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The Cauchy Problem for the Elastic Boltzmann Equation with an External Force

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Abstract: The Cauchy problem for the elastic Boltzmann equation with external force is considered for near vacuum data. We prove the existence of solutions for initial data small enough with respect to the sup norm by using the fixed point theorem of Krasnoselskii.

INTRODUCTION

The purpose of this study is to study the existence to the Cauchy problem with an external force. Let f(t, x, v) be the distribution function for particles at time $t\geq 0$ and position $x\varepsilon\mathbb{R}^3$ with velocity $v\varepsilon\mathbb{R}^3$ in the presence of the external force $\vec{F}=\vec{F}(t,x)=(F_1(t,x), F_2(t,x), F_3(t,x))$ that is differentiable with respect to the time. The Cauchy problem for Boltzmann equation with force term is Eq. 1:

$$\begin{cases} \dfrac{\partial f}{\partial t}+\vec{v}\cdot\nabla_x f+\vec{F}\cdot\nabla_v f=Q(f,f) \\ f(0,x,v)=f_0(x,v) \end{cases} \quad (1)$$

The term $f_0(x, v)$ is a non-negative function which represents the initial data. The explicit form of the collision operator Q(f, f) in the hard sphere case is Eq. 2:

$$Q(f,f)=\sigma\int_{S_+^2}\int_{\mathbb{R}^3}\omega\cdot(v-u)\begin{bmatrix} f(t,x,v')f(t,x,u')- \\ f(t,x,v)f(t,x,u)\end{bmatrix}du\, d\omega \quad (2)$$

Here:

$$S_+^2=\{\omega\in S^2:\omega\cdot v\geq\omega\cdot u\}$$

Both momentum and energy are conserved:

$$u'+v'=u+v$$

And:

$$|u'|^2+|v'|^2=|u|^2+|v|^2$$

Appropiate space for solution is proposed by Glassey[1]. We denote $C^0([0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3)$ by the set of all continuous functions in the space $([0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3)$. Given $\beta>0$, let:

$$X=\begin{Bmatrix} f\in C^0\left([0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3\right): \text{there exists} \\ c>0\ \text{s.t.}\ |f(t,x,v)|\leq ce^{-\beta\left(|x|^2+|v|^2\right)} \end{Bmatrix}$$

With norm:

$$\|f\|=\max\begin{Bmatrix} \sup\limits_{t,x,v}\left|f(t,x,v)\right|e^{\beta\left(|x|^2+|v|^2\right)}, \\ \sup\limits_{t,x,v}\left|\dfrac{\partial f(t,x,v)}{\partial v_i}\right|e^{\beta\left(|x|^2+|v|^2\right)} \end{Bmatrix}, i=1,2,3$$

The normed space (X, $\|.\|$) is a Banach space. We introduce the notation:

$$f^{\#}(t, x, v) = f(t, x, +vt, v+Ft)$$

Then, the Boltzmann equation can be written in Eq. 3:

$$\frac{df^{\#}(t, x, v)}{dt} = Q^{\#}(f, f) + t\frac{\partial F}{\partial t}\cdot\nabla_v f^{\#} \quad (3)$$

So, we integrate in time to get Eq. 4:

$$f^{\#}(t, x, v) = f_0(x, v) + \int_0^t Q^{\#}(f, f)d\tau + \int_0^t \tau\frac{\partial F}{\partial \tau}\cdot\nabla_v f^{\#}d\tau \quad (4)$$

Define the operator P on X by Eq. 5:

$$Pf^{\#} = f_0(x, v) + \int_0^t Q^{\#}(f, f)d\tau + \int_0^t \tau\frac{\partial F}{\partial \tau}\cdot\nabla_v f^{\#}d\tau \quad (5)$$

We claim that the operator has a fixed point. In other words, the operator equation Pf = f has a solution.

Theorem 1: Define the operators A and B on the space X Eq. 6:

$$\begin{cases} Af^{\#}(t, x, v) = f_0(x, v) + \int_0^t \tau\frac{\partial F}{\partial \tau}\cdot\nabla_v f^{\#}d\tau \\ Bf^{\#}(t, x, v) = \int_0^t Q^{\#}(f, f)d\tau \end{cases} \quad (6)$$

The operator A+B:X→X has a fixed point. This is there is fεX such that (A+B)(f) = f. Let us review some research on the Cauchy problem for the Boltzmann equation near vacuum by Bellomo *et al*.[2]. In the absence of an external force, the local existence was proved by Kaniel and Shinbrot[3] in terms of an iterative scheme. With the same idea, the first global existence proof was given by Illner and Shinbrot[3] for the hard sphere model and some generalization followed by Cercignani[5], Hamdache[6], Bellomo and Toscani[7], Polewczak[8], Alonso and Gamba[9]. For the Boltzmann equation with the external force[10], the local existence theorem to the Cauchy problem was given by Asano[11]. A local existence and uniqueness theorem is proved by Andrades[12] for the homogeneous Boltzmann equation with force term integrable with respect to time. The stationary case is also analyzed[13]. By Diperna and Lions[14] the existence and weak global stability of the Cauchy problem of the Boltzmann equation with general collision kernels is shown, proving that the bounded sequences of solutions that only satisfy natural physics converge weakly to a solution in L^1. By is mentioned a theorem of existence, uniqueness and positivity of the solution of the Boltzmann equation with force termand initial data near vacuum. For classical solutions, the first existence result was obtained by Guo[15] for soft potentials when the external force is small and decays in time with some rates. By Gressman and Strain [16] and Bardos *et al*.[17] the global existence is demonstrated with rapid decay to the equilibrium of classical Boltzmann equation solutions without angular breaks to the interactions of the Maxwell equilibrium states. By Galeano *et al*.[18] and Strain[19] the Cauchy problem is studied for the relativistic Boltzmann equation with initial data near the vacuum. By Burton[20] is showed that an operator consisting of the sum of a contractive operator with another one that is completely continuous has a solution in a Banach space.

MATERIALS AND METHODS

Let, X denote a Banach space with a norm $\|.\|$.

Definition 2.1: A set M in X is said to be convex, if for all u and v in M and all α in the interval [0, 1], the point αu (1-α) also belongs to M.

Theorem 2.1: Let (X, $\|.\|$) be a complete normed space and let F:X→X be a contraction with Lipschitzian constant L. Then F has a unique fixed point uεX.

Proof: By Agarwal *et al*.[21] two main results of fixed point theory are Schauder's theorem and the contraction mapping principle. Krasnoselskii combined them into the following result.

Theorem 2.2: Let M be a closed convex nonempty subset of a Banach space (X,$\|.\|$). Suppose that A and B map M into X such that:

- $Ax+By\in M(\forall x,y\in M)$
- A is a contraction with constant L<1
- B is continuous and BM is contained in a compact set

Then, there is a $y\in M$ with Ay+By = y.

RESULTS AND DISCUSSION

As a direct consequence of Krasnoselskii's theorem, we state that all the hypothesis are satisfied.

Proposition 3.1: Let X be a normed space and let $u_0\varepsilon X$ with $r\geq 0$ given. Then, the ball $X_R = \{u\varepsilon X:\|u-u_0\|\leq R\}$ is convex.

Proof: If $u,v\in X_R$ and $0\leq\alpha\leq 1$, then:

$$\begin{aligned}&\|\alpha u+(1-\alpha)v-u_0\| = \|\alpha(u-u_0)+(1-\alpha)(v-u_0)\| \leq \\ &\|\alpha(u-u_0)\|+\|(1-\alpha)(v-u_0)\| = \alpha\|u-u_0\|+ \\ &(1-\alpha)\|(v-u_0)\| \leq \alpha R+(1-\alpha)R = R\end{aligned}$$

Therefore:

$$\|\alpha u+(1-\alpha)v-u_0\| \le R. \text{ Then } \alpha u+(1-\alpha)v\in X_R$$

Lemma 3.2: Define the closed ball of radius R in X:

$$X_R = \{u\in X : \|u-u_0\| \le R\}$$

Let, A: $X_R \to X$. Define the operator in theorem 1:

$$Af^{\#}(t, x, v) = f_0(x, v) + \int_0^t \tau \frac{\partial F}{\partial \tau}\cdot \nabla_v f^{\#} d\tau$$

where, $\vec{F}$ is a diferentiable vector field Lipschitz continuous. Then A is a contraction.

Proof: Let, $f^{\#}(t,x,v)$ and $g^{\#}(t,x,v)$ be two elements of X:

$$\left|Af^{\#}-Ag^{\#}\right| = \left|\int_0^t \tau \frac{\partial F}{\partial \tau}\cdot\nabla_v\left(f^{\#}-g^{\#}\right)d\tau\right| \le \int_0^t |\tau|\left|\frac{\partial F}{\partial \tau}\cdot\nabla_v\left(f^{\#}-g^{\#}\right)\right|d\tau =$$

$$\int_0^t \tau\left|\frac{\partial F}{\partial \tau}\cdot\nabla_v\left(f^{\#}-g^{\#}\right)\right|d\tau = \int_0^t \tau\left|\sum_{i=1}^3 \frac{\partial F_i}{\partial \tau}\frac{\partial\left(f^{\#}-g^{\#}\right)}{\partial v_i}\right|d\tau$$

By the mean value theorem the components F_τ are bounded by $M_i>0$, Then, for all i = 1, 2, 3, we have:

$$\begin{cases}\left|\frac{\partial F_1}{\partial \tau}\right| \le M_1\\ \left|\frac{\partial F_2}{\partial \tau}\right| \le M_2\\ \left|\frac{\partial F_3}{\partial \tau}\right| \le M_3\end{cases}$$

$$\left|\frac{\partial F_i}{\partial \tau}\right| \le M' = \max\{M_1, M_2, M_3\}$$

Then:

$$\left|Af^{\#}-Ag^{\#}\right| \le \int_0^t \tau\left|\sum_{i=1}^3 M'\frac{\partial\left(f^{\#}-g^{\#}\right)}{\partial v_i}\right|d\tau \le \int_0^t \tau M'\sum_{i=1}^3\left|\frac{\partial\left(f^{\#}-g^{\#}\right)}{\partial v_i}\right|d\tau \le$$

$$\operatorname*{Sup}_{t,x,v} e^{-\beta\left(|x|^2+|v|^2\right)} e^{\beta\left(|x|^2+|v|^2\right)}\int_0^t \tau M'\sum_{i=1}^3\left|\frac{\partial\left(f^{\#}-g^{\#}\right)}{\partial v_i}\right|d\tau \le$$

$$e^{-\beta\left(|x|^2+|v|^2\right)}\int_0^t 3\tau M'\left\|f^{\#}-g^{\#}\right\|d\tau \le$$

$$e^{\beta\left(|x|^2+|v|^2\right)}\left(\int_0^t 3\tau M' dt\right)\left\|f^{\#}-g^{\#}\right\| =$$

$$e^{-\beta\left(|x|^2+|v|^2\right)}\frac{3}{2}T^2M'\cdot\left\|f^{\#}-g^{\#}\right\|$$

Finally:

$$\operatorname*{Sup}_{t,x,v}\left|Af^{\#}-Ag\right|e^{\beta\left(|x|^2+|v|^2\right)} = \left\|Af^{\#}-Ag^{\#}\right\| \le K\left\|f^{\#}-g^{\#}\right\|$$

For an interval [0,T] where $3T^2M'<2$. There exists K>0 such that $\|Af^{\#}-Ag^{\#}\| \le K\|f^{\#}-g^{\#}\|$. The operator A is a contraction.

Lemma 3.3: The operator B defined in theorem 1:

$$Bf^{\#}(t, x, v) = \int_0^t Q^{\#}(f, f)d\tau$$

Satisfies that $B:X\to X$:

Proof: Let, $f^{\#}\varepsilon X$:

$$\left|Bf^{\#}\right| = \left|\int_0^t Q^{\#}(f, f)d\tau\right| \le \int_0^t\left|Q^{\#}(f, f)\right|d\tau$$

By definition:

$$Q^{\#}(f, f) = \sigma\int_{S_+^2}\int_{\mathbb{R}^3}\omega\cdot(v-u)\begin{bmatrix}f^{\#}(t, x,v')f^{\#}(t, x, u')-\\ f^{\#}(t, x,v)f^{\#}(t, x, u)\end{bmatrix}du\,d\omega =$$

$$\pi\sigma\int_{\mathbb{R}^3}|v-u|\begin{bmatrix}f^{\#}(t, x,v')f^{\#}(t, x, u')-\\ f^{\#}(t, x,v)f^{\#}(t, x, u)\end{bmatrix}du$$

By triangular inequality:

$$\left|Q^{\#}(f, f)\right| \le \pi\sigma\int_{\mathbb{R}^3}|v-u|\begin{bmatrix}\left|f^{\#}(t, x, v')f^{\#}(t, x, u')\right|+\\ \left|f^{\#}(t, x, v)f^{\#}(t, x, u)\right|\end{bmatrix}du \le$$

$$\pi\sigma\int_{\mathbb{R}^3}|v-u|\left|f^{\#}(t, x,v')f^{\#}(t, x, u')\right|du+$$

$$\pi\sigma\int_{\mathbb{R}^3}|v-u|\left|f^{\#}(t, x,v)f^{\#}(t, x, u)\right|du \le$$

$$\pi\sigma\int_{\mathbb{R}^3}|v-u|\left|f^{\#}(t, x,v')\right|\left|f^{\#}(t, x, u')\right|du+$$

$$\pi\sigma\int_{\mathbb{R}^3}|v-u|\left|f^{\#}(t, x,v)\right|\left|f^{\#}(t, x,u)\right|du$$

Both integral are bounded, so, also the operator B is bounded. Integrating on both sides from 0-1:

$$\int_0^t\left|Q^{\#}(f,f)\right|d\tau \le 2\pi\sigma\left\|f^{\#}\right\|^2 e^{-\beta\left(|x|^2+|v|^2\right)}$$

$$\int_{\mathbb{R}^3}|v-u|\left[\int_0^t e^{-\beta\left(|x+\tau(v-u)|^2\right)}d\tau\right]e^{-\beta|u|^2}du \le$$

$$2\pi\sigma\left\|f^{\#}\right\|^2 e^{-\beta\left(|x|^2+|v|^2\right)}\int_{\mathbb{R}^3}|v-u|\left[\int_0^\infty e^{-\beta\left(|x+\tau(v-u)|^2\right)}d\tau\right]e^{-\beta|u|^2}du \le$$

$$2\pi\sigma\left\|f^{\#}\right\|^2 e^{-\beta\left(|x|^2+|v|^2\right)}\sqrt{\frac{\pi}{\beta}}\int_{\mathbb{R}^3}|v-u|e^{-\beta|u|^2}du \le$$

$$2\pi\sigma\left\|f^{\#}\right\|^{2} e^{-\beta\left(|x|^{2}+|v|^{2}\right)} \sqrt{\frac{\pi}{\beta}}\left(\sqrt{\frac{\pi}{\beta}}\right)^{3} \leq 2\pi\sigma\left\|f^{\#}\right\|^{2} e^{-\beta\left(|x|^{2}+|v|^{2}\right)} \frac{\pi^{3}}{\beta^{2}}$$

Where:

$$\int_0^t \left|Q^{\#}(f, f)\right| d\tau \leq 2\sigma\left\|f^{\#}\right\|^{2} e^{-\beta\left(|x|^{2}+|v|^{2}\right)} \frac{\pi^{3}}{\beta^{2}}$$

Thus:

$$\sup_{t, x, v}\left\{\left|\int_0^t Q^{\#}(f, f) d\tau\right| e^{-\beta\left(|x|^{2}+|v|^{2}\right)}\right\} \leq 2\sigma\left\|f^{\#}\right\|^{2} \frac{\pi^{3}}{\beta^{2}}$$

We conclude that:

$$\int_0^t Q^{\#}(f, f)\, d\tau \in X$$

This shows that $Bf^{\#} \varepsilon X$, then $B: X \to X$:

Lemma 3.4: Let, $B: X_R \to X$ an operator defined as:

$$Bf^{\#} = \left(Bf^{\#}\right)(t, x, v) = \int_0^t Q^{\#}(f, f) d\tau$$

Then, B is continuous.

Proof: From Lemma 3.3 it follows that:

$$\sup_{t, x, v}\left\{\left|\int_0^t Q^{\#}(f, f) d\tau\right| e^{\beta\left(|x|^{2}+|v|^{2}\right)}\right\} =$$

$$\sup_{t, x, v}\left\{\left|Bf^{\#}\right| e^{-\beta\left(|x|^{2}+|v|^{2}\right)}\right\} = \left\|Bf^{\#}\right\| \leq 2\sigma\left\|f^{\#}\right\|^{2} \frac{\pi^{3}}{\beta^{2}} =$$

$$\left(2\sigma \frac{\pi^{3}}{\beta^{2}}\left\|f^{\#}\right\|\right)\left\|f^{\#}\right\| = C \cdot \left\|f^{\#}\right\|$$

$$\left\|Bf^{\#}\right\| \leq C \cdot \left\|f^{\#}\right\|$$

With constant:

$$C = 2\sigma \frac{\pi^{3}}{\beta^{2}}\left\|f^{\#}\right\|$$

Theorem 3.1. (Main theorem): Let X_R be a convex nonempty subset of a Banach space $(X, \|.\|)$. Suppose that $A, B: X_R \to X$ such that:

- $Ax + By \in X_R (\forall x, y \in X_R)$
- Under Lemma 3.2 the operator A is a contraction
- Under the Lemma 3.4 the operator B is continuous

Then, there is a $y^{\#} \in X_R$ such that:

$$Ay^{\#} + By^{\#} = y^{\#}$$

In other words, there is a $y^{\#} \in X_R$ such that:

$$Py^{\#} = Ay^{\#}By^{\#} = y^{\#} = y_0(x, v) + \int_0^t Q^{\#}(y, y) d\tau + \int_0^t \tau \frac{\partial F}{\partial \tau} \cdot \nabla_v y^{\#} d\tau$$

CONCLUSION

In this study, it has been proved that the operator P has a fixed point. In other words, the Boltzmann equation with or without an external force has a solution in a Banach space. This is an alternative way to show existence via. the Krasnoselskii fixed point theorem. Furthermore, when there is no force or when it is constant in time it can be guaranteed that the solution found is unique.

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