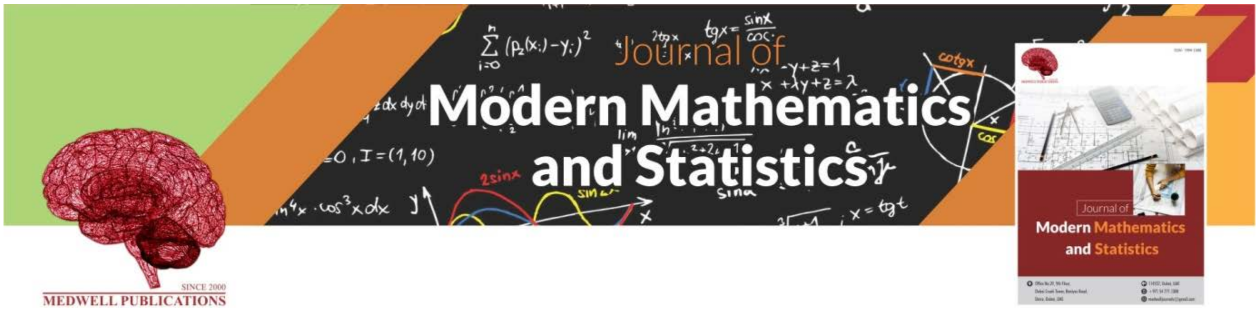

Lower Order Perturbations of Critical Fractional Laplacian Equations*

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Abstract: We give sufficient conditions for the existence of nontrivial solutions to a class of critical nonlocal problems of the Brezis-Nirenberg type. Our result extends some results in the literature for the local case to the nonlocal setting. It also complements the known results for the nonlocal case.

Key words: Fractional laplacian, critical problems, nontrivial solutions, conditions, existence, complements

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INTRODUCTION

Nonlinear elliptic equations involving critical Sobolev exponents have been extensively studied in the literature, beginning with the following celebrated result of Brezis and Nirenberg[1].

Theorem 1.1: Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$ and consider the problem:

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2} u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where, $\lambda > 0$ is a parameter and $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Let $\lambda_1 > 0$ be the first dirichlet eigenvalue of $-\Delta$ in Ω.

- If $n \geq 4$, then problem (1.1) has a solution for all $\lambda \in (0, \lambda_1)$
- If $n = 3$, then there exists $\lambda_* \in [0, \lambda_1]$ such that problem Eq. 1 has a solution for all $\lambda \in (\lambda_*, \lambda_1)$
- If $n = 3$ and $\Omega = B_1(0)$ is the unit ball, then $\lambda_* = \lambda_1/4$ and problem Eq. 1 has no solution for $\lambda \leq \lambda_1/4$

Following[1], Gazzola and Ruf[2] considered the more general problem:

$$\begin{cases} -\Delta u = g(x, u) + |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2)$$

where, g is a Caratheodory function on $\Omega \times \mathbb{R}$ with sub critical growth:

$$\lim_{|t| \to +\infty} e \frac{g(x, t)}{|t|^{2^*-1}} = 0$$

uniformly a.e., on Ω. Let $0 < \lambda_1 < \lambda_2 \leq, \ldots, \to +\infty$ be the sequence of Dirichet eigenvalues of $-\Delta$ in Ω, repeated according to multiplicity. The following extensions of Theorem 1.1 were obtained by Gazzola and Ruf[2].

Theorem 1.2: Assume the following conditions on g; for all $\epsilon > 0$, there exists $a_\epsilon \in L^{2n/(n+2)}(\Omega)$ such that $|g(x, t)| \leq a_\epsilon(x) + \epsilon |t|^{2^*-1}$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$. $G(x, t) := \int_0^t g$ (x,

$\tau)d\tau \geq 0$ for a.a$x\in\Omega$ and all $t\in\mathbb{R}$; there exist $k\in\mathbb{N}$, δ, $\sigma>0$ and $\mu\in(\lambda_k, \lambda_{k+1})$ such that $1/2(\lambda_k+\sigma)t^2 \leq G(x, t) \leq 1/2\ \mu\ t^2$ for a.a. $x\in\Omega$ and $|t|\leq\delta$; $G(x, t)\geq 1/2\ (\lambda_k+\sigma)t^2 - \frac{1}{2^*}t^{2^*}$ for a.a $x\in\Omega$ and all $t\in\mathbb{R}$; if n = 3, there exists a nonempty open subest Ω_0 of Ω such that:

$$\operatorname{lim e}_{t\to+\infty} \frac{G(x,t)}{t^4} = +\infty$$

uniformly a.e. on Ω_0. Then problem (2) has a nontrivial solution.

Theorem 1.3: Assume conditions (1), (2) and there exists $\delta>0$, $k\in\mathbb{N}$ and $\mu\in(\lambda_k, \lambda_{k+1})$ such that $1/2\ \lambda_k\ t^2 \leq 1/2\mu t^2$ for a.a. $x\in\Omega$ and $|t|\leq\delta$; there exists $\sigma\in(0, 1/2^*)$ such that $G(x, t)\geq 1/2\mu\ t^2-(1/2^*-\sigma)\ |t|^{2^*}$ for a.a. $x\in\Omega$ and all $t\in\mathbb{R}$; there exists a nonempty open subset Ω_0 of Ω such that:

$$\operatorname{lim e}_{t\to+\infty} \frac{G(x,t)}{t^{8n/(n^2-4)}} = +\infty$$

uniformly a.e. on Ω_0. Then, problem (1.2) has a nontrivial solution. Other extensions and generalizations can be found, e.g., by Capozzi *et al*.[3], Cerami *et al*.[4] and Tarantello[5]. More recently, Servadei and Valdinoci[6, 7] considered the nonlocal critical problem:

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{2_s^*-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega \end{cases} \quad (3)$$

where, $s\in(0, 1)$, Ω is a bounded domain in $\mathbb{R}^n$, $n>2s$ with Lipschitz boundary, $\lambda>0$ is a parameter and $2^*_s = 2n/(n\ 2s)$ is the fractional critical Sobolev exponent. Here $(\Delta)^s$ is the fractional Laplacian operator, defined, up to a normalization factor, on smooth functions by:

$$(-\Delta)^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \backslash B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \quad x\in\mathbb{R}^n$$

Let us recall the definition of a weak solution of problem Eq. 3. Let:

$$H^s(\mathbb{R}^n) = \left\{ u\in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dx\ dy < +\infty \right\}$$

be the usual fractional Sobolev space endowed with the Gagliardo norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \left(\|u\|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dx\ dy \right)^{1/2}$$

and let:

$$H^s_0(\Omega) = \left\{ u\in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n\backslash\Omega \right\}$$

Then, $H^s_0(\Omega)$ is a closed linear subspace of $H^s(\mathbb{R}^n)$, equivalently renormed by the Gagliardo seminorm:

$$[u]_s := \left(\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}} dx\,dy \right)^{1/2}$$

and the imbedding $H^s_0(\Omega) \hookrightarrow L^r(\Omega)$ is continuous for $r\in[1, 2^*_s]$ and compact for $r\in[1, 2^*_s]$ [8]. A weak solution of problem Eq. 3 is a function $u\in H^s_0(\Omega)$ satisfying:

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\upsilon(x)-\upsilon(y))}{|x-y|^{n+2s}} dx\ dy = \\ &\int_\Omega \left(\lambda u(x) + |u(x)|^{2_s^*-2} u(x)\right)\upsilon(x) dx \end{aligned} \quad (4)$$

Let $0<\lambda_1<\lambda_2\leq, \ ..., \ \to +\infty$ denote the sequence of eigenvalues of the nonlocal eigenvalue problem:

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n\backslash\Omega \end{cases}$$

repeated according to multiplicity (Proposition)[9]. Servadei and Valdinoci[6, 7] obtained the following results.

Theorem 1.4: If $n\geq 4s$, then problem (3) has a nontrivial weak solution for each $\lambda>0$ that is not an eigenvalue of (4).

Theorem 1.5: If $2s<n<4s$, then there exists $\lambda s>0$ such that problem Eq. 3 has a nontrivial weak solution for each $\lambda>\lambda s$ that is not an eigenvalue of Eq. 4. By Servadei and Valdinoci[10], they also considered the more general problem:

$$\begin{cases} (-\Delta)^s u = \lambda u + |u|^{2_s^*-2}u + f(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n\backslash\Omega \end{cases} \quad (5)$$

where, f is a Caratheodory function on $\Omega\times R$ and obtained the following result.

Theorem 1.6: Assume the following conditions:

- For all $M>0$, $\sup\{|f(x, t)|: x\in\Omega, |t|\leq M\} < +\infty$
- $\lim_{|t|\to+\infty} \frac{f(x, t)}{t} = 0$ uniformly a.e. on Ω
- $\lim_{|t|\to+\infty} \frac{f(x, t)}{|t|^{2_s^*-1}} = 0$ uniformly a.e. on Ω

If $n \geq 4s$, then problem Eq. 5 has a nontrivial weak solution for all $\lambda \in (0, \lambda_1)$. In the present paper we consider the problem:

$$\begin{cases} (-\Delta)^s u = g(x, u) + |u|^{2_s^*-2} u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \backslash \Omega \end{cases} \tag{6}$$

where $s \in (0, 1)$, Ω is a bounded domain in $\mathbb{R}^n$, $n>2s$ with Lipschitz boundary and g is a Caratheodory function on $\Omega \times \mathbb{R}$. Our main result is the following theorem.

Theorem 1.7: Assume the following conditions:

- H_1 there exist $p \in [1, 2_s^*)$ and $C>0$ such that $|g(x, t)| \leq C(|t|^{p-1}+1)$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$
- H_2 $G(x, t) \int_0^t g(x, \tau) d\tau \geq 0$ fora.a. $x \in \Omega$ and all $t \in \Omega$ and all $t \in \mathbb{R}$
- H_3 there exist $k \in \mathbb{N}$, $\delta, \sigma>0$ and $\mu \in (\lambda_k, \lambda_{k+1})$ such that $1/2\ (\lambda_k+\sigma)t^2 \leq G(x, t) \leq /2\mu\ t^2$ for a.a. $x \in \Omega$ and $|t| \leq \delta$
- H_4 $G(x, t) \geq 1/2\ (\lambda_k+\sigma)\ t^2 - \frac{1}{2_s^*}|t|^{2_s^*}$ for a.a. $x \in \Omega$ and all $t \in \mathbb{R}$
- H_5 there exists a nonempty open subset Ω_0 of Ω such that $\lim_{|t| \to +\infty} \frac{G(x, t)}{t^{(n+2s)/(n-2s)}} = +\infty$ uniformly a.e. on Ω_0

Then problem Eq. 6 has a nontrivial weak solution. Theorem 1.7 extends the results of Gazzola and Ruf[2] to the nonlocal case and complements the results of Servadei and Valdinoci[6, 7, 10]. This theorem will be proved after some preliminaries in the next section.

PRELIMINARIES

A function $u \in H_0^s(\Omega)$ is a weak solution of problem Eq. 6 if:

$$\int_{\mathbb{R}^{2n}} \frac{\left(u(x)-u(y)(u(x)-u)(y)\right)}{|x-y|^{n+2s}} dx\ dy = \int_\Omega \left(g(x, u) + |u(x)|^{2_s^*-2} u(x)\right) \upsilon(x) dx$$

for all $u \in H_0^s(\Omega)$. Weak solutions coincide with critical points of the C^1-functional:

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{\left(u(x)-u(y)\right)^2}{|x-y|^{n+2s}} dx\ dy - \int_\Omega \left(G(x, u) + \frac{1}{2_s^*}|u|^{2_s^*}\right) dx,\ \ u \in H_0^s(\Omega)$$

Recall that E satisfies the Palais-Smale compactness condition at the level $c \in \mathbb{R}$ or the $(PS)_c$ condition for short, if every sequence $(u_j) \subset H_0^s(\Omega)$ such that $E(u_j) \to c$ and $E'(u_j) \to 0$, called a $(PS)_c$ sequence has a convergent subsequence. Let:

$$S = \inf_{u \in H_0^s(\Omega) \backslash \{0\}} \frac{\int_{\mathbb{R}^{2n}} \frac{\left(u(x)-u(y)\right)^2}{|x-y|^{n+2s}} dx\ dy}{\left(\int_\Omega |u|^{2_s^*} dx\right)^{2/2_s^*}} \tag{7}$$

be the best constant for the fractional Sobolev imbedding $H_0^s(\Omega) \to L^{2_s^*}(\Omega)$. Proof of theorem 1.7 will be based on the following proposition.

Proposition 2.1: If $0 < c < s/n\ S^{n/2s}$, then every $(PS)_c$ sequence has a subsequence that converges weakly to a nontrivial critical point of E.

Proof: Let (u_j) be a $(PS)_c$ sequence. Then:

$$E(u_j) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \frac{\left(u_j(x)-u_j(y)\right)^2}{|x-y|^{n+2s}} dx\ dy - \int_\Omega \left(G(x, u_j) + \frac{1}{2_s^*}|u_j|^{2_s^*}\right) dx = c + o(1) \tag{8}$$

and

$$E(u_j)u_j = \int_{\mathbb{R}^{2n}} \frac{\left(u_j(x)-u_j(y)\right)^2}{|x-y|^{n+2s}} dx\ dy - \int_\Omega \left(u_j g(x, u_j) + |u_j|^{2_s^*}\right) dx = o(1)\|u_j\| \tag{9}$$

Dividing Eq. 9 by 2 and subtracting from Eq. 8 gives:

$$\int_\Omega \left[\frac{1}{2} u_j g(x, u_j) - G(x, u_j) + \frac{s}{n}|u_j|^{2_s^*}\right] dx = o(1)\|u_j\| + O(1)$$

which together with (H_1) and the Holder and Young's inequalities gives:

$$\int_\Omega |u_j|^{2_s^*} dx \leq o(1)\|u_j\| + O(1)$$

This together with (H_1) and Eq. 8 implies that (u_j) is bounded in $H_0^s(\Omega)$. So, a renamed subsequence converges to some u weakly in $H_0^s(\Omega)$ strongly in $L^q(\Omega)$ for all $q \in [1, 2_s^*]$ and a.e. in Ω. Then, u is a critical point of E by the weak continuity of E'. Suppose u = 0. Since, (u_j) is bounded in $H_0^s(\Omega)$ and converges to 0 in $L^p(\Omega)$, Eq. 9, (H_1), and Eq. 7 give:

$$O(1) = \int_{\mathbb{R}^{2n}} \frac{\left(u_j(x)-u_j(y)\right)^2}{|x-y|^{n+2s}} dx\ dy - \int_\Omega |u_j|^{2_s^*} dx \geq \|u_j\|^2 \left(1 - \frac{\|u_j\|^{2_s^*-2}}{S^{2_s^*/2}}\right)$$

If $\|u_j\| \to 0$, then $E(u_j) \to 0$, contradicting $c > 0$, so, this implies:

$$\|u_j\|^2 \geq S^{n/2s} + o(1)$$

for a renamed subsequence. Dividing Eq. 9 by 2*s and subtracting from Eq. 8 then gives:

$$c = \frac{s}{n}\int_{\mathbb{R}^{2n}} \frac{(u_j(x)-u_j(y))^2}{|x-y|^{n+2s}}\,dx\,dy + o(1) \geq \frac{s}{n}S^{n/2s} + o(1)$$

contradicting $c < \frac{s}{n}S^{n/2s}$. To produce $(PS)_c$ sequences with $0<x<s/n\ S^{n/2s}$, we will use the following linking theorem of Rabinowitz[11, 12].

Theorem 2.2: Let E be a C^1 functional on a Banach space V and let $V = V^- \oplus V^+$ be a direct sum decomposition with $\dim V^- < \infty$. Assume that there exist $R>\rho>0$ and $w_0 \in V^+$ with $\|w_0\| = 1$ such that:

$$\max_{u\in\partial Q} E(u) < \inf_{u\in\partial B_\rho \cap V^+} E(u)$$

where:

$$Q = \{u + two: \upsilon \in V^-, \|\upsilon\| \leq R, t \in [0, R]\}$$

Let $\Gamma = \{h \in C(Q, V): h|_{\partial Q} = id\}$ and set:

$$c := \inf_{h\in\Gamma} \max_{u\in h(Q)} E(u)$$

Then:

$$\inf_{u\in\partial B_\rho\cap V^+} E(u) \leq c \leq \max_{u\in Q} E(u)$$

and E has a $(PS)_c$ sequence.

Proof of Theorem 1.7: In this section we prove Theorem 1.7. Let $e_1, ..., e_k$ be L^2-orthonormal eigenfunc-tions for $\lambda_1, ..., \lambda_k$, let $H^- = \text{span}\{e_1, ..., e_k\}$ and let $H^+ = (H^-)^\perp$. Without loss of generality we may assume that $0 \in \Omega_0$. For $m \in \mathbb{N}$, so, large that $B4/m := \{x\in\mathbb{R}^n: |x|<4/m\} \subset \Omega_0$, let:

$$\zeta_m(x) = \begin{cases} 0, & x \in B_{1/m} \\ m|x|-1, & x \in A_m = B_{2/m}\backslash B_{1/m} \\ 1, & x \in \Omega \setminus B_{2/m} \end{cases}$$

It is easily seen that:

$$|\zeta_m(x) - \zeta_m(y)| \leq m|x-y| \quad \forall x, y \in \Omega \tag{10}$$

Let $e_j^m = \zeta_m e_j$, $j = 1, ..., k$ and let $H_m^- = \text{span}\{e_1^m, ..., e_k^m\}$

Lemma 3.1: Let $f \in L^\infty(\Omega)$ and let $u \in H_0^s(\Omega)$ be a weak solution of $(-\Delta)^s u = f$ in Ω. Then:

$$\|\zeta_m u\|^2 \leq \|u\|^2 + \frac{C|f|_\infty^2}{m^{n-2s}}$$

where, $C = C(n, \Omega, s)>0$. To prove this lemma we will need the following estimates from[13].

Lemma 3.2; ([6], Lemma 2.3): Let $f \in L^q(\Omega)$, $1<q\leq\infty$ and let $u \in H_0^s(\Omega)$ be a weak solution of $(-\Delta)^s u = f$ in Ω. Then $|u|_r \leq C|f|_q$ where:

$$r = \begin{cases} nq/(n-2sq), & 1<q<n/2s \\ \infty, & n/2s<q\leq\infty \end{cases}$$

and $C = C(n, \Omega, s, q)>0$. In particular, if $f \in L^\infty(\Omega)$, then $|u|_\infty = C|f|_\infty$.

Lemma 3.3 (Lemma 2.5)[13]: Let $f \in L^q(\Omega)$, $n/2s<q\leq\infty$ and let $u \in H_0^s(\Omega)$ be a weak solution of $(-\Delta)^s u = f$ in Ω. Then:

$$\|\varphi u\|^2 \leq C|f|_q^2\left(|\varphi|_{2q'}^2 + \|\varphi\|^2\right) \quad \forall \varphi \in L^{2q'}(\Omega) \cap H_0^s(\Omega)$$

where, $C = C(n, \Omega, s, q)>0$ and $q' = q/(q-1)$.

Proof of Lemma 3.1: We have:

$$\begin{aligned} \|\zeta_m u\|^2 \leq &\int_{A1} \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dx\,dy + \\ &\int_{A2} \frac{|\zeta_m(x)u(x) - \zeta_m(y)u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \\ &2\int_{A3} \frac{(\zeta_m(x)u(x)-u(y))^2}{|x-y|^{n+2s}}\,dx\,dy =: I_1 + I_2 + I_3 \end{aligned}$$

where, $A_1 = B_{2/m}^c \times B_{2/m}^c$, $A_2 = B_{3/m} \times B_{3/m}$ and $A_3 = B_{2/m} \times B_{3/m}^c$ we have $I_1 \leq \|u\|^2$. To estimate I_2, let:

$$\varphi_m(x) = \begin{cases} \zeta_m(x), & x \in B_{3/m} \\ 4-m|x|, & x \in B_{4/m}\backslash B_{3/m} \\ 0, & x \in B_{4/m}^c \end{cases}$$

Applying Lemma 3.3 to φ_m with $q = \infty$:

$$I_2 \leq \|\varphi_m u\|^2 \leq C|f|_\infty^2\left(|\varphi_m|_2^2 + \|\varphi_m\|^2\right)$$

where, $C = C(n, \Omega, s)>0$. Since, $\varphi_m(x) = \varphi_1(mx)$:

$$|\varphi_m|_2^2 = \int_{\mathbb{R}^n} \varphi_m(x)^2\,dx = \int_{\mathbb{R}^n} \varphi_1(mx)^2\,dx = \frac{|\varphi_1|_2^2}{m^2}$$

and:

$$\|\varphi_m\|^2 = \int_{\mathbb{R}^{2n}} \frac{|\varphi_m(x)-\varphi_m(y)|^2}{|x-y|^{n+2s}} dx\,dy = \int_{\mathbb{R}^{2n}} \frac{|\varphi_1(mx)-\varphi_1(my)|^2}{|x-y|^{n+2s}} dx\,dy = \frac{\|\varphi_1\|^2}{m^{n-2s}}$$

So:

$$I_2 \le \frac{C|f|_\infty^2}{m^{n-2s}}$$

For $(x, y)\in A_3$, $|x-y|\ge|y|-|x|>|y|-2/m\ge|y|-(2/3)|y| = |y|/3$, so:

$$I_3 \le C|u|_\infty^2 \int_{A_3} \frac{1}{|y|^{n+2s}} dx\,dy \le \frac{C|f|_\infty^2}{m^{n-2s}}$$

by Lemma 3.2. The desired conclusion follows.

Lemma 3.4: We have $e_j^m \to e_j$ in $H_0^s(\Omega)$ as $m\to\infty$ and:

$$\max_{\left\{u\in H_m^-:\int_\Omega u^2 dx=1\right\}} \|u\|^2 \le \lambda_k + \frac{C}{m^{n-2s}} \tag{11}$$

for some constant C>0.

Proof: We have:

$$\|e_j^m-e_j\|^2 = \int_{\mathbb{R}^{2n}} \frac{\left[\begin{array}{l}(\zeta_m(x)e_j(x)-e_j(x))-\\(\zeta_m(y)e_j(y)-e_j(y))\end{array}\right]^2}{|x-y|^{n+2s}} dx\,dy = \int_{\mathbb{R}^{2n}} \frac{\left|\begin{array}{l}e_j(x)[\zeta_m(x)-\zeta_m(y)]+\\ {[\zeta_m(y)-1][e_j(x)e_j(y)]}\end{array}\right|^2}{|x-y|^{n+2s}} dx\,dy \le$$
$$2\int_{\mathbb{R}^{2n}} \frac{e_j(x)^2[\zeta_m(x)-\zeta_m(y)]^2}{|x-y|^{n+2s}} dx\,dy + \int_{\mathbb{R}^{2n}} \frac{[\zeta_m(y)-1]^2[e_j(x)-e_j(y)]^2}{|x-y|^{n+2s}} dx\,dy \le 2\left(|e_j|_\infty^2 I_1+I_2\right) \tag{12}$$

Where:

$$I_1 = \int_{\mathbb{R}^{2n}} \frac{[\zeta_m(x)-\zeta_m(y)]^2}{|x-y|^{n+2s}} dx\,dy,$$
$$I_2\int_{\mathbb{R}^{2n}} \frac{[\zeta_m(y)-1]^2[e_j(x)-e_j(y)]^2}{|x-y|^{n+2s}} dx\,dy$$

We will show that I_1 and I_2 go to 0 as $m\to\infty$. Since, $\zeta_m = 1$ in $B_{2/m}^c$:

$$I_1 = \int_{B_{2/m}\times B_{2/m}} \frac{[\zeta_m(x)-\zeta_m(y)]^2}{|x-y|^{n+2s}} dx\,dy+2I_1 = \int_{B_{2/m}\times B_{2/m}^c} \frac{[1-\zeta_m(x)]^2}{|x-y|^{n+2s}} dx\,dy =: I_3+2I_4$$

Write:

$$\int_{B_{2/m}\times B_{2/m}^c} \frac{[1-\zeta_m(x)]^2}{|x-y|^{n+2s}} dx\,dy + \int_{B_{2/m}\times(B3_{/m}\setminus B_{2/m})} \frac{[1-\zeta_m(x)]^2}{|x-y|^{n+2s}} dx\,dy =: I_5+I_6$$

Clearly, I_3 and I_6 are less than or equal to:

$$\int_{B_{2/m}\times B3_{/m}} \frac{[\zeta_m(x)-\zeta_m(y)]^2}{|x-y|^{n+2s}} dx\,dy =: I_7$$

so, $I_1 = 2I_5+3I_7$. To estimate I_5 and I_7, we change variables from (x, y) to (x, ζ) where, $\zeta = x-y$. For $(x, y) \in B_{2/m}\times B_{3/m}^c$, $|\xi|\ge|y|-|x|>1/m$ and hence:

$$I_5 \le \int_{B_{2/m}\times B_{2/m}^c} \frac{dx\,dy}{|x-y|^{n+2s}} \le \int_{B_{2/m}\times B_{1/m}^c} \frac{dx\,dy}{|\xi|^{n+2s}} \le \frac{C}{m^{n-2s}} \tag{13}$$

For $(x, y)\in B_{2/m}\times B_{3/m}$, $|\xi|\le|x|+|y|<5/m$ and hence (11) gives:

$$I_7 \le m^2\int_{B_{2/m}\times B_{3/m}} \frac{dx\,dy}{|x-y|^{n-2(1-s)}} \le m^2\int_{B_{2/m}\times B_{5/m}} \frac{dx\,dy}{|\xi|^{n-2(1-s)}} \le \frac{C}{m^{n-2s}}$$

Thus, $I_1\le C/m^{n-2s}$. Now we estimate I_2.We have:

$$I_2 = \int_{\mathbb{R}^n\times B_{2/m}} \frac{[1-\zeta_m(y)]^2[e_j(x)-e_j(y)]^2}{|x-y|^{n+2s}} dx\,dy \le I_8+4|e_j|_\infty^2 I_9$$

Where:

$$I_8\int_{B_{2/m}\times B_{2/m}} \frac{[e_j(x)-e_j(y)]^2}{|x-y|^{n+2s}} dx\,dy, \quad = I_9\int_{B_{3/m}^c\times B_{2/m}} \frac{dx\,dy}{|x-y|^{n+2s}}$$

Since, $e_j \in H_0^s(\Omega)$ and $|B_{3/m}\times B_{2/m}|\to 0$, $I_8\to 0$. As in Eq. 13, $I_9\le C/m^{n-2s}$. Thus, $I_2\le C/m^{n-2s}+o(1)$. To prove Eq. 11, let $\upsilon = \sum_{j=1}^k \alpha_j e_j \in H^-$. By Lemma 3.1:

$$\|\zeta_m\upsilon\|^2 \le \|\upsilon\|^2 + \frac{C|f|_\infty^2}{m^{n-2s}} \tag{14}$$

Where:

$$f = (-\Delta)^s \upsilon = \sum_{j=1}^k \lambda_j\alpha_j e_j \in H^-$$

Since, $\dim H^-<\infty$:

$$|f|_\infty^2 \le c_1|f|_2^2 = c_1\sum_{j=1}^{k}\lambda_j^2\alpha_j^2 \le c_1\lambda_k^2\sum_{j=1}^{k}\alpha_j^2 = c_2|\upsilon|_2^2$$

for some constants c_1, $c_2>0$. Since, $\|\upsilon\|^2\le\lambda_k|\upsilon|^2_2$, this together with Eq. 14 gives:

$$\|\zeta_m\upsilon\|_2^2 \le \left(\lambda_k \frac{C}{m^{n-2s}}\right)|\upsilon|_2^2 \tag{15}$$

On the other hand:

$$\|\zeta_m\upsilon\|_2^2 = \int_{\Omega\setminus B_2/m}\upsilon^2\,dx + \int_{B_2/m}(\zeta_m\upsilon)^2\,dx \ge \int_\Omega \upsilon^2 dx - \int_{B_2/m}\upsilon^2 dx$$

and:

$$\int_{B_2/m}\upsilon^2\,dx \ge c_3\frac{|\upsilon|_\infty^2}{m^n} \le c_4\frac{|\upsilon|_2^2}{m^n}$$

for some constants c_3, $c_4>0$, so:

$$\|\zeta_m\upsilon\|_2^2 \ge \left(1-\frac{c_4}{m^n}\right)|\upsilon|_2^2 \tag{16}$$

Combining Eq. 15 and 16 gives:

$$\|\zeta_m\upsilon\|^2 \le \left(\lambda_k + \frac{C}{m^{n-2s}}\right)|\zeta_m\upsilon|_2^2$$

Since, Eq. 11 follows from this.

Lemma 3.5: For all sciently large m, $H_0^s(\Omega) = H_m^- \oplus H^+$.

Proof: Let $P: H_0^s(\Omega)\to H^-$ be the orthogonal projection. First we show that $PH_m^- = H^-$ for all sufficiently large m. Since, $PH_m^-\subset H^-$ and dim H^- = k, it suffices to show that $Pe_1^m, ..., Pe_k^m$ are linearly independent. Suppose not. Then there exists $\alpha^m = (\alpha_1^m, ..., \alpha_k^m)\in S^{n-1}$ such that:

$$\sum_{j=1}^{k}\alpha_j^2 Pe_j^m = 0 \tag{17}$$

where, S^{n-1} is the unit sphere in $\mathbb{R}^n$. Passing to a subsequence, we may assume that $\alpha^m\to\alpha = (\alpha_1, ..., \alpha_n)\in S^{n-1}$. Since, $Pe_j^m \to Pe_j = e_j$ by Lemma 3.4, then passing to the limit is Eq. 17 gives:

$$\sum_{j=1}^{k}\alpha_j e_j = 0$$

Since, e_1, ..., e_k are linearly independent, then $\alpha_1 = \cdots = \alpha_k = 0$, contradicting $\alpha\in S^{n-1}$.Given $u\in H_0^s(\Omega)$, write $u = \upsilon+w$ with $\upsilon\in H^-$, $w\in H^+$. Since, PH_m^- = H^-, there exists $z\in H_m^-$ such that $Pz = \upsilon$. Then $u = z+(\upsilon-z+w)$ and $\upsilon-z+w\in H^+$ since, $P(\upsilon-z+w) = 0$. Finally, suppose $u\in H_m^-\cap H^+$. Since, $u\in H_m^-$:

$$u = \sum_{j=1}^{k}\alpha_j e_j^m$$

for some α_1, ..., $a_k\in\mathbb{R}$. Since, $u\in H^+$:

$$P_u = \sum_{j=1}^{k}\alpha_j Pe_j^m = 0$$

Since, $Pe_1^m, ..., Pe_k^m$ are linearly independent for sufficiently large m, then $\alpha_1 = \cdots = \alpha_k = 0$ and hence, u = 0. As by Rabinowitz[11], set:

$$U_\varepsilon(x) = \frac{c(n, s)\varepsilon^{(n-2s)/2}}{\left(\varepsilon^2+|x|^2\right)^{(n-2s)/2}},\ \varepsilon>0$$

where, c(n, s)>0 is such that:

$$\|U_\varepsilon\|^2 = |U_\varepsilon|_{2_s^*}^{2_s^*} = S^{n/2s}$$

Then take a smooth function $\eta_m:\mathbb{R}^n\to[0, 1]$ such that $\eta_m = 1$ in $B_1/_{4m}$ and $\eta = 0$ outside $B_1/_{2m}$ and set $u_\varepsilon^m = \eta_m U_\varepsilon$. The following estimates were obtained Rabinowitz[11]:

$$\|u_\varepsilon^m\|^2 = S^{n/2s}+O\left(\varepsilon^{n-2s}\right),\quad |u_\varepsilon^m|_{2_s^*}^{2_s^*} = S^{n/2s}+O\left(\varepsilon^n\right) \tag{18}$$

as $\varepsilon\to0$. We prove Theorem 1.7 by applying Theorem 2.2 using the direct sum decomposition $H_0^s(\Omega) = H_m^-\oplus H^+$ and taking $w_0 = u_\varepsilon^m$. We will show that:

$$\max_{u\in\partial Q_\varepsilon^m} E(u) \le 0 < \inf_{u\in\partial B_\rho\cap H^+} E(u)$$

if $\rho,\varepsilon>0$ are sufficiently small and m, $R>\rho$ are sufficiently large where:

$$Q_\varepsilon^m = \left\{\upsilon+tu_\varepsilon^m : \upsilon\in H_m^-, \|\upsilon\|\le R,\ t\in[0, R]\right\}$$

Let $\Gamma = \left\{h\in C\left(Q_\varepsilon^m, H_0^s(\Omega)\right): h|_{\partial Q_\varepsilon^m} = id\right\}$ and set:

$$c := \inf_{h\in\Gamma}\max_{u\in h\left(Q_\varepsilon^m\right)} E(u)$$

Then Theorem 2.2 gives a $(PS)_c$ sequence with:

$$\inf_{u\in\partial B_\rho\cap H^+} E(u) \le c \le \max_{u\in Q_\varepsilon^m} E(u)$$

We will show that:

$$\max_{u\in Q_\varepsilon^m} E(u) < \frac{S}{n} S^{n/2s} \quad (19)$$

if ε is sufficiently small and apply Proposition 2.1 to obtain a nontrivial critical point of E.

Lemma 3.6: If ρ>0 is sufficiently small, then:

$$\inf_{u\in\partial B_\rho \cap H^+} E(u) > 0$$

Proof: By (H_1) and (H_3), $G(x, t) \le 1/2\mu t^2 + c_5|t|^p$ for a.a.$x\in\Omega$ and all $t\in\mathbb{R}$ for some constant $c_5>0$.

For $u\in H^+$, this together with the fact that $\frac{\|u\|^2}{|u|_2^2} \ge \lambda_{k+1}$ and the fractional Sobolev embedding theorem gives:

$$E(u) \ge \frac{1}{2}\|u\|^2 - \int_\Omega \left(\frac{1}{2}\mu u^2 + c_5|u|^p + \frac{1}{2_s^*}|u|^{2_s^*}\right)dx \ge$$
$$\frac{1}{2}\left(1-\frac{\mu}{\lambda_{k+1}}\right)\|u\|^2 - c_6\left(\|u\|^p + \|u\|^{2_s^*}\right)$$

for some constant $c_6>0$. Since, $\mu<\lambda_{k+1}$ and $2<p<2^*_s$, the desired conclusion follows from this for sufficiently small ρ.

Lemma 3.7: If m and R>ρ are sufficiently large and ε>0 is sufficiently small, then:

$$\max_{u\in\partial Q_\varepsilon^m} E(u) \le 0 \quad (20)$$

Proof: For $\upsilon \in H_m^-$ with $\|\upsilon\|\le R$ and $t\in[0, R]$:

$$E(\upsilon + tu_\varepsilon^m) = E(\upsilon) + E(tu_\varepsilon^m) - 4t\int_{B_{1/m}^c \times B_{1/2m}} \frac{\upsilon(x)u_\varepsilon^m(y)}{|x-y|^{n+2s}}dx\,dy \quad (21)$$

since, $\upsilon = 0$ in $B_{1/m}$ and $u_\varepsilon^m = 0$ outside $B_{1/2m}$. By Lemma 3.4 and (H_4):

$$E(\upsilon) \le \frac{1}{2}\left(\lambda_k + \frac{C}{m^{n-2s}}\right)\int_\Omega \upsilon^2 dx - \frac{1}{2}(\lambda_k+\sigma)\int_\Omega \upsilon^2\,dx =$$
$$-\frac{1}{2}\left(\sigma - \frac{C}{m^{n-2s}}\right)\int_\Omega \upsilon^2\,dx \le -\frac{\sigma}{4}\int_\Omega \upsilon^2 dx$$

for sufficiently large m. Since, H_m^- is finite dimensional, it follows from this that:

$$E(\upsilon) \le -c_7\|\upsilon\|^2 \quad (22)$$

for some constant $c_7>0$ in particular, $E(\upsilon)\le 0$. By (H_2) and Eq. 18:

$$E(tu_\varepsilon^m) \le \frac{t^2}{2}\|u_\varepsilon^m\|^2 - \frac{t^{2_s^*}}{2_s^*}|u_\varepsilon^m|_{2_s^*}^{2_s^*} \ge \left(\frac{t^2}{2} - \frac{t^{2_s^*}}{2_s^*}\right)S^{n/2s} + c_8 R^{2_s^*}\varepsilon^{n-2s} \quad (23)$$

for some constant $c_8>0$. The last integral in Eq. 21 is bounded by:

$$c(n, s)|\upsilon|_\infty\, \varepsilon^{(n-2s)/2}\int_{B_{1/m}^c \times B_{1/2m}} \frac{dx\,dy}{|x-y|^{n+2s}\left(\varepsilon^2 + |y|^2\right)^{(n-2s)/2}}$$

Changing variables from (x, y)-(ζ, y) where ζ = x-y, $|\zeta| \ge |x|-|y| > 1/2m$ and hence, the integral on the right is bounded by:

$$\int_{B_{1/2m}^c \times B_{1/2m}} \frac{d\zeta\,dy}{|\zeta|^{n+2s}|y|^{n-2s}}$$

and the scaling $(\zeta, y)\mapsto(m\zeta, my)$ shows that this integral is independent of m. Since, $|\upsilon|\le R$, it now follows that:

$$\left|\int_{B_{1/2m}^c \times B_{1/2m}} \frac{\upsilon(x)u_\varepsilon^m(y)}{|x-y|^{n+2s}}dx\,dy\right| \le c_9 R\,\varepsilon^{(n-2s)/2} \quad (24)$$

for some constant $c_9>0$. Combining Eq. 21-24 gives:

$$E(\upsilon + tu_\varepsilon^m) \le -c_7\|\upsilon\|^2 + \left(\frac{t^2}{2} - \frac{t^{2_s^*}}{2_s^*}\right)S^{n/2s} + c_8 R^{2_s^*}\varepsilon^{n-2s} + c_{10}R^2\varepsilon^{(n-2s)/2}$$

where $c_{10} = 4c_9$. For $\upsilon + tu_\varepsilon^m \in \partial Q_\varepsilon^m \setminus H_m^-$, either $\|\upsilon\| = R$ or $t = R$, so, it follows from this that there exists R>ρ such that Eq. 20 holds for all sufficiently small ε. Turning to Eq.19 by contradiction, suppose:

$$\max_{u\in Q_{\varepsilon_j}^m} E(u) \ge \frac{s}{n}S^{n/2s}$$

for some sequence $\varepsilon_j \searrow 0$. Since, H_m^- is finite dimensional, $Q_{\varepsilon_j}^m$ is compact and hence, the above maximum is attained at some point $u_j = \upsilon_j + t_j u_{\varepsilon_j}^m \in Q_{\varepsilon_j}^m$. Then:

$$\frac{s}{n}S^{n/2s} \le E(u_j) = E(\upsilon_j) + E(t_j u_{\varepsilon_j}^m) -$$
$$4t_j\int_{B_{1/2m}^c \times B_{1/2m}} \frac{\upsilon(x)u_{\varepsilon_j}^m(y)}{|x-y|^{n+2s}}dx\,dy \le \frac{t_j^2}{2}\|u_{\varepsilon_j}^m\|^2 - \frac{t_j^{2_s^*}}{2_s^*}|u_{\varepsilon_j}^m|_{2_s^*}^{2_s^*} - \quad (25)$$
$$\int_\Omega G(x, t_j u_{\varepsilon_j}^m)dx + c_{11}\,\varepsilon_j^{(n-2s)/2}$$

for some constant $c_{11}>0$ as in the proof of Lemma 3.7. The estimates in Eq. 18 give:

$$\frac{t_j^2}{2}\|u_{\varepsilon_j}^m\|^2 - \frac{t_j^{2_s^*}}{2_s^*}|u_{\varepsilon_j}^m|_{2_s^*}^{2_s^*} \le \left(\frac{t_j^2}{2} - \frac{t_j^{2_s^*}}{2_s^*}\right)S^{n/2s} + c_{12}\varepsilon_j^{n-2s} \quad (26)$$

$$\leq \max_{t\in[0,\infty)}\left(\frac{t^2}{2}-\frac{t^{2_s^*}}{2_s^*}\right)S^{n/2s}+c_{12}\,\varepsilon_j^{n-2s}=\frac{s}{n}S^{n/2s}+c_{12}\,\varepsilon_j^{n-2s} \quad (27)$$

for some constant $c_{12}>0$, so, Eq. 25 gives:

$$\int_\Omega G\left(x,\, t_j u_{\varepsilon_j}^m\right)dx \leq c_{13}\,\varepsilon_j^{(n-2s)/2} \quad (28)$$

for some constant $c_{13}>0$. Since, $t_j\in[0, R]$, t_j converges to some $t_0\in[0, R]$ for a renamed subsequence. In Eq. 25 and 26 (H_2):

$$\frac{s}{n}S^{n/2s} \leq \left(\frac{t_j^2}{2}-\frac{t_j^{2_s^*}}{2_s^*}\right)S^{n/2s}+c_{14}\varepsilon_j^{(n-2s)/2}$$

for some constant $c_{14}>0$ and passing to the limit gives:

$$\frac{t_0^2}{2}-\frac{t_0^{2_s^*}}{2_s^*}\geq\frac{s}{n}$$

Since, the function $[0, \infty)\to\mathbb{R}$, $t\mapsto\frac{t^2}{2}-\frac{t^{2_s^*}}{2_s^*}$ attains its maximum value of s/n only at t = 1, it follows that $t_0 = 1$. We now show that (28) together with (H_2) and (H_5) leads to a contradiction. For j, so, large that $B_{\varepsilon j}\subset B_{4/m}$, ($H_2$) gives:

$$\int_\Omega G\left(x,\, t_j u_{\varepsilon_j}^m\right)dx \geq \int_{B_{\varepsilon_j}} G\left(x,\, t_j U_{\varepsilon_j}\right)dx \quad (29)$$

since, $\eta_m = 1$ in $B_{1/4m}$. Set:

$$\varphi(t) = \inf_{x\in\Omega_0,\tau\geq t}\frac{G(x,\,\tau)}{\tau^{(n+2s)/(n-2s)}},\; t\geq 0$$

Then φf is nondecreasing:

$$\lim_{t\to+\infty}\varphi(t) = +\infty \quad (30)$$

by (H_5) and $G(x, t)\geq\varphi(t)t^{(n+2s)/(n-2s)}$ for a.a. $x\in\in\Omega_0$ and$t\geq 0$. Since, $B_{\varepsilon j}\subset B_{4/m}\subset\Omega_0$, this together with (29) gives:

$$\int_\Omega G\left(x,\, t_j u_{\varepsilon_j}^m\right)dx \geq \int_{B_{\varepsilon_j}} G\left(t_j U_{\varepsilon_j}\right)dx\left(t_j U_{\varepsilon_j}\right)^{(n+2s)/(n-2s)}dx \quad (31)$$

For $x\in B_{\varepsilon j}$:

$$U_{\varepsilon_j}(x) = U_{\varepsilon_j}\left(\left|\varepsilon_j\right|\right)\geq U_{\varepsilon_j}\left(\varepsilon_j\right)= c_{15}\varepsilon_j^{-(n-2s)/2}$$

for some constant $c_{15}>0$. Since, $t_j\to 1$ and φ is nondecreasing, this together with Eq. 31 gives:

$$\int_\Omega G\left(x,\, t_j u_{\varepsilon_j}^m\right)dx \geq c_{16}\int_{B_{\varepsilon_j}}\varphi\left(c_{17}\,\varepsilon_j^{-(n-2s)/2}\right)\varepsilon_j^{-(n+2s)/2}\,dx =$$
$$c_{18}\varphi\left(c_{17}\varepsilon_j^{-(n-2s)/2}\right)\varepsilon_j^{(n-2s)/2}$$

for some constants c_{16}, c_{17}, $c_{18}>0$ and all sufficiently large j. This together with (28) implies that $f(c_{17}\varepsilon_j^{-(n-2s)/2})$ is bounded, contradicting (30). This completes the proof of Theorem 1.7.

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