverse Power Gamma Shape distributions (IPGSM) to analyze positive data. Basic structural properties such raw and central moments, hazard rate function and order statistics are obtained. Different estimation methods are studied to estimate the proposed model parameters. Simulation studies is done to present the performance and behavior of the different estimates of the proposed model parameters. Two real data sets are provided to compare the reliability of the new model with other models.

INTRODUCTION

If we have positive data, right-skewed and assumed to come from a mixture distribution, then the use of a Gamma density is a logical choice. There are many papers discussed finite mixtures of Gamma. John[1] discussed nite mixtures of Gamma of a two-component model with both the method of moments and maximum likelihood. Gharib[2] studied two characterizations for a mixture of two Gamma distributions. Huang and Chang[3] showed that the Lukacstype characterization for the sum of independent Gamma random variables can be represented as a particular mixture of Gamma. Mixtures of Gamma have also been presented as applied models for various applications for example, characterizing rates across sites of molecular sequence evolution Mayrose *et al*.[4], modeling internet traffic Almhana *et al*.[5] and modeling extremes in various hydrological phenomena Evin *et al*.[6]. The objective of this paper is to consider seven different estimators for the parameters of our proposed distribution and evaluates their performance in simulation and applications studies. Many authors have compared several classical estimation methods for estimating the parameters of well-known distributions. For example, Rodrigues *et al*.[7] for Poisson-exponential distribution, Karamikabir *et al*.[8] for a new extended generalized Gompertz distribution, Dey *et al*.[9] for exponentiated Chen distribution and Sharma *et al*.[10] for the generalized inverse Lindley distribution.

In this study, we are motivated to introduce the IPGSM model because it contains a mixture of another lifetime sub model; this model reveals upside down bathtub-shaped hazard rate which occurs in most real life systems and very useful in survival analysis; the proposed model can be considered as a suitable model for fitting the positive data with a longer right tail which can be used in various fields such as survival analysis and biomedical studies and the IPGSM model outperforms most well-known lifetime models with respect to two real data sets.

For our paper, let X is continuous random variable follow the Gamma distribution with parameters λ and θ, then the probability density function (pdf) is given by:

$$f(y; \lambda, \theta) = \frac{\theta^{\lambda}}{\Gamma(\lambda)} y^{\lambda-1} e^{\theta y}; \ y > 0, \lambda, \theta > 0 \tag{1}$$

where, $\Gamma(a) = \int_0^{\infty} y^{a-1} e^{-y} dy$ is (complete) Gamma function. Here, λ is a shape parameter and θ is a an inverse scale parameter called a rate parameter for the gamma density. We denote this distribution by $G(\lambda, \theta)$ and the cumulative distribution function (cdf) can be written as:

$$F(y; \lambda, \theta) = \frac{\gamma(\lambda, \theta y)}{\Gamma(\lambda)} \tag{2}$$

where, $\gamma(a, x) = \int_0^{x} y^{a-1} e^{-y} dy$ is the lower incomplete gamma function. Let Y be a random variable having pdf (1), then the random variable $X = Y^{-1/\alpha}$ = is said to follow an Inverse Power Gamma (IPG) distribution, shown as X ~ IPG (α, λ, θ) with pdf and corresponding cdf defined, respectively by:

$$f(x; \alpha, \lambda, \theta) = \frac{\alpha\theta^{\lambda}}{\Gamma(\lambda)} x^{-(1+\alpha\lambda)} e^{-\theta x^{-\alpha}}; \ x > 0, \alpha, \lambda, \theta > 0 \tag{3}$$

$$F(x; \lambda, \theta, \alpha) = \frac{\Gamma(\lambda, \theta x^{-\alpha})}{\Gamma(\lambda)} \tag{4}$$

where, $\Gamma(a, x) = \int_x^{\infty} y^{a-1} e^{-y} dy$ is the upper incomplete gamma function. It can be noticed that the inverse Gamma (IG) distribution is a special case of IPG when $\alpha = 1$.

Suppose a mixture distribution consisting of k components (i = 1, 2, ..., k) and the distribution of the ith individual component follows an IPG distribution. The generated mixture distribution represents the Inverse Power Gamma Shape Mixture (IPGSM) distribution with pdf and cdf defined, respectively by:

$$f(y; \pi, \alpha, \theta) = \sum_{i=1}^{k} \pi_i f_i(x; \alpha, \theta) \tag{5}$$

$$F(y; \pi, \alpha, \theta) = \sum_{i=1}^{k} \pi_i F_i(x; \alpha, \theta) \tag{6}$$

where f_i (x; α, θ) = f (x; α, i, θ) and F_i (x; i, α, θ) denote respectively, the pdf and cdf of an inverse power Gamma IPG (α, i, θ) random variable. Let k is known and fixed, whereas $\pi = (\pi_1, ..., \pi_k)$ is a vector of mixture weights (proportions) that satisfy the conditions:

- $0<\pi_i<1 \ \forall i = 1, 2, ..., k$
- $\sum_{i=1}^{k} \pi_i = 1$

The aim of this study is to define and study a new finite mixture distribution called the Inverse Power Gamma Shape Mixture (IPGSM) distribution with its mathematical properties. These include the reliability measurers such as survival, reverse survival and hazard rate function. The moments and moment generating function are provided. Maximum likelihood estimation of the model parameters and confidence interval are derived. Application of the model to a real data set is finally presented and compared to the fit attained by some other well-known distributions.

The IPGSM distribution and statistical properties: Equation 5 and 6 can simply be rewritten using next theorem.

Theorem 1: Let X be a random variable that follows the IPGSM distribution, then the pdf and cdf can be written, respectively as:

$$f(x) = \frac{\alpha\theta^{k}}{\Gamma(k, \theta)} \frac{(1+x^{\alpha})^{k-1}}{x^{\alpha k+1}} e^{-(1+x^{-\alpha})^{\theta}}; \ y > 0, \theta, \alpha > 0 \tag{7}$$

$$F(x) = \frac{\Gamma(k, (1+x^{-\alpha})\theta)}{\Gamma(k, \theta)} \tag{8}$$

where k = 1, 2, 3, ...

Proof:

$$f(x) = \frac{\alpha\theta^{k}}{\Gamma(k, \theta)} \frac{(1+x^{\alpha})^{k-1}}{x^{\alpha k+1}} e^{-\theta(x^{-\alpha}+1)} = \frac{\alpha\theta^{k} e^{-\theta}}{\Gamma(k, \theta) x^{\alpha k+1}} \sum_{i=1}^{k} \binom{k-1}{i-1} x^{\alpha(k-i)} e^{-\theta x^{-\alpha}}$$

where, $(1+x^{\alpha})^{k-1} = \sum_{i=1}^{k} \binom{k-1}{i-1} x^{\alpha(k-1)}$

$$f(x) = \frac{\Gamma(k) e^{-\theta}}{\Gamma(k, \theta)} \sum_{i=1}^{k} \frac{\theta^{k-i}}{(k-i)!} \frac{\alpha\theta^{i}}{\Gamma(i)} x^{-(\alpha i+1)} e^{-\theta x - \alpha} = \frac{\Gamma(k) e^{-\theta}}{\Gamma(k, \theta)} \sum_{i=1}^{k} \frac{\theta^{k-i}}{(k-i)!} f_i(x; \alpha, \theta) \tag{9}$$

From the definition of the upper incomplete gamma function, we have:

$$\Gamma(k,\theta) = \int_0^\infty (\theta+u)^{k-1} e^{-(\theta+u)}\, du$$

where, k = 1, 2, 3, ...

$$\Gamma(k,\theta) = \int_0^\infty \sum_{i=1}^{k} \binom{k-1}{i-1} \theta^{k-i} u^{i-1} e^{-u} du = e^{-\theta}\sum_{i=1}^{k}\binom{k-1}{i-1}\theta^{k-i}\int_0^\infty u^{i-1}e^{-u}du = e^{-\theta}\sum_{i=1}^{k}\frac{\Gamma(k)\theta^{k-i}}{(k-i)!\Gamma(i)}\Gamma(i)$$

so, we have:

$$\Gamma(k,\theta) = \Gamma(k)e^{-\theta}\sum_{i=1}^{k}\frac{\theta^{k-i}}{(k-i)!} \tag{10}$$

From Eq. 9 and 10, we have:

$$\pi_i = \frac{\Gamma(k)\theta^{k-i}e^{-\theta}}{\Gamma(k,\theta)(k-i)!} \tag{11}$$

where, π_i is restricted to be positive and sum to unity (π_i i>0 and $\Sigma^k_{i=1}\pi_i = 1$). The cumulative distribution function (cdf) of the IPGSM distribution is given by:

$$F(x) = \int_0^x f(z) = \frac{\alpha\theta^k}{\Gamma(k,\theta)}\int_0^x \frac{(1+z^\alpha)^{k-1}}{z^{\alpha k+1}} e - (1+z^{-\alpha})^\theta dz$$

letting $y = (1+z^{-\alpha})\theta$ and after simplification the expression, we get the following:

$$F(x) = \frac{1}{\Gamma(k,\theta)}\int_{(1+x^{-\alpha})\theta}^\infty y^{k-1}e^{-y}dy = \frac{\Gamma(k,(1+x^{-\alpha})\theta)}{\Gamma(k,\theta)}$$

Behavior of the density function: The behaviors of the density function of the IPGSM distribution at x = 0 and x = ∞, respectively are given by:

$$\lim_{x\to 0} f(x) = \frac{\alpha\theta^k}{\Gamma(k,\theta)}\frac{\left(\lim_{x\to 0}\left(1+x^\alpha\right)^{k-1}\right)\left(\lim_{x\to 0} e - \left(1+x^{-\alpha}\right)^\theta\right)}{\left(\lim_{x\to 0} x^{\alpha k+1}\right)}$$

putting $y = x^{-1}$:

$$\lim_{x\to 0} f(x) = \lim_{y\to\infty} f(y) = \frac{\alpha\theta^k}{\Gamma(k,\theta)}\lim_{x\to\infty}\left(\frac{y^{\alpha k+1}}{e^{(1+y\alpha)\theta}}\right) = 0$$

and:

$$\text{mathop}\lim_{x\to\infty} f(x) = \lim_{y\to 0} f(y) = \frac{\alpha\theta^k}{\Gamma(k,\theta)}\lim_{x\to 0}\left(\frac{y^{\alpha k+1}(1+y^\alpha)^{k-1}}{e(1+y^\alpha)^\theta y^{\alpha(k-1)}}\right) = \frac{\alpha\theta^k}{\Gamma(k,\theta)}\lim_{x\to 0}\left(\frac{y^\alpha(1+y^\alpha)^{k-1}}{e(1+y^\alpha)^\theta}\right) = 0$$

Theorem 2: The probability density function of the MIPGSD model is unimodal shaped in x.

Proof: The first derivative of f(x)is given by:

$$f'(x) = -\frac{\psi(x)}{g(x)} f(x)$$

where:

$$\psi(x) = ax^{2\alpha} + bx^\alpha + c,\ g(x) = (+x^\alpha)x^{\alpha+1}$$

with:

$$a = 1+\alpha,\ b = 1+(k-\theta)\alpha,\ c = -\alpha\theta$$

It is clear that ψ(x) is a unimodal quadratic function and that the mode of f(x) implies ψ(x) = 0. Let $D = (b^2-4ac)$ be the discriminant of ψ(x), the second derivative of f(x) given by:

$$f''(x) = -\frac{1}{g(x)}[(g'(x)+\psi(x))f'(x)+\psi'(x)f(x)]$$

where, $g'(x) = (1+\alpha+(1+2\alpha)\,x^\alpha$ and $\psi'(x) = 2a\alpha x^{2\alpha-1}+b\alpha x^{\alpha-1}$. Clearly, D>0 and ψ(x) has maximum value at the point x_0 where:

$$x_0 = \left(\frac{-b+\sqrt{D}}{2a}\right)^{\frac{1}{\alpha}}$$

since, $f''(x_0) = -\left(\sqrt{D}/g(x_0)\right)f(x_0) < 0$, f(x) has a global maximum at x_0, hence, the mode of f(x) is given by:

$$x_0 = \left(\frac{-1-(k-\theta)\alpha+\sqrt{(1+(k-\theta)\alpha)^2+4\alpha\theta(1+\alpha)}}{2(1+\alpha)}\right)^{\frac{1}{\alpha}}$$

In Fig. 1, we plot the behavior of pdf for the IPGSM distribution for some values of θ, α.

Behavior of the hazard rate function: The hazard rate function (hf) of the proposed model is obtained as:

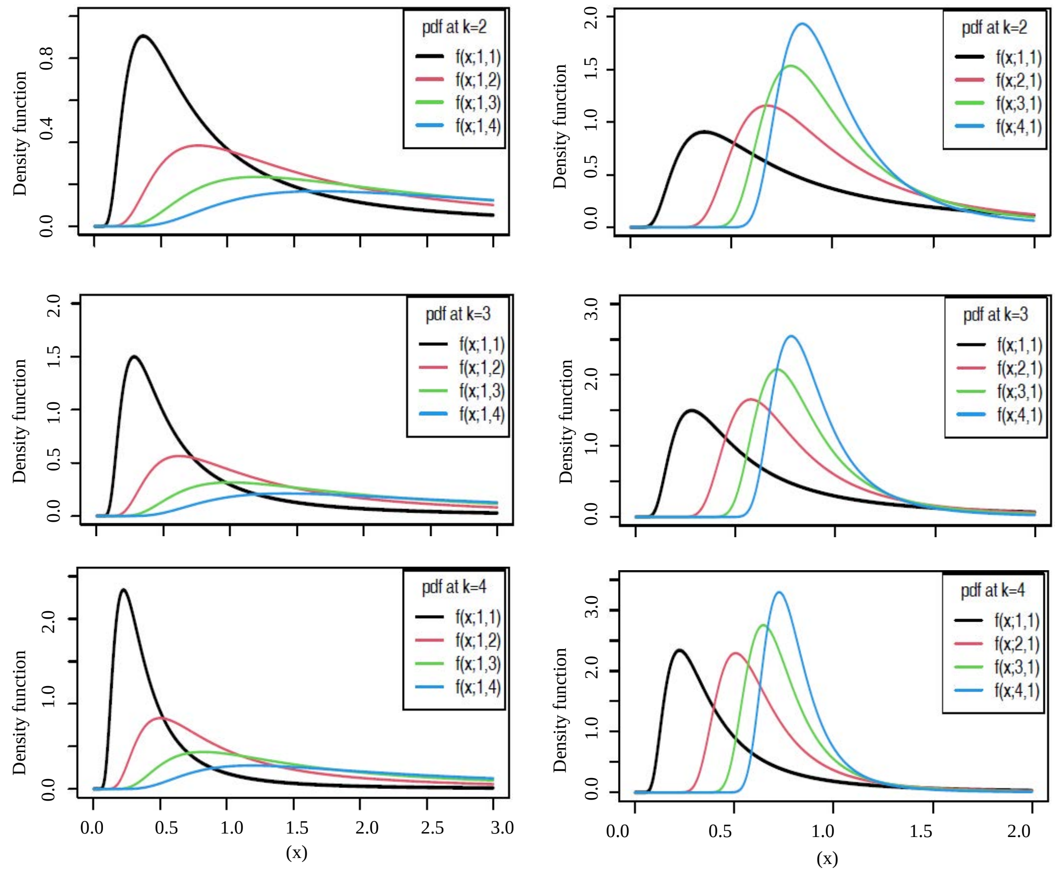

Fig. 1: Plots of the probability density function of the MIPGSD model for different parameter values

$$h(x) = \frac{\alpha\theta^k}{\Gamma(k,\theta) - \Gamma(k,(1+x^{-\alpha})\theta)} \frac{(1+x^{\alpha})^{k-1}}{x^{\alpha k+1}} e^{-(1+x^{-\alpha})\theta} \quad (12)$$

where, $x>0$ and α, $\theta>0$. Figure 2 shows the hf plots of the MIPGSD model for different values of the parameters and Fig. 2 reveals that the hf of proposed model is upside down bathtub shaped.

Moments and related measures: Let X be a random variable that follows the MIPGSD model with pdf as in Eq. 11, then the rth raw moment (about the origin) is given by:

$$\mu'_r = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)} \sum_{i=1}^{k} \frac{\theta^{k+\frac{r}{\alpha}-i}}{(k-i)!\Gamma(i)} \Gamma\left(\frac{i\alpha - r}{\alpha}\right), i > \frac{r}{\alpha} \quad (13)$$

The mean of the MIPGSD model is given by:

$$\mu = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)} \sum_{i=1}^{k} \frac{\theta^{k+\frac{r}{\alpha}-i}}{(k-i)!\Gamma(i)} \Gamma\left(\frac{i\alpha - r}{\alpha}\right), i > \frac{r}{\alpha}$$

The nth central moments of the proposed model are given by:

$$\mu_n = E(X-\mu)^n = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)} \sum_{j=0}^{n} \sum_{i=1}^{k} \binom{n}{j} \frac{(\mu)^{n-j}\theta^{k+\frac{j}{\alpha}-i}}{(k-i)!\Gamma(i)} \Gamma\left(\frac{i\alpha - j}{\alpha}\right), i > \frac{j}{\alpha} \quad (14)$$

The variance, coefficient of skewness, kurtosis and variation measures can be obtained from the expressions:

$$\sigma^2 = \mu_2 = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)} \sum_{j=0}^{2} \sum_{i=1}^{k} \binom{n}{j} \frac{(\mu)^{n-j}\theta^{k+\frac{j}{\alpha}-i}}{(k-i)!\Gamma(i)} \Gamma\left(\frac{i\alpha - j}{\alpha}\right), i > \frac{j}{\alpha}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \beta_2 = \frac{\mu_4}{\mu_2^2} \text{ and } CV = \frac{\sigma}{\mu} \times 100$$

upon substituting for the central moments in Eq. 14. Raw moment of the MIPGSD model will exist only when $i>r/\alpha$.

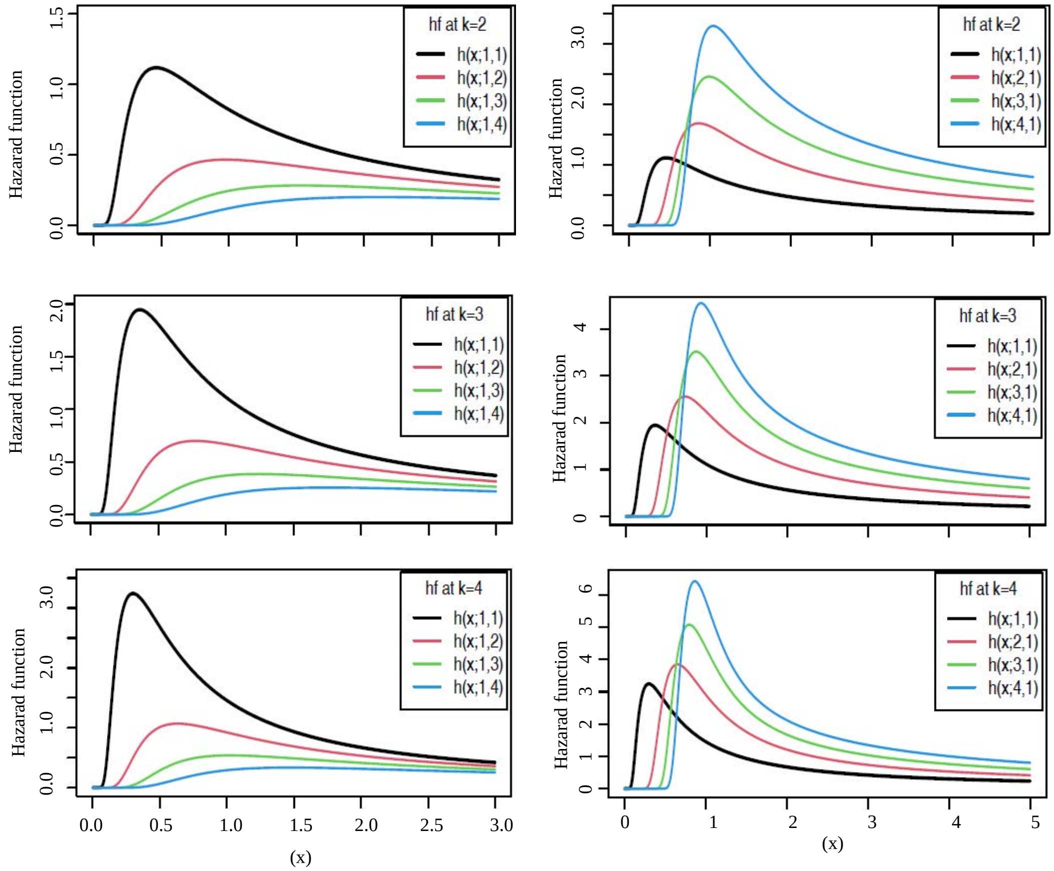

Fig. 2: Plots of the hazard function of the MIPGSD model for different parameter values

Therefore, the evaluation of inverse moments may be of interest. The rth raw inverse moment (about the origin) is given by:

$$\mu'_{r^{-1}} = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)}\sum_{i=1}^{k}\frac{\theta^{k-\left(\frac{r}{\alpha}+i\right)}}{(k-i)!\Gamma(i)}\Gamma\left(\frac{i\alpha+r}{\alpha}\right) \tag{15}$$

The harmonic mean of the proposed distribution is obtained by:

$$H = \frac{\Gamma(k)e^{-\theta}}{\Gamma(k,\theta)}\sum_{i=1}^{k}\frac{\theta^{k-\frac{1}{\alpha}-i}}{(k-i)!\Gamma(i)}\Gamma\left(\frac{i\alpha+1}{\alpha}\right) \tag{16}$$

From the empirical relation among mean, median and mode, the median (M)of the proposed distribution can be written as:

$$M = \frac{1}{3}x_0 + \frac{2}{3}\mu \tag{17}$$

Table 1 shows some important measures of the IPGM distribution at different parameter combination and it is observed that the shape of the proposed distribution is right skewed for values of k, α and θ.

Order statistics: Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are the n ordered random sample drawn from pdf (7). Then, the density of the rth order statistic follows from Arnold *et al*.[11] with the pdf of $X_{(r)}$ is given:

$$f_{r:n}(x) = \frac{(r-1)!(n-r)}{(n)!}!\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)_j[F(x)]^{r+j-1}f(x),\ x>0$$

and the rth order cdf $F_{r:n}(x)$ is:

$$F_{r:n}(x) = \sum_{i=0}^{n}\sum_{j=0}^{n-r}\binom{n}{i}\binom{n-i}{j}(1-)_j[F(x)]_{i+j}$$

Hence, using (Eq. 7, 8), the pdf and the cdf of rth order statistics are, respectively, given by:

$$f_{r:n}(x) = \frac{(r-1)!(n-r)!\alpha\theta^k}{(n)!\Gamma(k,\theta)}\sum_{l=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{n-r}\binom{k-1}{i-1}\binom{n-r}{j}\left[\frac{\Gamma(k,(1+x^{-\alpha})\theta)}{\Gamma(k,\theta)}\right]^{r+j-1}\frac{x^{-i\alpha+1}(-1)^l[1|(1+x^{-\alpha})\theta]^l}{l!} \tag{18}$$

Table 1: Values of some important measures of the MIPGSD model

Moments	μ	σ^2	β_1	β_2	x_0	M	CV	H
k (α = 6, θ = 4)								
2	1.3748	0.1162	8.1008	25.1965	1.1870	1.3122	24.7971	0.7609
4	1.2685	0.0906	9.0295	27.4190	1.1010	1.2127	23.7307	0.8211
6	1.1538	0.0606	11.0618	32.8759	1.0225	1.1100	21.3309	0.8954
8	1.0476	0.0327	14.1981	43.8147	0.9608	1.0187	17.2496	0.9754
10	0.9662	0.0148	14.6772	55.0649	0.9143	0.9489	12.6023	1.0479
k = 4, θ = 2								
6	1.0501	0.0479	11.1658	33.6334	0.9370	1.0124	20.8408	0.9825
8	1.0338	0.0236	7.0516	19.3308	0.9546	1.0074	14.8540	0.9843
10	1.0253	0.0141	5.5140	14.9817	0.9646	1.0050	11.5615	0.9862
12	1.0202	0.0093	4.7200	12.9152	0.9710	1.0038	9.4705	0.9878
14	1.0168	0.0067	4.2376	11.7156	0.9755	1.0030	8.0227	0.9891
k = 2, α = 6								
1	1.0347	0.0578	9.2082	28.1615	0.9058	0.9917	23.2315	1.0050
2	1.1966	0.0846	8.4914	26.1580	1.0365	1.1432	24.3055	0.8724
3	1.2991	0.1025	8.2261	25.4955	1.1225	1.2402	24.6464	0.8047
4	1.3748	0.1162	8.1008	25.1965	1.1870	1.3122	24.7971	0.7609
5	1.4351	0.1274	8.0318	25.0363	1.2388	1.3697	24.8768	0.7291

$$F_{r:n}(x) = \sum_{j=0}^{n}\sum_{k=0}^{n-r}\binom{n}{j}\binom{n-j}{k}(-1)^k\left[\frac{\Gamma(k,(1+x^{-\alpha})\theta)}{\Gamma(k,\theta)}\right]^{j+k} \quad (19)$$

Estimation and inference of the parameters: The main aim of this section is to study different estimation methods of the unknown parameters of the MIPGSD model.

Maximum likelihood method: The most widely method used for parameter estimation is maximum likelihood method. Let $x_1, x_2, ..., x_n$ be a random sample from the MIPGSD model with pdf (Eq. 11). The log-likelihood function is given by:

$$L = -n\theta + n\ln(\alpha) + kn\ln(\theta) - n\ln\Gamma(k,\theta) - (1+\alpha)\sum_{i=1}^{n}x_i + (k-1)\sum_{i=1}^{n}\ln(1+x_i^{-\alpha}) - \theta\sum_{i=1}^{n}\ln\left(x_i^{-\alpha}\right)$$

The maximum likelihood estimators (MLEs) of α, θ denoted by $\hat{\alpha}_{MLE}$ and $\hat{\theta}_{MLE}$ can be obtained by solving the following system of non-linear equations:

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\ln(x_i) - \theta\sum_{i=1}^{n}-x_i^{-\alpha}\ln(x_i) + (k-1)\sum_{i=1}^{n}\frac{x_i^{-\alpha}\ln(x_i)}{1+x_i^{-\alpha}} = 0$$

$$\frac{\partial L}{\partial \theta} = -n + \frac{kn}{\theta} + \frac{n\theta^{k-1}e^{-\theta}}{\Gamma(k,\theta)} - \sum_{i=1}^{n}x_i^{-\alpha} = 0$$

We used non-linear maximization techniques to get the solution of the MLE's of the parameters. For interval estimation of the parameter vector $\Theta = (\alpha, \theta)^T$, we derive Fisher information matrix for constructing 100 (1-ψ)% asymptotic confidence interval for the parameters using large sample theory. The Fisher information matrix can be obtained by using log-likelihood function as:

$$I(\hat{\alpha}, \hat{\theta}) = -E\begin{pmatrix} \frac{\partial^2 L}{\partial \alpha^2} & \frac{\partial^2 L}{\partial \alpha\theta} \\ \frac{\partial^2 L}{\partial \alpha\theta} & \frac{\partial^2 L}{\partial \theta^2} \end{pmatrix}$$

where:

$$\frac{\partial^2 L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \theta\sum_{i=1}^{n}(\ln(x_i))\,x_i^{-\alpha} + (k-1)\sum_{i=1}^{n}\left(\frac{x_i^{-\alpha}(\ln(x_i))^2}{1+x_i^{-\alpha}} - \frac{x_i^{-2\alpha}(\ln(x_i))^2}{(1+x_i^{-\alpha})^2}\right)$$

$$\frac{\partial^2 L}{\partial \theta^2} = \frac{kn}{\theta^2} - n\left(-\frac{\theta^{2k-2}e^{-2\theta}}{\Gamma(k,\theta)} - \frac{(k-1)\theta^{k-2}e^{-\theta}}{\Gamma(k,\theta)}\right)$$

$$\frac{\partial^2 L}{\partial \alpha\theta} = \frac{\partial^2 L}{\partial \theta\alpha} = -\sum_{i=1}^{n}x_i^{-\alpha}\ln(x_i)$$

The diagonal elements of the inverse of the Fisher information matrix $I^{-1}\left(\hat{\alpha}, \hat{\theta}\right)$ provide asymptotic variance of α and θ, respectively. The corresponding asymptotic 100 (1-ψ)% confidence interval of θ and α are given by:

$$\hat{\alpha} \mp Z_{1-\frac{\psi}{2}}\sqrt{\mathrm{Var}(\hat{\alpha})},\ \hat{\theta} \times Z_{1-\frac{\psi}{2}}\sqrt{\mathrm{Var}(\hat{\theta})}$$

respectively.

Least squares and weighted least squares methods: The least squares (LSE) and the weighted least squares (WLSE) methods are used to find the minimum distance between theoretical cumulative distribution and the empirical cumulative distribution.

These methods were introduced by Swain *et al*.[12] to estimate the parameters of Beta distributions. Let $F(X_{(i)})$ be the distribution function of the ordered random variables $X_{(1)}<X_{(2)}, ..., X_{(n)}$ where $\{X_1, X_2, ..., X_2\}$ is a random sample of size n from a distribution function F(.).

Then, the expectation of the empirical cumulative distribution function is defined as:

$$E[F(X_{(i)})] = \frac{i}{n+i}; \; i = 1, 2, ... n$$

The LSEs of α and θ denoted by $\hat{\alpha}_{LSE}$ and $\hat{\theta}_{LSE}$ can be obtained by minimizing the following function:

$$LS(\alpha, \theta) = \sum_{i=1}^{n}\left(F(x_{(i)}; \alpha, \theta) - \frac{i}{n+i}\right)^2 \quad (20)$$

with respect to α and θ where F(.) is given by Eq. 8. Therefore, $\hat{\alpha}_{LSE}$ and $\hat{\theta}_{LSE}$ can be obtained as the solution of the following system of non-linear equations:

$$\frac{\partial LS(\alpha, \theta)}{\partial \alpha} = \sum_{i=1}^{n}\left(F(x_{(i)}; \alpha, \theta) - \frac{i}{n+i}\right)F'_{\alpha}(x; \alpha, \theta) = 0 \quad (21)$$

$$\frac{\partial LS(\alpha, \theta)}{\partial \theta} = \sum_{i=1}^{n}\left(F(x_{(i)}; \alpha, \theta) - \frac{i}{n+i}\right)F'_{\theta}(x; \alpha, \theta) = 0 \quad (22)$$

Gupta and Kundu[13] introduced the following weighted function:

$$w_i = \frac{1}{Var(x_{(i)})} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$$

The WLSEs of α and θ denoted by $\hat{\alpha}_{WLSE}$ and $\hat{\theta}_{WLSE}$ can be obtained by minimizing:

$$WLS(\alpha, \theta) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(x_{(i)}; \alpha, \theta) - \frac{i}{n+i}\right)^2 \quad (23)$$

with respect to α and θ, therefore, these estimators can also be obtained by solving:

$$\frac{\partial WLS(\alpha, \theta)}{\partial \alpha} = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(x_{(i)}; \alpha, \theta) - \frac{i}{n+i}\right) F'_{\alpha}(x; \alpha, \theta) = 0 \quad (24)$$

$$\frac{\partial WLS(\alpha, \theta)}{\partial \theta} = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(F(x_{(i)}; \alpha, \theta) - \frac{i}{n+i}\right) F'_{\theta}(x; \alpha, \theta) = 0 \quad (25)$$

where:

$$F'_{\alpha}(x; \alpha, \theta) = \frac{\theta^k}{\Gamma(k, \theta)}\frac{(1+x^{\alpha})^{k-1}}{x^{\alpha k}}e^{-(1+x-\alpha)\theta}\ln(x)$$

and:

$$F'_{\theta}(x; \alpha, \theta) = \frac{\theta^{k-1}\left(\Gamma(k, \theta) + \Gamma(k, (1+x^{-\alpha})\theta) - 1 + x^{-\alpha})^k e^{-\theta x - \alpha}\right)e^{-\theta}}{[\Gamma(k, \theta)]^2}$$

Cramer-von-Mises estimator: The Cramer-von Mises (CME) method is a type of minimum distance estimation method introduced by Choi and Bulgren[14]. This method based on the Cramer-von Mises statistics given by:

$$W^2 = n\int_0^{\infty}[F(x_i) - E[F(x_{(i)})]]^2 dF(x_i)$$

Boos[15] proved that the Cramer-von Mises statistics can be given by:

$$C(\alpha, \theta) = \frac{1}{12n} + \sum_{i=1}^{n}\left(F(x_{(i)}; \alpha, \theta) - \frac{2i-1}{2n}\right)^2 \quad (26)$$

Then the CME estimators $\hat{\alpha}_{CME}$ and $\hat{\theta}_{CME}$ of α and θ are obtained by minimizing (Eq. 26) with respect to α and θ. These estimators can also be obtained by solving the following non-linear equations:

$$\frac{\partial C(\alpha, \theta)}{\partial \alpha} = \sum_{i=1}^{n}\left(F(x_{(i)}; \alpha, \theta) - \frac{2i-1}{2n}\right)F'_{\alpha}(x; \alpha, \theta) = 0 \quad (27)$$

$$\frac{\partial C(\alpha, \theta)}{\partial \theta} = \sum_{i=1}^{n}\left(F(x_{(i)}; \alpha, \theta) - \frac{2i-1}{2n}\right)F'_{\theta}(x; \alpha, \theta) = 0 \quad (28)$$

Maximum product spacing method: Cheng and Amin[16] introduced the maximum product spacing (MPS) and showed that the MPS method can be used as an alternative to MLE to estimate the parameters of continuous univariate distributions. This method assumes that differences (spacings) between the cdf values should be identically distributed at consecutive data points. Let the difference is defined as:

$$D_i(\alpha, \theta) = F(x_{(i)}; \alpha, \theta) - F(x_{(i-1)}; \alpha, \theta), \; i = 1, 2, ..., n \quad (29)$$

where, $F(x_{(0)}; \alpha, \theta) = 0$ and $F(x_{(n+1)}; \alpha, \theta) = 1$. The geometric mean of the differences can be written as:

$$G(\alpha, \theta) = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i(\alpha, \theta)} \quad (30)$$

Substituting (Eq. 29) in (Eq. 30) and maximizing the above expression, we have:

$$g(\alpha, \theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log(F(x_{(i)}; \alpha, \theta) - F(x_{(i-1)}; \alpha, \theta) \quad (31)$$

Cheng and Stephens[17] showed that finding the maximum of the geometric mean of the spacings is the same as finding the minimum of the Moran's statistics, the Moran's statistics given by:

$$M(\alpha, \theta) = \sqrt[n+1]{\prod_{i=1}^{n+1} D_i(\alpha, \theta)} \quad (32)$$

The MPSEs $\hat{\alpha}_{MPS}$ and $\hat{\theta}_{MPS}$ of α and θ are obtained as the simultaneous solution of the following non linear equations:

$$\frac{\partial \log G(\alpha, \theta)}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{F'_\alpha(x_{(i)}; \alpha, \theta) - F'_\alpha(x_{(i-1)}; \alpha, \theta)}{F(x_{(i)}; \alpha, \theta) - F(x_{(i-1)}; \alpha, \theta)}\right) = 0 \quad (33)$$

$$\frac{\partial \log G(\alpha, \theta)}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{F'_\theta(x_{(i)}; \alpha, \theta) - F'_\theta(x_{(i-1)}; \alpha, \theta)}{F(x_{(i)}; \alpha, \theta) - F(x_{(i-1)}; \alpha, \theta)}\right) = 0 \quad (34)$$

where, $F'_\alpha(x, \alpha, \theta)$ and $F'_\theta(x, \alpha, \theta)$ are defined above.

Anderson-Darling and right-tail Anderson-Darling methods: Another type of minimum distance estimation method is the method of Anderson-Darling (AD). This method was introduced by Al-Zahrabi[18] and Anderson and Darling[19] and is based on an Anderson-Darling statistic. The Anderson-Darling statistic is given by:

$$A^2 = n\int_0^\infty \frac{\left(F_{(xi)} - F[F(x_{(i)})]\right)^2}{F(x_i)(1 - F(xi_i))} dF(x_i)$$

Boos[15] proved that the Anderson-Darling statistic has computational form which is given by:

$$A(\alpha, \theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_{(i)}; \alpha, \theta) + \log(1 - F(x_{(i)}; \alpha, \theta))] \quad (35)$$

Therefore, the ADs $\hat{\alpha}_{AD}$ and $\hat{\theta}_{AD}$ of α and θ can be determined by minimizing (Eq. 35) with respect to α and θ. These estimators can also be obtained by solving the non-linear equations:

$$\frac{\partial A(\alpha, \theta)}{\partial \alpha} = -\frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\left(\begin{array}{c}\dfrac{F''_\alpha(x_{(i)}; \alpha, \theta)}{F(x_{(i)}; \alpha, \theta)} - \\ \dfrac{F''_\alpha(x_{(n-i+1)}; \alpha, \theta)}{1 - F(x_{(n-i+1)}; \alpha, \theta)}\end{array}\right) = 0 \quad (36)$$

$$\frac{\partial A(\alpha, \theta)}{\partial \theta} = -\frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\left(\begin{array}{c}\dfrac{F''_\theta(x_{(i)}; \alpha, \theta)}{F(x_{(i)}; \alpha, \theta)} - \\ \dfrac{F''_\theta(x_{(n-i+1)}; \alpha, \theta)}{1 - F(x_{(n-i+1)}; \alpha, \theta)}\end{array}\right) = 0 \quad (37)$$

Luceno[20] provides some motivation about AD statistics and also introduces a modification, namely Right-tail Anderson-Darling statistics. The Right-tail AD statistics given by:

$$RA^2 = n\int_0^\infty \frac{\left(F(x_i) - E[F(x_{(i)})]\right)^2}{1 - F(x_i)} dF(x_i)$$

Also, the Right-tail AD has computational form which is given by:

$$RA = (\alpha, \theta) = \frac{n}{2} - \sum_{i=1}^{n} F(x_{(i)}; \alpha, \theta) - \frac{1}{n} - \sum_{i=1}^{n}(2i-1)\log(1 - F(x_{(i)}; \alpha, \theta)) \quad (38)$$

Hence, the RADs $\hat{\alpha}_{RAD}$ and $\hat{\theta}_{RAD}$ of α and θ are obtained by minimizing (Eq. 38) with respect to α and θ. These estimators can also be determined by solving the non-linear equations:

$$\frac{\partial RA(\alpha, \theta)}{\partial \alpha} = -n\sum_{i=1}^{n+1}\left(\begin{array}{c}\dfrac{F''_\alpha(x_{(i)}; \alpha, \theta)}{F(x_{(i)}; \alpha, \theta)} + \dfrac{1}{n}\sum_{i=1}^{n+1}(2-i) \\ \dfrac{F''_\alpha(x_{(n-i+1)}; \alpha, \theta)}{1 - F(x_{(n-i+1)}; \alpha, \theta)}\end{array}\right) = 0 \quad (39)$$

$$\frac{\partial RA(\alpha, \theta)}{\partial \theta} = -n\sum_{i=1}^{n+1}\left(\begin{array}{c}\dfrac{F''_\theta(x_{(i)}; \alpha, \theta)}{F(x_{(i)}; \alpha, \theta)} + \dfrac{1}{n}\sum_{i=1}^{n+1}(2-i) \\ \dfrac{F''_\theta(x_{(n-i+1)}; \alpha, \theta)}{1 - F(x_{(n-i+1)}; \alpha, \theta)}\end{array}\right) = 0 \quad (40)$$

Simulation: Here, a simulation study is performed to examine the performance of the different estimates presented above. The following procedure for evaluating the efficiency of the estimators is adopted as follow:

- Generate random sample with size n from the MIPGSD model
- The values obtained in step 1 are used to compute the $\hat{\Theta} = (\hat{\alpha}, \hat{\theta})$ considering the MLE, LSE, WLSE, CME, MPS, AD and RAD estimators
- Repeat the steps 1 and 2 N times
- Using $\hat{\Theta} = (\hat{\alpha}, \hat{\theta})$ and $\Theta = \alpha, \theta$ compute the Bias and the Mean Square Errors (MSE)

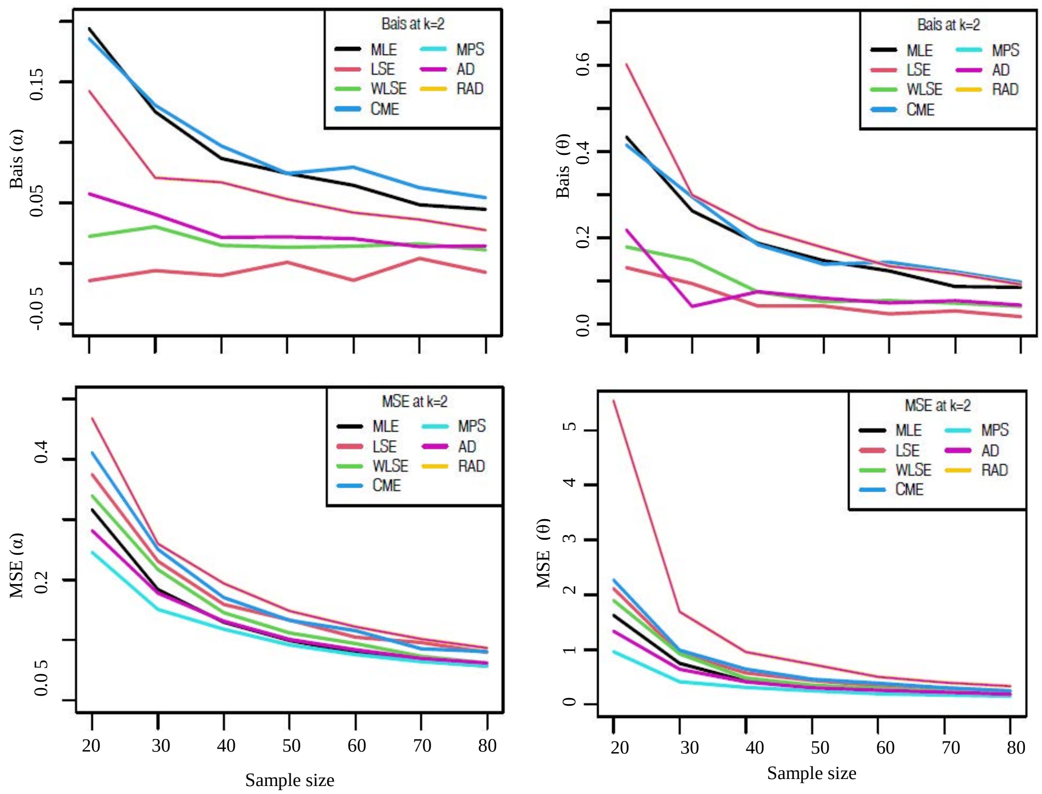

Fig. 3: Bias and MSEs, for the estimates of α = 2:5 and θ= 3:5 versus n when k = 2 for the estimation methods

The results are computed using the nlminb function (in the stat package) and Nelder-Mead method in R software. The chosen values to perform this procedure are Θ = (1:5, 0:8), N = 5,000 and n = (50, 80, 120, 200, 300). The simulation studies are put under the same conditions (initial values and random samples) for different estimation methods.

Figure 3-5 show how the seven biases, mean squared errors vary with respect to sample size for k = 2, 3 and 4. As expected, the Biases and MSEs of estimated parameters converge to zero as n increases.

Real data application: In this study, we use maximum likelihood estimate of the parameters to perform the goodness of fit of the MIPGSD model for two different data sets to know the potentiality of the new model as compared to some other existing models.

The first data set represent the relief times (in minutes) of 20 patients receiving an analgesic and reported by Clark and Gross[21]. The observed values are:

- 1.1 1.4 1.3 1.7 1.9
- 1.8 1.6 2.2 1.7 2.7
- 4.1 1.8 1.5 1.2 1.4
- 3.0 1.7 2.3 1.6 2.0

The second data set was originally reported by Nassar and Nada[22]. This data set represents the monthly actual taxes revenue in Egypt (in 1000 million Egyptian pounds) between January 2006 and November 2010. The observed values are:

- 5.9 20.4 14.9 16.2 17.2 7.8 6.1 9.2 10.2 9.6
- 13.3 8.5 21.6 18.5 5.1 6.7 17 8.6, 9.7 39.2
- 35.7 15.7 9.7 10 4.1 36 8.5 8 9.2 26.2
- 21.9 16.7 21.3 35.4 14.3 8.5 10.6 19.1 20.5 7.1
- 7.7 18.1 16.5 11.9 7 8.6 12.5 10.3 11.2 6.1
- 8.4 11 11.6 11.9 5.2 6.8 8.9 7.1 10.8

The two data sets are used to compare the MIPGSD model for values of k = 2 and 10 with four competitive models such as: Inverted Exponentiated Gamma (IEG) model[23]:

$$f(x) = \frac{\theta}{x^3}\left(1-\left(1+\frac{1}{x}\right)\exp\left(\frac{-1}{x}\right)\right)^{\theta-1}\exp\left(\frac{-1}{x}\right)$$

where, x, θ>0. Inverse Gompertz (IG) model[24]:

$$f(x) = \frac{\alpha}{x^2}\exp\left(\frac{-\alpha}{\beta}\left(\exp\left(\frac{\beta}{x}\right)-1\right)+\frac{\beta}{x}\right)$$

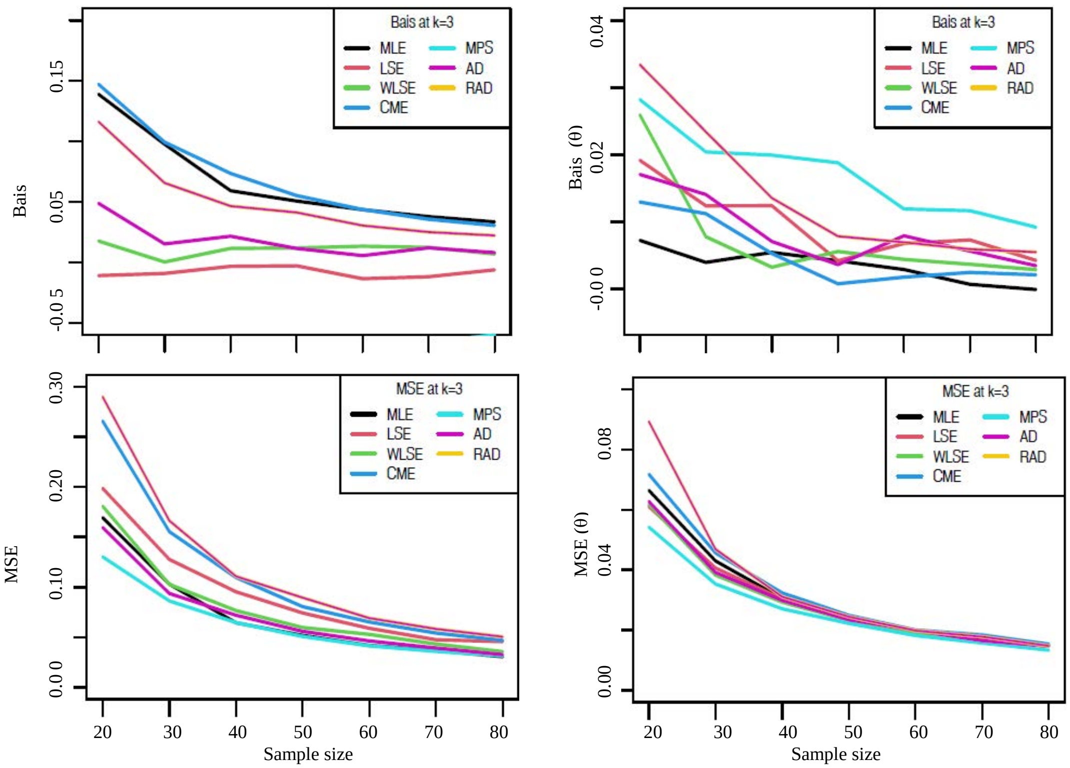

Fig. 4: Bias and MSEs, for the estimates of α = 2 and θ= 1:5 versus n when k = 3 for the estimation methods

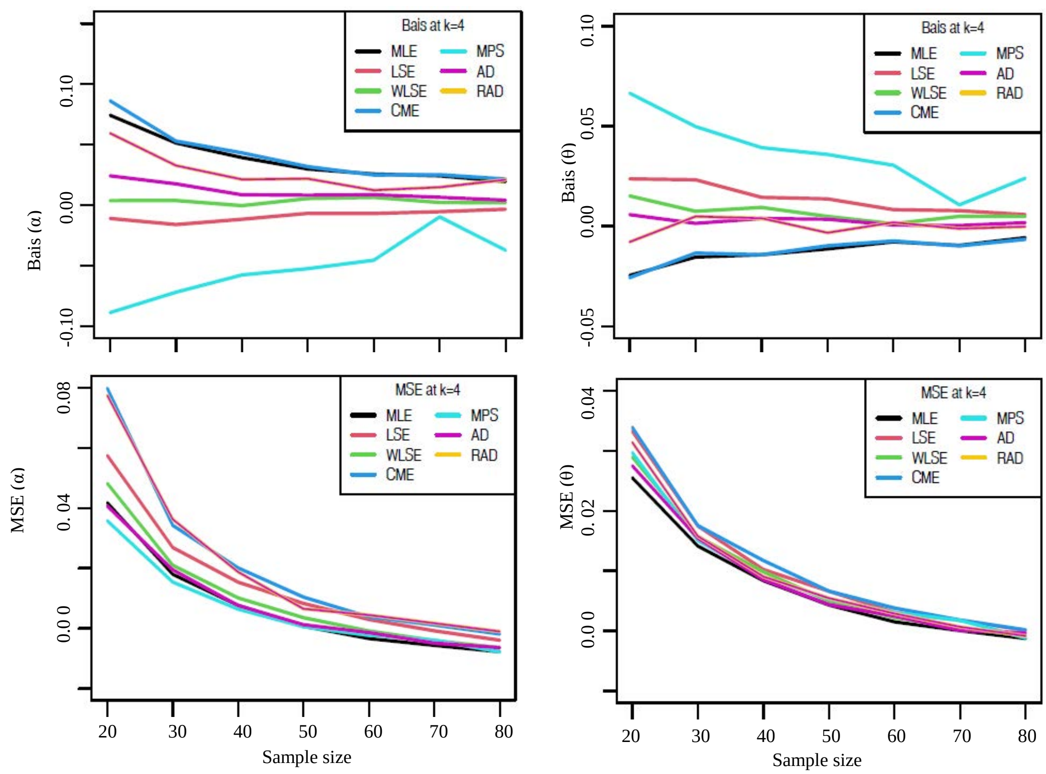

Fig. 5: Bias and MSEs, for the estimates of α= 1:5 and θ = 1 versus n when k = 4 for the estimation methods

Table 2: The goodness of fit measures for the first data set

	Measures						
Models	MLEs	-	L	AIC	BIC	HQIC	SS
$PM_{k=2}$ (α, θ)	3.9812	6.7190	15.4131	34.8263	36.8178	35.2150	0.0219
$PM_{k=10}$ (α, θ)	3.6825	12.2518	15.4776	34.95514	36.9466	35.34389	0.0241
IEG(θ)	-	0.4449	38.1938	80.3875	82.3790	80.7763	1.1791
IG(α, β)	0.11034	6.1454	16.3915	36.7830	38.7745	37.1718	0.0454
IXG(θ)	-	2.7245	33.6363	71.2727	73.2641	71.6614	1.0252
EIR((α, θ))	1.3176	2.0952	21.1825	46.3650	48.3564	46.7537	0.3791

Table 3: The goodness of fit measures for the second data set

	Measures						
Models	MLEs	-	L	AIC	BIC	HQIC	SS
$PM_{k=2}$	2.2466	145.6021	188.9398	381.8796	386.0347	383.5016	0.0315
$PM_{k=10}$ (α, θ)	2.2463	153.4062	188.9413	381.8826	386.0377	383.5046	0.0316
IEG(θ)	-	0.1762	267.5936	539.1872	543.3423	540.8092	3.9434
IG(α, β)	3.5305	14.1447	194.2590	392.5179	396.6730	394.1399	0.1334
IXG(θ)	-	11.7903	212.8971	429.7941	433.9492	431.4161	1.5017
EIR((α, θ))	7.8664	11.1360	189.5877	383.1754	387.3304	384.7973	0.0596

where x, α, β>0. Inverted Xgamma (IXG) model[25]:

$$f(x) = \frac{\theta^2}{(1+\theta)}.\frac{1}{x^2}\left(1+\frac{\theta}{2}.\frac{1}{x^2}\right)\exp\left(\frac{-\theta}{x}\right)$$

where, x, θ>0. Exponentiated Inverse Rayleigh (EIR) model[26]:

$$f(x) = \frac{2\alpha\theta}{x^3}\exp\left(\frac{-\alpha\theta}{x^2}\right)$$

where x, α, θ>0. For more simplification, let $MIPGSD_k$ (α, θ) = PM_k(α, θ) and to compare the models, we take the following goodness of fit measures into consideration: the log likelihood function (-L), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hanna-Quinn Information Criterion (HQIC) and the sum of squares (SS) defined by:

$$AIC = -2L + 2q$$

$$BIC = -2L + q\ln(n)$$

$$HQIC = -2L + 2q[\ln(n)]$$

$$SS = \sum_{i=1}^{n}\left(F\left(x_i;\hat{\Theta}\right) - \frac{i-0.375}{n+0.25}\right)^2$$

where q is the number of parameters, n is the sample size and $F(x_i;\hat{\Theta})$ estimated cumulative distribution function of theoretical models. The model with the lowest values of goodness of fit measures provides the best fit for dataset.

Tests statistics such as Cramer-von Mises W^2_n, Anderson-Darling A^2_n, Watson U^2_n, Liao-Shimokawa L_n and Kolmogrov-Smirnov K-S with its respective p-value are considered in order to verify which distribution fits better to each data set. These tests display the differences between the proposed cumulative distribution function and the empirical cumulative distribution function from the data to verify the fit of the distributions (p>0.05). For more details about above tests statistics see Al-Zahrani[18].

Table 2 and 3, provide the values of the goodness of fit measures for the fitted models to both data sets. The MIPGSD model provides the lowest values for all measures among all fitted models to both data sets. The tests shown in Table 4-5% that the proposed model, IG model and EIR model fit the two first data set (p>0.05) and the proposed model shows the lowest test statistics with the largest p-values. Thus, The MIPGSD model fits well two data sets and can be considered as a good competitor against the other models.

Furthermore, seven estimation methods are used to estimate the unknown parameters of IPGM distribution. Table 6-7 display the estimates of the MIPGSD parameters using these estimation methods with its rank and the values of SS, K-S and its p-value for the two data sets, respectively. Based on the values of SS, K-S and p-value in Table 6-7 , the CME estimation method is recommended to estimate the MIPGSD parameters for first data set whereas the LSE estimation methods are recommended to estimate the MIPGSD parameters for second data set.

Figure 6 and 7 show the Probability-Probability (P-P) plot of the fitted models for the first data set and the second data set respectively whereas Fig. 8 and 9 display the plots of fitted cdfs for the first data set and the second data set respectively. These plots provide that the MIPGSD model obtain a greater approximation between the empirical and the theoretical curves and reveal that the MIPGSD model provides a better fit than other models for

Table 4: The goodness of fit test statistics for the first data set

Models	W^2_n	A^2_n	U^2_n	L_n	K -S	p-value
	Statistics					
$PM_{k=2}$ (α, θ)	0.0269	0.1560	4.5412	0.6137	0.1031	0.9836
$PM_{k=10}$ (α, θ)	0.0294	0.1707	4.5417	0.6200	0.1110	0.9661
IEG(θ)	1.2132	5.7234	5.6100	2.1319	0.4791	0.0002
IG(α, β)	0.0527	0.3230	4.5590	0.7783	0.1412	0.8198
IXG(θ)	1.0562	5.0874	5.6879	2.0657	0.4038	0.0029
EIR(α, θ)	0.3976	2.0654	4.9926	1.4152	0.2566	0.1436

Table 5: The goodness of fit test statistics for the second data set

Models	W^2_n	A^2_n	U^2_n	L_n	K -S	p-values
	Statistics					
$PM_{k=2}$ (α, θ)	0.0333	0.2508	14.2837	0.5818	0.0647	0.9658
$PM_{k=10}$ (α, θ)	0.0334	0.2510	14.2837	0.5820	0.0648	0.9657
IEG(θ)	3.9773	18.6459	18.1075	3.5552	0.4958	5.062E-13
IG(α, β)	0.1405	1.1213	14.3999	0.9853	0.1180	0.3842
IXG(θ)	1.5210	7.9638	15.8701	2.4513	0.3004	4.758E-5
EIR (α, θ))	0.0635	0.4637	14.3339	0.7045	0.0822	0.8203

Table 6: The parameter estimates of the MIPGSD model, SS, K-S and p-value for first dataset at k = 2

Est. Meth.	$\hat{\alpha}$	$\hat{\theta}$	SS	K-S	p-value s	Rank
	Est. Par.					
MLE	3.9812	6.7190	0.0219	0.1031	0.9836	5
LSE	3.8771	6.4680	0.0228	0.1007	0.9873	4
WLSE	3.6478	5.8106	0.0280	0.1052	0.9797	6
CME	4.2343	7.6692	0.0211	0.0930	0.9952	1
MPS	3.4120	5.2305	0.0370	0.1074	0.9753	7
AD	3.9517	6.6765	0.0219	0.1006	0.9875	3
RAD	4.0641	7.0510	0.0210	0.0977	0.9910	2

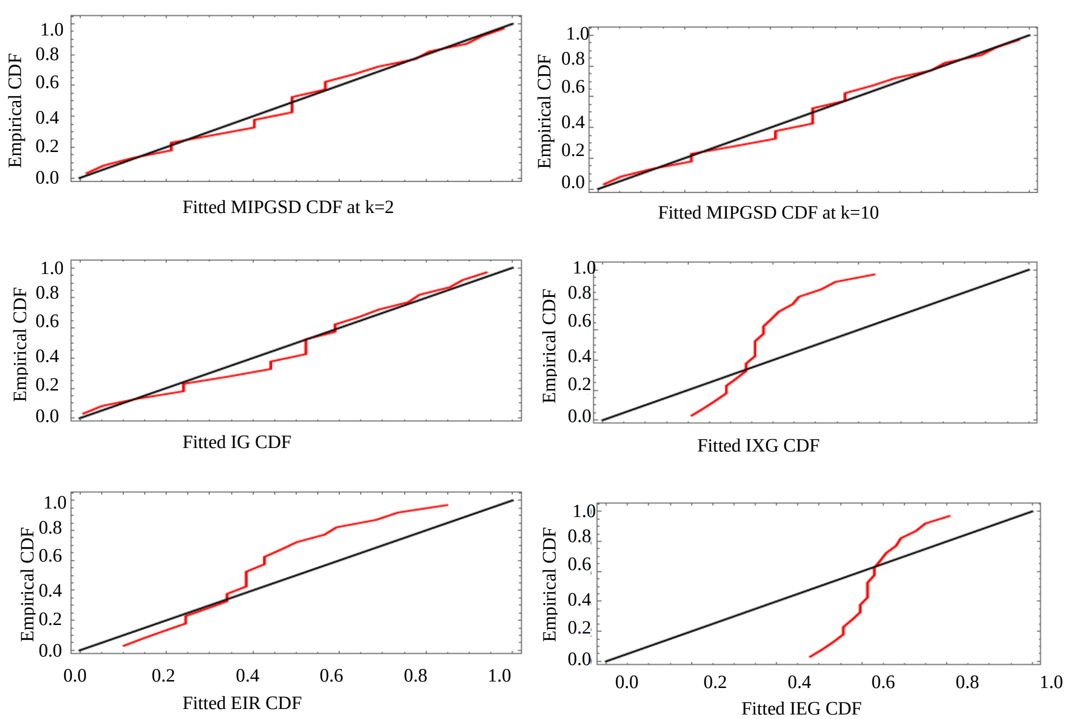

Fig. 6: P-P plots for the first data set

Table 7: The parameter estimates of the MIPGSD model, SS, K-S and p-value for second dataset at k = 2

Est. Meth.	Est. Par. $\hat{\alpha}$	$\hat{\theta}$	SS	K-S	p-value	Rank
MLE	2.2466	145.6021	0.0315	0.0647	0.9658	6
LSE	2.1891	130.1747	0.0301	0.0567	0.9914	1
WLSE	2.2274	141.6413	0.0295	0.0591	0.9861	2
CME	2.2453	148.4132	0.0295	0.0596	0.9848	3
MPS	2.1106	106.4929	0.0374	0.0694	0.9385	7
AD	2.2468	148.6367	0.0295	0.0602	0.9830	4
RAD	2.2811	161.7276	0.0304	0.0623	0.9759	5

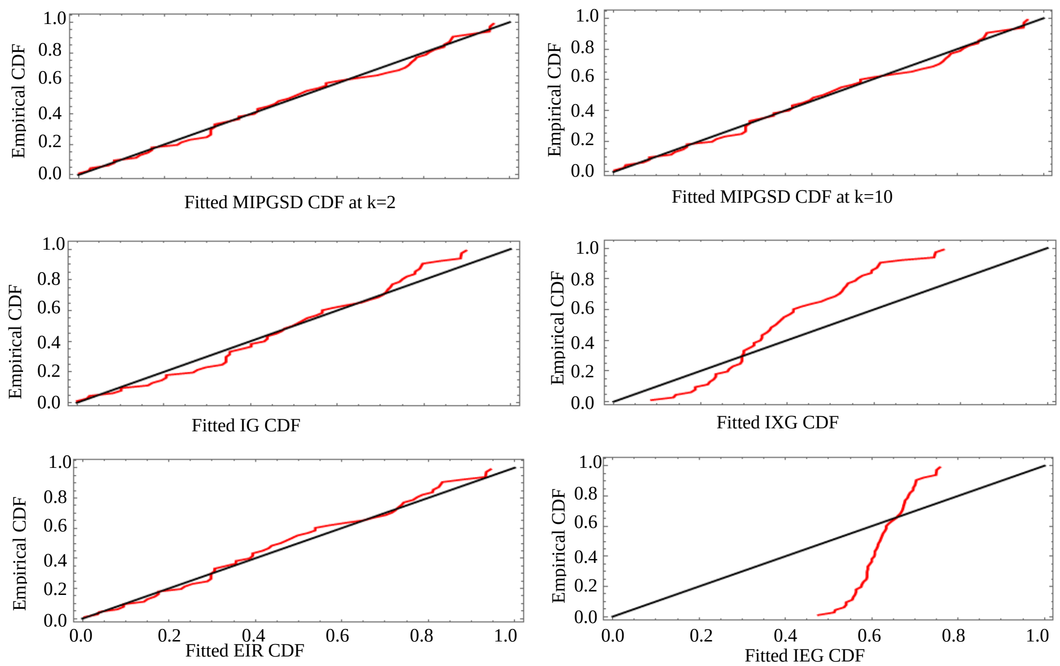

Fig. 7: P-P plots for the second data set

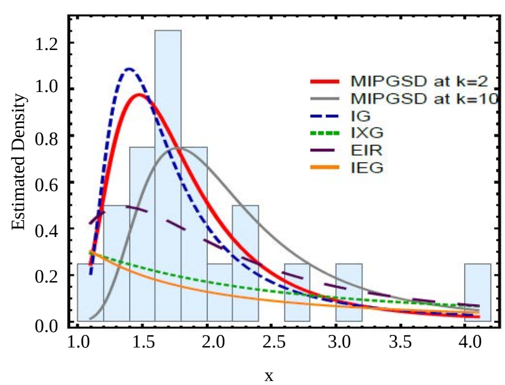

Fig. 8: Estimated pdfs for the first data set

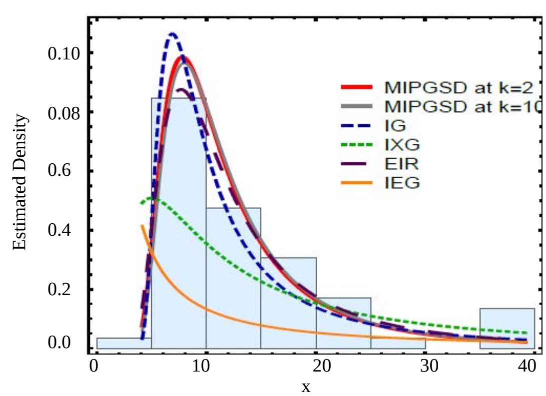

Fig. 9: Estimated pdfs for the second data set

both data sets. We can conclude that the proposed distribution was the one which best adjusted to the two data sets.

CONCLUSION

In this study, we proposed a mixture model of inverse power Gamma shape distributions and studied in detail.

Some statistical expression for its properties are obtained. The estimation of distribution parameters by using seven estimation methods are performed. We present a simulation study to illustrate the performance of the estimates. Two data sets also presented for the demonstration of enhanced flexibility and better fit of the observed model as compared to some other well-known existing models.

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