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Study of Viscoelastic Model for Longitudinal Wave Propagation in a Non-Homogeneous Viscoelastic Filament

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Abstract: This study is concerned to check the validity of a five parameter viscoelastic model for longitudinal wave propagating in the non-homogeneous viscoelastic rods. In this study, it is assumed that density (ρ), rigidity (G) and viscosity (η) of the specimen, i.e., rod are space dependent and obey the laws $\rho = \rho_0 e^{2\alpha_1 x}$, $G = G_0 e^{2\alpha_2 x}$ and $\eta = \eta_0 e^{2\alpha_3 x}$. The method of non-linear partial differential equation (Eikonal equation) has been used for finding the dispersion equation of longitudinal waves.

Key words: Longitudinal waves, variable density, viscoelastic media, Friedlander series, India

INTRODUCTION

The viscoelasticity theory is used in the field of solid mechanics, seismology, exploration geophysics, acoustics and engineering. The solutions of many problems related with wave-propagation for homogeneous media are available in many literatures of continuum mechanics of solids. However in the recent years, the interest has arisen to solve the problems connected with non-homogeneous bodies. These problems are useful to understand the properties of polymeric materials and industrial related applications.

Modeling and model parameter estimation is of great importance for a correct prediction of the foundation behavior. Many researchers Alfrey (1944), Barberan and Herrera (1966), Achenbach and Reddy (1967), Bhattacharya and Sengupta (1978) and Acharya *et al*. (2008) formulated and developed this theory. Further, Bert and Egle (1969), Abd-Alla and Ahmed (1996) and Batra (1998) successfully applied this theory to wave propagation in homogeneous, elastic media. Murayama and Shibata (1961) and Schiffman *et al.* (1964) have proposed higher order viscoelastic models of five and seven parameters to represent the soil behavior. Recently, Kakar *et al*. (2012) and Kaur *et al*. (2012) analyzed various viscoelastic models under dynamic loading.

In most of the literature, the problems of non-homogeneity are taken as independent of space coordinate. But in this study, researchers consider the wave propagation in non-homogeneous media when density (ρ), rigidity (G) and viscosity (η) of the material are space dependent such that the wave velocity is also space dependent. The problem is solved with Eikonal equation when the wave equation is approximated using WKB theory. The displacements assumed in the problem are so small that under isothermal conditions, the linear constitutive laws hold. The displacement and stress expressions are solved for time dependent displacement and stress boundary conditions.

METHODOLOGY

Formulation of problem: Researchers consider the five parameter model with three springs $S_1(G_1)$, $S_2(G_2)$, $S'_2(G'_2)$ and two dash-pots D'_2 and D_3 with viscoelasticity η'_2 and η_3, respectively (Fig. 1). It has 3 sections, section 1 contains one spring $S_1(G_1)$, section 2 contains three elements two springs $S_2(G_2)$, $S'_2(G'_2)$ and one dash-pots $D_2(\eta'_2)$ where $S'_2(G'_2)$ spring and dash-pot $D_2(\eta'_2)$ are in series forming Maxwell Model. The spring $S_2(G_2)$ is parallel to the Maxwell element. The section 3 contains one dash pot $D_3(\eta_3)$. The springs represent recoverable elastic response and dash pot represents elements in the structure giving rise to viscous drag. For section 1:

$$e = e_1 + e_2 + e_3 \quad (1)$$

$$\sigma = G_1 e_1 \quad (2)$$

For section 2:

$$e = e_{21} + e_{22} \quad (3)$$

$$\sigma = \sigma_1 + \sigma_2 \quad (4)$$

$$\sigma_1 = G_2 e_2 \quad (5)$$

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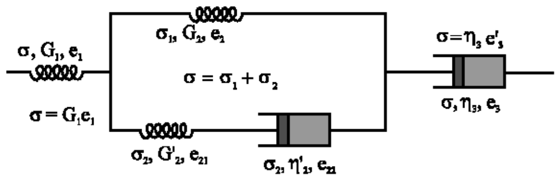

Fig. 1: Five parameter viscoelastic model

$$\sigma_2 = G'_2 e_{21} \tag{6}$$

$$\sigma_2 = \eta'_2 \dot{e}_{22} \tag{7}$$

From Eq. 4 and 5:

$$\sigma_2 = \sigma - G_2 e_2 \tag{8}$$

Taking derivative on both sides of Eq. 3 and using Eq. 6-8, researchers get:

$$\left(1 + \frac{G_2}{G'_2}\right)\dot{e}_2 + \frac{G_2 e_2}{\eta'_2} = \frac{\dot{\sigma}}{G'_2} + \frac{\sigma}{\eta'_2} \tag{9}$$

For section 3:

$$\sigma = \eta_3 \dot{e}_3 \tag{10}$$

Therefore, the behavior of viscoelastic materials regarding to this five parameter model is given by Eq. 2, 9 and 10. Where σ is the normal stress and e is the corresponding overall normal strain of the five parameter model, e_1-e_3 are the normal strains associated with the Maxwell spring, the three parameter model and the Maxwell dashpot, respectively. Also $G_1 = \lambda_1 + 2\mu_1$, $G_2 = \lambda_2 + 2\mu_2$, $G'_2 = \lambda'_2 + 2\mu'_2$ are the modulli of elasticity associated with Maxwell and three parameter element, respectively and η'_2, η_3 are Newtonian viscosities coefficients are all taken as functions of x in the non-homogeneous case considered here. Eliminating e_1-e_3 from Eq. 2, 9 and 10, researchers get:

$$\left\{\frac{1}{G_1}\left(1 + \frac{G_2}{G'_2}\right) + \frac{1}{G'_2}\right\}\ddot{\sigma} + \left\{\frac{1}{\eta_3}\left(1 + \frac{G_2}{G'_2}\right) + \frac{1}{\eta'_2}\left(1 + \frac{G_2}{G_1}\right)\right\}$$
$$\dot{\sigma} + \frac{G_{02}}{\eta'_2 \eta_3}\sigma = \left(1 + \frac{G_2}{G'_2}\right)\ddot{e} + \frac{G_2}{\eta'_2}\dot{e} \tag{11}$$

or;

$$P(D)\sigma = Q(D)e \tag{11a}$$

where;

$$P(D) = \delta_2 D^2 + \delta_1 D + \delta_0 \tag{11b}$$

$$Q(D) = \gamma_2 D^2 + \gamma_1 D \tag{11c}$$

$$\delta_0 = \frac{G_2}{\eta'_2 \eta_3} \tag{11d}$$

$$\delta_1 = \frac{1}{\eta_3}\left(1 + \frac{G_2}{G'_2}\right) + \frac{1}{\eta'_2}\left(1 + \frac{G_2}{G_1}\right) \tag{11e}$$

$$\delta_2 = \frac{1}{G_1}\left(1 + \frac{G_2}{G'_2}\right) + \frac{1}{G'_2} \tag{11f}$$

$$\gamma_1 = \frac{G_2}{\eta'_2} \tag{11g}$$

$$\gamma_2 = 1 + \frac{G_2}{G'_2} \tag{11h}$$

The equation of motion is:

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 U}{\partial t^2} \tag{12}$$

$$e = \frac{\partial U}{\partial x} \tag{13}$$

Where, ρ = ρ (x) is the variable density of the material. Using Eq. 12 and 13, Eq. 11 leads to:

$$\delta_2 \sigma_{ttt} + \delta_1 \sigma_{tt} + \delta_0 \sigma_t = \frac{1}{\rho}\left\{\gamma_2 \sigma_{xxt} - \gamma_2 (\log \rho)_x \sigma_{xt} + \gamma_1 \sigma_{xx} - \gamma_1 (\log \rho)_x \sigma_x\right\} \tag{14}$$

METHOD OF SOLUTION

Researchers assume that the solution σ = (x, t) of Eq. 14 may be represented by the series (Friedlander, 1947):

$$\sigma(x,t) = \sum_{n=0}^{\infty} A_n(x) F_n\{t - h(x)\}; \quad A_0 \neq 0 \tag{15}$$

Where:

$$F'_n = F_{n-1} \quad (\text{where } n = 1, 2, 3, \ldots) \tag{16}$$

and it is assumed that $A_n \approx 0$ for n<0. Researchers further assume that the derivatives of σ may be obtained by term-wise differentiation of Eq. 15, the prime in Eq. 16 denotes differentiation with respect to the argument concerned and by using Eq. 16, researchers relate all F'_ns to F_0 by successive integrations.

The solution of Eq. 14 in the form of Eq. 15 can be obtained by taking a phase function h(x), h(x) satisfies the Eikonal equation of geometrical optics (Karl and Keller, 1959):

$$\left(\frac{dh(x)}{dx}\right)^2 = \frac{\rho}{G} = \frac{1}{c^2} \tag{17}$$

Where, c = c(x) is the variable wave speed for elastic longitudinal waves in a medium whose modulus of elasticity is G.

Using Eq. 15, the successive derivatives of σ (x, t) w.r.t. t and x in Eq. 14, researchers get:

$$\left(\delta_2 A_n - \frac{\gamma_2}{\rho} A_n h'^2\right) F_{n-3} + \left\{ \begin{array}{l} \delta_1 A_n + \frac{\gamma_2}{\rho}\left(2A'_n h' + A_n h''\right) - \\ \frac{\gamma_2}{\rho}(\log\rho)_x A_n h' - \frac{\gamma_1}{\rho} A_n h'^2 \end{array} \right\} F_{n-2}$$

$$= \left\{ \begin{array}{l} -\delta_0 A_n + \frac{\gamma_2}{\rho} A''_n - \frac{\gamma_2}{\rho}(\log\rho)_x A'_n - \\ \frac{\gamma_1}{\rho}\left(2A'_n h' + A_n h''\right) + \frac{\gamma_1}{\rho}(\log\rho)_x A_n h' \end{array} \right\} F_{n-1} +$$

$$\left\{ \frac{\gamma_1}{\rho} A''_n - \frac{\gamma_1}{\rho}(\log\rho)_x A'_n F_n \right\} \tag{18}$$

To obtain the solution, i.e., to determine A'_ns and phase velocity function h(x), researchers take the coefficient of $F_{n-3} = 0$ where $A_n \neq 0$ and get the Eikonal equation of geometrical optics as:

$$\delta_2 - \frac{\gamma_2}{\rho} h'^2 = 0 \tag{19}$$

$$\Rightarrow h'^2 = \frac{\delta_2 \rho}{\gamma_2} = \rho \left\{ \frac{\frac{1}{G_1}\left(1 + \frac{G_2}{G'_2}\right) + \frac{1}{G'_2}}{1 + \frac{G_2}{G'_2}} \right\} \tag{20}$$

$$= \rho \left[\frac{G_1(G_2 + G'_2)}{G_1 + G_2 + G'_2} \right]^{-1} = \frac{\rho}{G}$$

Where:

$$G = \frac{G_1(G_2 + G'_2)}{G_1 + G_2 + G'_2}$$

From Eq. 17 and 20, researchers get:

$$h'^2 = \frac{\rho}{G} = \frac{1}{c^2(x)} \tag{21}$$

Using Eq. 14 and 15, the amplitude function satisfy the equation:

$$2h'(x)A'_n(x) + \frac{\rho}{\gamma_2}\left\{ \begin{array}{l} \delta_1 - \frac{\gamma_2}{\rho}(\log\rho)_x h'(x) - \\ \frac{\gamma_1}{\rho} A_n h'^2(x) + \frac{\gamma_2}{\rho} h''(x) \end{array} \right\} A_n(x) = X_n; \quad (n = 0,1,2,\ldots) \tag{22}$$

Where:

$$X_n = A''_{n-1} - \left\{ (\log\rho)_x + 2\frac{\gamma_1}{\gamma_2} h' \right\} A'_{n-1} - \left\{ \frac{\rho\delta_0}{\gamma_2} + \frac{\gamma_1}{\gamma_2} h'' - \frac{\gamma_1}{\gamma_2}(\log\rho)_x h' \right\} A_{n-1} + \frac{\gamma_1}{\gamma_2} A''_{n-2} - \frac{\gamma_1}{\gamma_2}(\log\rho)_x A'_{n-2} \tag{23}$$

Since, the wave is travelling along x-axis; therefore integrating Eq. 17, researchers get:

$$h(x) = h(0) \pm \int_0^x \frac{ds}{c(s)} \tag{24}$$

Where the plus (+) sign is associated with wave traveling in the positive direction of x and the minus (-) sign is associated with the waves travelling in the negative direction of x. Solution of Eq. 22 can be obtained as:

$$A_n(x) = A_n(0)\left\{\frac{1(x)}{1(0)}\right\}^{\frac{1}{2}} \exp\left\{\mp\int_0^x m(s)ds\right\} \pm \frac{1}{2}\int_0^x c(s)\left\{\frac{1(x)}{1(s)}\right\}^{\frac{1}{2}} \exp\left\{\pm\int_x^z m(z)dz\right\} X_n^{\pm}(s)ds \quad (n = 0,1,2,\ldots) \tag{25}$$

Where:

$$1(x) = \rho c$$

And:

$$m(x) = \frac{\rho c}{2}\left(\frac{\delta_1}{\gamma_2} - \frac{\gamma_1}{\gamma_2}.\frac{1}{\rho c^2}\right)$$

Let, an impulse of magnitude σ_0 suddenly applied at the end x = 0 of the rod and thereafter steadily maintained that is:

$$\sigma(0,t) = \sigma_0 H(t) \tag{26}$$

From Eq. 15 and 26, researchers have:

$$\sum_{n=0}^{\infty} A_n(0) F_n\{t - h(0)\} = \sigma_0 H(t) \tag{27}$$

Thus, researchers choose (Moodie, 1973):

$$A_n(0) = \begin{vmatrix} \sigma_0 \cdots\cdots\cdots \text{if } n = 0 \\ 0 \cdots\cdots\cdots \text{if } n < 0 \text{ or } n > 0 \end{vmatrix} \tag{28}$$

$$h(0) = 0 \quad \text{and} \quad F_0 = H(t) \tag{29}$$

The solution of Eq. 14, for the waves travelling in the positive direction of x is generated by boundary stress, therefore Eq. 26 becomes:

$$\begin{aligned} \sigma(x,t) &= \sum_{n=0}^{\infty} A_n(x) \frac{\{t - h(x)\}^n}{n!} H\{t - h(x)\} \\ &= \sum_{n=0}^{\infty} A_n(x) \frac{\{t - h(x)\}^n}{n!} H\left\{t - \int_0^x \frac{ds}{c(s)}\right\} \end{aligned} \tag{30}$$

Where:

$$h(x) = \int_0^x \frac{ds}{c(s)} \tag{31}$$

Where, $A_n(x)$ are given recursively by Eq. 25 (with upper signs) in conjunction with Eq. 28. The first-term approximation leads to Eq. 30 as:

$$\sigma(x,t) = \sigma_0 \left\{\frac{1(x)}{1(0)}\right\}^{\frac{1}{2}} \exp\left\{-\int_0^x m(s) ds\right\} H\left\{t - \int_0^x \frac{ds}{c(s)}\right\} \tag{32}$$

Equation 32 represents a transient stress wave which starts from the end x = 0 with amplitude σ_0 and moves in the positive direction of x with velocity c(x). Hence, it is modulated by the factor:

$$\left\{\frac{1(x)}{1(0)}\right\}^{\frac{1}{2}} \exp\left\{-\int_0^x m(s) ds\right\} \tag{33}$$

Further, terms in the approximate solution may be obtained recursively from Eq. 25. The solution of Eq. 32 applies until the wave moving in the positive direction of x strikes either an interface (in the case of a composite rod) or at end (in the case of a finite rod). Researchers will show that reflected waves are produced at the other end of the finite rod while both reflected and transmitted waves are produced at an interface between two dissimilar media.

VISCOELASTIC MODEL APPLIED TO A PARTICULAR CASE

For the sake of concreteness and for studying the qualitative effect of non-homogeneity on the longitudinal wave propagation in non-homogeneous five parameter viscoelastic rods, it is assumed that density (ρ), rigidity (G) and viscosity (η) of the specimen, i.e., rod are space dependent and obey the laws:

$$\rho = \rho_0 e^{2\alpha_1 x}, G = G_0 e^{2\alpha_2 x}, \eta = \eta_0 e^{2\alpha_3 x} \tag{34}$$

If:

$$\alpha_1 = \alpha_2 = \alpha_3, \text{i.e., density} \geq \text{rigidity} \geq \text{viscosity} \tag{35}$$

Case 1: When $\alpha_1 = \alpha_2 = \alpha_3$, then from Eq. 34, researchers get:

$$\rho = \rho_0 e^{2\alpha x}, G = G_0 e^{2\alpha x}, \eta = \eta_0 e^{2\alpha x} \tag{36}$$

Therefore, from Eikonal equation of geometric optics:

$$\left(\frac{dh(x)}{dx}\right)^2 = \frac{\rho}{G} = \frac{\rho_0 e^{2\alpha x}}{G_0 e^{2\alpha x}} = \frac{\rho_0}{G_0} = \frac{1}{c_0^2} = \text{Constant} \tag{37}$$

$$\Rightarrow c = \sqrt{\frac{G_0}{\rho_0}} \tag{38}$$

Since, the exponential variation of modulus of rigidity (G) and density (ρ) is similar, therefore sound speed is constant, i.e., non-homogeneous has no effect on speed and phase of the wave is given $h(x) = x/c_0$. So, it becomes the case of semi non-homogeneous medium (a medium when characteristics are space dependent while the speed is independent of space variable). The amplitude function $A_n(x)$ satisfies the equation:

$$\begin{aligned} &2h'(x)A'_n(x) + \left\{\frac{\rho_0}{\gamma_{02}}\delta_{01} - \alpha h'(x) - \frac{\gamma_{01}}{\gamma_{02}} h'^2(x)\right\} \\ &A_n(x) = Y_n; \qquad (n = 0,1,2\ldots) \end{aligned} \tag{39}$$

Where:

$$\delta_{01} = \frac{1}{\eta_{03}}\left(1 + \frac{G_{02}}{G'_{02}}\right) + \frac{1}{\eta'_{02}}\left(1 + \frac{G_{02}}{G_{01}}\right) \tag{40a}$$

$$\gamma_{01} = \frac{G_{02}}{\eta'_{02}} \tag{40b}$$

$$\gamma_{02} = 1 + \frac{G_{02}}{G'_{02}} \tag{40c}$$

And:

$$Y_n = A''_{n-1} - \left\{\alpha + 2\frac{\gamma_{01}}{\gamma_{02}}h'\right\}A'_{n-1} - \left\{\frac{\rho_0\delta'_0}{\gamma_{02}} - \frac{\gamma_{01}}{\gamma_{02}}\alpha h'\right\} A_{n-1} + \frac{\gamma_{01}}{\gamma_{02}}A''_{n-2} - \frac{\gamma_{01}}{\gamma_{02}}\alpha A'_{n-2} \quad (40)$$

Where:

$$\delta'_0 = \frac{G_{02}}{\eta'_{02}\eta_{03}}$$

And solution of Eq. 39 is given as:

$$A_n(x) = A_n(0)\exp\left\{\int_0^x \frac{\alpha}{2}dx\right\}\exp\left\{\pm\int_0^x m_0 ds\right\} \pm \frac{1}{2}\int_0^x c_0 \exp\left\{\int_0^x \frac{\alpha}{2}dx\right\}\exp\left\{\pm\int_x^s m_0 dz\right\} Y_n^{\pm}(s)ds \quad (n = 0,1,2,\dots) \quad (41)$$

Where:

$$m_0 = \frac{\rho_0 c_0}{2}\left\{\frac{1}{\eta_{03}} + \frac{1}{\eta'_{02}}\frac{\left(1+\frac{G_{02}}{G_{01}}\right)}{\left(1+\frac{G_{02}}{G'_{02}}\right)} - \frac{1}{\rho}\frac{G_{02}}{\eta'_{02}}\left(1+\frac{G_{02}}{G'_{02}}\right)^{-1}\right\} \quad (42)$$

For this case, the value of first term approximation, the stress function is given by:

$$\sigma(x,t) = \sigma_0\left\{\exp\int_0^x \frac{\alpha}{2}dx\right\}\exp\left\{-\int_0^x m_0 ds\right\}H\{t - h(x)\} \quad (43)$$

It is modulated by the factor:

$$\left\{\exp\int_0^x \frac{\alpha}{2}dx\right\}\exp\left\{-\int_0^x m_0 ds\right\} \quad (44)$$

Case 2: $\alpha_1>\alpha_2>\alpha_3$, i.e., density>rigidity>viscosity then from Eq. 34, researchers get:

$$\rho = \rho_0 e^{2\alpha_1 x}, G = G_0 e^{2\alpha_2 x}, \eta = \eta_0 e^{2\alpha_3 x}$$

From Eikonal equation of geometric optics:

$$\left(\frac{dh(x)}{dx}\right)^2 = \frac{\rho}{G} = \frac{\rho_0 e^{2\alpha_1 x}}{G_0 e^{2\alpha_2 x}} = \frac{\rho_0}{G_0}e^{2(\alpha_1-\alpha_2)x} = \frac{1}{c^2} \quad (45)$$

Here:

$$c = \sqrt{\frac{G_0}{\rho_0}}e^{(\alpha_2-\alpha_1)x} \quad (46)$$

The amplitude function $A_n(x)$ satisfies the equation:

$$2h'(x)A'_n(x) + \left\{\begin{array}{l} e^{2(\alpha_1-\alpha_3)x}\frac{\rho_0}{\gamma_{02}}\delta_{01} - 2\alpha_1 h'(x) - \\ e^{2(\alpha_2-\alpha_3)x}\frac{\gamma_{01}}{\gamma_{02}}h'^2(x) + h'' \end{array}\right\}A_n(x) = Y'_n \quad (n = 0,1,2,\dots) \quad (47)$$

Where:

$$Y'_n = Y'_n A''_{n-1} - \{2\alpha_1 + 2Kh'\}A'_{n-1} - \left\{e^{2(\alpha_1+\alpha_2-2\alpha_3)x}\frac{\rho_0\delta'_0}{\gamma_{02}} + Kh'' - 2K\alpha_1 h'\right\}A_{n-1} + KA''_{n-2} - 2K\alpha_1 A'_{n-2} \quad (n = 0,1,2,\dots)$$

And:

$$K = e^{2(\alpha_2-\alpha_3)x}\frac{\gamma_{01}}{\gamma_{02}}$$

And its solution is obtained as:

$$A_n(x) = A_n(0)\left\{\frac{1(x)}{1(0)}\right\}^{\frac{1}{2}}\exp\left\{\int_0^x \alpha_1 dx\right\}\exp\left\{\mp\int_0^x m(s)ds\right\} \pm \frac{1}{2}\int_0^x c(s)\left\{\frac{1(x)}{1(s)}\right\}^{\frac{1}{2}} xp\left\{\int_0^x \alpha_1 dx\right\}\exp\left\{\pm\int_x^z m(z)dz\right\}X_n^{\pm}(s)ds \quad (48)$$

Where:

$$1(x) = \rho c$$

And:

$$m(x) = \frac{\rho c}{2}\left\{\frac{1}{\eta_3} + \frac{1}{\eta'_2}\frac{\left(1+\frac{G_2}{G_1}\right)}{\left(1+\frac{G_2}{G'_2}\right)} - \frac{1}{\rho}\frac{G_2}{\eta'_2}\left(1+\frac{G_2}{G'_2}\right)^{-1}\right\}$$

For this case, the value of first term approximation, the stress function is given by:

$$\sigma(x,t) = \sigma_0\left\{\frac{1(x)}{1(0)}\right\}^{\frac{1}{2}}\exp\left\{\int_0^x \alpha_1 dx\right\}\exp\left\{-\int_0^x m(s)ds\right\} \quad (49)$$

It is modulated by the factor:

$$\left\{\frac{l(x)}{l(0)}\right\}^{\frac{1}{2}} \exp\left\{\int_0^x \alpha_1 dx\right\} \exp\left\{m\int_0^x m(s)ds\right\} \qquad (50)$$

CONCLUSION

- When the density, rigidity and viscosity all are equal for the first material specimen, the sound speed is constant, i.e., non-homogeneous has no effect on speed and phase of the wave is given:

$$h(x) = \frac{x}{c_0}$$

So, it becomes the case of semi non-homogeneous medium (a medium when characteristics are space dependent while the speed is independent of space variable). The longitudinal speed will be equal to:

$$c = \sqrt{\frac{G_0}{\rho_0}}$$

- When the density, rigidity and viscosity are not equal for the second material specimen, the speed of sound varies exponential as:

$$c = \sqrt{\frac{G_0}{\rho_0}} e^{(\alpha_2 - \alpha_1)x}$$

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