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Analytic Bounded Point Evaluation Over Crescent Regions

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Abstract: In this study, I will prove that if $0\in abpe(P^t(\mu))$, then under certain conditions over the region G, we conclude that $0\in abpe(P^t(\mu\,|_{G\setminus\Delta(\alpha;\,\delta)}))$ for some $\delta>0$.

INTRODUCTION

Recall that G is a crescent region if $G = W/\bar{V}$ where, V and W are Jordan regions such that $V\subseteq W$ and $\bar{V}\cap\partial W$ is a single point (the multiple boundary point of G). Now let, $\partial_o G$ be the inner boundary of the region G (that is ∂V) and let $\partial_\infty G$ be the outer boundary of G (that is ∂W). For any crescent G, we let V_G denote the boundary component of $C/\bar{G}$ and we let mbp(G) denote the multiple boundary point of G. Throughout the work that follows we let G be a crescent region such that $\partial_\infty G = \partial D$ and mbp(G) = 1. This assumption on $\partial_\infty G$ simplifies our main result, even though our main result carry through for general crescents. By a Mobuis transformations of the disk, we may assume that $0\in V_G$.

A complex number z is called a bounded point evaluation for $P^t(\mu)$ if there is a constant M such that $|p(z)|\leq M.\|p\|L^t(\mu)$ for all polynomials p; the collection of all such points is denoted $bpe(P^t(\mu))$. If $z\in C$ and there are positive constants M and r such that $|p(w)|\leq M.\ \|p\|L^t(\mu)$ whenever p is a polynomial and $|w-z|<r$, then, we call z an analytic bounded point evaluation for $P^t(\mu)$; the set of all points z of this type is denoted by $abpe(P^t(\mu))$. Notice that $abpe(P^t(\mu))$ is an open subset of $bpe(P^t(\mu))$ and by the maximum modulus theorem, each component of $abpe(P^t(\mu))$ is simply connected. If $z\in abpe(P^t(\mu))$, then by the Hahn-Banach and Riesz representation theorems, there exists K_z in $L^s(\mu)$ such that $(1/s+1/t = 1)$ such that $p(z) = \int p(\zeta)K_z(\zeta)d(\zeta)$ for each poly-nomial p. For f in $P^t(\mu)$, define $\hat{f}$ on $bpe(P^t(\mu))$ by $f^\wedge(z) = \int f(\zeta)K_z(\zeta)d(\zeta)$. Observe that $\hat{f} = f$ a.e.μ on $bpe(P^t(\mu))$ and in fact $z\rightarrow\hat{f}(z)$ is analytic on $abpe(P^t(\mu))$. The set $abpe(P^t(\mu))$ support (μ) can be thought of as a set of over-convergence for $P^t(\mu)$. I will start rst by stating two important results that appears by Al-Hami (2015) and Akeroyd and Alhami (2002), (respectively) and needed for the proof of Theorem 3.

Theorem 1: Let μ be a finite, positive Borel measure with compact support in C such that $D\subseteq abpe(P^t(\mu))$. If K is a compact subset of D, then $D\subseteq abpe(P^t(\mu|_{(C\setminus K)}))$.

Theorem 2: Let μ be any finite, positive Borel measure with compact support in C and choose λ in C/support(μ). Then $\lambda\in abpe(P^t(\mu))$ if and only if $1/z-\lambda\notin P^t(\mu)$.

MAIN RESULTS

Theorem 3: Let μ be a finite, positive Borel measure with support in $\bar{D}$ such that $\partial\mu = \omega dA$ (dA denotes area measure on C) where $\omega L^\infty(dA)$. If $0\in abpe(P^t(\mu))$ and $1\le t<\infty$, then for any point α in ∂ D and 0<r<1, $0\in abpe(P^t\mu|_{\bar{D}/\overline{\Delta(\alpha;r)}}+|s/_{\gamma_r})$ where $\Delta(\alpha; r) := \{z:|z-\alpha|<r\}$, $\Gamma_r := \partial\Delta(\alpha; r)$ $\gamma_r := \Gamma_r\cap\bar{D}$ and s denotes normalized arclength measure on Γ_r.

Proof: Choose α in ∂D and 0<r<1. Let $\Delta(\alpha; r) := \{z: |z-\alpha| <r\}$, let $\Gamma_r := \partial\Delta(\alpha; r)$ and let $K_r = (\bar{D}/\Delta(\alpha; r))\cup\Gamma_r$. Let η denote the sweep of $\mu|_{D\cap\Delta(\alpha;\ r)}+\delta\alpha$ to Γ_r and let $v=\mu|_{(\bar{D}/\Delta(\alpha;r))}+\eta$; observe that $\|p\|_{L^t(\mu)}\le\|p\|_{L^t(v)}$ $\|p\|$ for any p in P and so $0\in abpe(P^t(v))$. Let $R^t(K_r, v)$ denote the closure of $Rat(K_r)$ in $L^t(v)$.

Claim: $0\in abpe(R^t(K_r, v))$. Let P denote the collection of poly-nomials and let $R = \{p(1/z-\alpha): p\in P \text{ and } p(0) = 0\}$. Now, $v|_{\Gamma_r}\ge\omega(.,\Delta(\alpha; r), \alpha)$ harmonic measure on $\partial\Delta(\alpha; r)$ evaluated at α which is normalized arclength measure s on Γ_r. Indeed, $v|\Gamma_r$ is boundedly equivalent to s, since, $d\mu = \omega dA$ ($\omega\in L^\infty(dA)$) and so, we assume, for our purposes, that $v|\Gamma_r\equiv s$. Suppose $\{p_n\}\subseteq P$, $\{q_n\}\subseteq R$ and $\|p_n+q_n\|_{L^t(v)}\to 0$ as n→∞. Then $\|p_n+q_n\|_{L^t(v)|\Gamma_r)}\to 0$ as n→∞.

Case 1: The 1<t<∞; from a theorem of M. Riesz it follows that $\|p_n\|_{L^t(v)|\Gamma_r)}\to 0$ and $\|q_n\|_{L^t(v)|\Gamma_r)}\to 0$ as n→∞. Via. the Mobius transformation $S(z) = r/z-\alpha$ and the fact that $v|_{\Gamma_r}\equiv\omega(., \Delta; r, \alpha)$ and $dv|_{D/\overline{\Delta(\alpha;r)}} = \omega dA|_{D/\overline{\Delta(\alpha;r)}}$ where $\omega\in L^\infty(dA)$, one can conclude that: $\|q_n\|_{L^t(v)}\to 0$ as n→∞ and there is a neighborhood W_1 of 0 such that $\|q_n\|_{\overline{W_1}}\to 0$ as n→∞. It follows that $\|q_n\|_{L^t(v)} = \|(p_n+q_n)-q\|p_n+q_n\|_{L^t(v)}$ + $\|q_n\|L^t(v)\to 0$ as $n\to\infty$.

Therefore, since, $0\in abpe(P^t(v))$, there exists a neighborhood W_2 of 0 such that $\|p_n\|_{\overline{W_2}}\to 0$ as n→∞; let $W = W_1\cap W_2$. We now have that $\|p_n+q_n\|_{\overline{W}}\to 0$ as n→∞ and so, our claim holds for 1<t<∞.

Case 2; t = 1: As before, we have that $\|p_n+q_n\|_{L^t(v|\Gamma_r)}\to 0$ as n→∞. Applying the Cauchy integral to p_n+q_n over Γ_r with evaluation at ζ in $C\backslash\overline{\Delta(\alpha, r)}$, we get that: $q_n\to 0$ uniformly on compact subsets of $C\backslash\overline{\Delta(\alpha, r)}$ and indeed $\|q_n\|_{L^t(\mu|D/\overline{\Delta(\alpha;r)})}\to 0$ as n→∞, since, $d\mu = \omega dA$ and $\omega\in L^\infty(dA)$. It follows from and our assumption about the convergence of $\|p_n+q_n\|_{L^t(v)}$ to zero that $\|p_n\|_{L^t(\mu|D\backslash\overline{\Delta(\alpha;r)})}\to 0$ as n→∞. Again applying the Cauchy integral to p_n+q_n over Γ_r but this time with evaluation at ζ in Δ(α, r), we get that $p_n\to 0$ uniformly on compact subsets of Δ(α, r) and indeed $\|p_n\|_{L^t(\mu|_{\Delta(\alpha,r)})}\to 0$ as n→∞, since, $d\mu = \omega dA$ and $\omega\in L^\infty(dA)$. An earlier observation we now conclude that $\|p_n\|_{L^t(\mu)}\to\infty$ as n→∞. Since, $0\in abpe(P^t(\mu))$, it follows that $p_n\to 0$ uniformly in some neighborhood of 0. This along gives us that $0\in abpe(R^t(K_r, v))$. Evidently, our claim holds for 1 = t<∞. Now by Theorem 1, we may assume that $0\notin$/support(v). So, by our claim, $1/z\notin R^t(K_r, v)$ and hence, there exists g in $L^s(v)$ (1/s+1/t = 1) such that $g\perp R^t(K_r, v)$ and yet $\int g(z)/z\, dv(z)\neq 0$.

Let, T be the Mobius transformation $T(z) = 1/r(z-\alpha)$ (Observe that $T(\Delta(a, \alpha)) = D$) and define τ and h by $\tau := voT^{-1}$ and $h := oT^{-1}$. Now $h\in L^s(\tau)$ and $g\perp R^t(T(K_r), \tau)$, yet $\int h(z)/z+\alpha/r\, d(\tau\neq 0)$. So, the Cauchy transform

$$\hat{h}(\zeta) := \int\frac{h(z)}{z-\zeta}d\tau(z)$$

(which is defined and analytic off the support of tau) is identically zero in D and in the unbounded component of $C/T(k_r)$ and yet is nonzero in a neighborhood of -α/r. Let $\gamma = T(\Gamma_r/\gamma_r)$ and notice that if $e^{i\varphi}\in\gamma$ and R>1, then $Re^{i\varphi}$ is in the unbounded component of $C/T(K_r)$. Therefore, if $e^{i\varphi}\in\gamma$, 0<ρ<1 and ρ is sufficiently near 1, then (for $\zeta = \rho e^{i\varphi}$):

$$0=\int\left(\frac{1}{z-\zeta}-\frac{1}{z-1/\bar{\zeta}}\right)h(z)d\tau(z)=\int_{T(\bar{D}\backslash\overline{\Delta(\alpha,r)})}\left(\frac{1}{z-\zeta}-\frac{1}{z-1/\bar{\zeta}}\right)h(z)d\tau(z)+\int P_\zeta(z)\bar{z}h(z)d\tau(z)$$

Now:

$$\int_{T(\bar{D}\backslash\overline{\Delta(\alpha,r)})}\left(\frac{1}{z-\zeta}-\frac{1}{z-1/\bar{\zeta}}\right)h(z)d\tau(z)\to 0$$

as ρ→1. Since, $\tau|\partial_{D=m}$ by (H, first corollary, page 38), we therefore have that $0=\lim_{\rho\to 1}\int P_\zeta(z)\bar{z}h(z)dm(z) = e^{-i\varphi}.h(e^{i\varphi})$ a.e.m on γ. So, h = 0 a.e τ on γ and hence, g = 0 a.e. v on Γ_r/γ_r. It follows that $1/z\notin P^t(\mu|_{\bar{D}\backslash\overline{\Delta(\alpha,r)}}+s|_{\gamma r}$ and so by Theorem 2, $0\in abpe(P^t(\mu|_{\bar{D}\backslash\overline{\Delta(\alpha,r)}+s|\gamma_r)})$. The proof is now complete. This coming result is not totally new but the approach is quite different.

Corollary 4: Let G be a crescent such that $\partial_\infty G = \partial D$, 1 = mbp(G) and $0\notin\bar{G}$. Define μ on G by $d\mu = |f|^t dA|_G$ and f is never zero in G. if $0\in abpe(P^t(\mu))$, $1\le t<\infty$, then for each α in ∂D with α ≠ 1, there exists δ>0 such that $0\in abpe(P^t(\mu|_{G\backslash\Delta(\alpha,\delta)}))$

Proof: Choose α in ∂D where α ≠ 1. Choose r>0 such that r<dist(α, $\partial_o G$), let $\Delta(\alpha; r) = \{z:|z-\alpha|<r\}$ let $\Gamma_r = \partial\Delta(\alpha; r)$ and let $\gamma_r = \Gamma_r\cap\bar{D}$. Now, Γ_r meets ∂D at two distinct points-call these points a and b. Since, $f\in H^\infty(G)$, f has nontangential limits a.e.m on ∂D and so, we may assume (with a slight alteration in r if needed) that f has nonzero nontangential limits at both a and b. Choose ∈>0, so that,

$\in$<r/30 and r+$\in$<dist(α, $\partial_o G$). Let {z$\in$D: r-$\in$<|z-α|<r+$\in$} and let F = {z$\in$E: dist(z; {a, b})<4:dist(z; ∂D)}. Now by our construction of F and the fact that f has nonzero nontangential limits at both a and b, there is a positive constant c_1 such that $|f(z)|^t \geq c1$ whenever z$\in$F. Furthermore, there is another constant c_2 such that $\Delta 2$: = {ξ: $|\xi$-z$|<c_2|$(z-a)(z-b)$|$}$\subseteq$F whenever z$\in\gamma_r\cap$D. So, for any polynomial p:

$$\int_{\gamma r}|p(z)|^t|(z\text{-}a)(z\text{-}b)|^2\, ds(z) = \int_{\gamma r}|\frac{1}{\pi|c_2(z\text{-}a)(z\text{-}b)|^2}$$

$$\int \Delta_z p(\xi) dA(\xi)|^t|(z\text{-}a)(z\text{-}b)|^2\, ds(z) \leq \int_{\gamma r}\frac{1}{\pi(c_2)^2}\int_{\Delta z}|p(\xi)|^t \quad (1)$$

$$dA(\xi)ds(z) \leq \frac{s(\gamma_r)}{\pi(c_2)^2}\int_F |p|^t\, dA \leq c\int_F |p|^t\, du$$

Since, 0$\in$abpe($P^t(\mu)$), we have by Theorem 3 that 0$\in$abpe($P^t(v)$) where $v = \mu|_{(\overline{G}\setminus\overline{\Delta(\alpha;r)})} + s|\gamma_r$. Hence, there exist positive constants ρ, M and N such that:

$$|(z\text{-}a)|(z\text{-}b)| \geq c_3 > 0 \quad (2)$$

and

$$|p(z)|(z\text{-}a)(z\text{-}b))^{\frac{2}{t}} \leq M.\|p.((z\text{-}a)(z\text{-}b)^{\frac{2}{t}}\|_{L^t(v)} \leq N$$
$$\{\int_{\overline{G}\setminus\overline{\Delta(\alpha,r)}}|p|^t\, d\mu + \int_{\gamma r}|p|^t\,(z\text{-}a)(z\text{-}b)|^2\, ds\}$$

whenever p$\in$P and z$\in\Delta$(0; ρ). Therefore, by Eq. 1 and 2, we conclude that $|p(z)|\ \text{Const.}\{\int_{\overline{G}\setminus\overline{\Delta(\alpha;r)}}|p|^t\, d\mu + \int_F |p|^t\, d\mu\}$ for every p in P and hence there exists δ>0 (we may choose δ = r-$\in$) such that 0$\in$abpe($P^t(d\mu|_{G\setminus\Delta(\alpha;\,\delta)})$).

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