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Teaching Conic Sections and Their Applications

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Abstract: Teachers' mastery of subject areas and ability to effectively impact the knowledge on the students are the major rudiments expected from an effective teacher. Conic sections in coordinate geometry are regarded by students as difficult. Conic sections is a very powerful conceptual framework for bringing algebra, geometry, history of Mathematics, applications and use in many other fields of knowledge together. It is thus, a rich point of departure for the idea of integration or interconnectedness in and around Mathematics which is making the learning of Mathematics more meaningful and further higher-order thinking skills. This study presents a problem solving approach to the teaching of identified difficult topic in Tertiary mathematics. Varied applications of Conic sections in Physical Sciences and Engineering were discussed.

Key words: Conic sections, hyperbola, ellipse, parabola, partial differential equations, laplace, diffusion and wave equations, South Africa

INTRODUCTION

Mathematics is a study of relationship among quantities, magnitudes and properties and of logical operations by which unknown quantities, magnitudes and properties may be deduced. Mathematics was in the past regarded as the science of quantity whether of magnitudes as in geometry or of numbers as in arithmetic or the generalization of these two fields as in algebra. Mathematics came to be regarded as the science of relations towards the middle of the 19th century or as the science that draws necessary conclusions. The latter view encompasses mathematical or symbolic logic; the science of using symbols to provide an exact theory of logical deduction and inference based on definitions, axioms, postulates and rules for transforming primitive elements into more complex relations and theorems.

Mathematics is perceived to be a difficult subject to understand. This has resulted into the poor performances of students at both internal and external examinations. A large number of students seem to learn very little at school indicating among other things that they find learning difficult (Salau, 1996). Adepoju (1991) and Ivowi *et al*. (1992) among others have attributed the low percentage passes in physical sciences and mathematics to dissatisfaction with the syllabus; teachers' qualification, workload, experience and disposition; general lack of teaching skills as measured by knowledge of subject-matter and the ineffective style of delivery of subject-matter. Salau (1995) in particular was of the view that as long as Mathematics is taught by whatever method to large groups of heterogeneous students in terms of ability, so that individual attention and a corresponding focus on analysis of individual students' problems are difficult, the problem of poor performance will remain.

Van de Walle (2007) reiterated NCTM (2000) call for a major shift from the traditional method of teaching to a modern method that will allow each child to construct his or her own schema (Dubinsky, 1991). The National Universities Commission (NUC) survey of labour market's expectations from products of faculties of education in the Nigerian Universities shows that graduate teachers are largely characterized by shallow subject-matter knowledge, very poor pedagogic skills; poor communication skills, poor classroom management and control, lack of professionalism; lack of entrepreneurial skills and generally poor attitude to research.

This study therefore, aims at enabling Senior Secondary School and Tertiary Institution, Mathematics teachers acquire some rudimentary knowledge of how some difficult concepts in tertiary mathematics can be meaningfully introduced to the students. The method of achieving this are those of problem-solving, Problem-based Learning (PBL) and demonstration, all of which are activity oriented. Particularly PBL is any learning environment in which the problem drives the learning. The West African Examinations Council marks and attendance sheet reveals the spread of questions attempted by candidates. Mathematics topics that are rarely attempted or not attempted at all are classified as difficult topics. Researchers are confronted with many

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questions such as why are students running away from some questions? Were those topics not taught? Did the students understand the concepts very well? This study focused on the teaching of conic sections in coordinate geometry and its applications in Tertiary mathematics. Further, Mathematics is one of the subjects to be offered only at the senior secondary schools. It is a bridge between secondary school mathematics and university mathematics. Questions from conic sections are rarely attempted at the Senior School Certificate Examinations (SSCE) and General Certificate Examinations (GCE). Some undergraduates still express fears on this topic in their 100 level Mathematics courses.

Definition of conic section: Conic sections in geometry are curves formed by the intersection of a plane with the surface of a right circular cone extended infinitely far on both sides of the vertex. The surface of the cone on either side of the vertex is called a nappe of the cone. Conic sections are 2-dimensional or plane curves and therefore, a desirable definition of conics avoids the notion of a cone which is 3-dimensional. A conic may be defined as the set of points of which the distances from some fixed point called the focus are in a constant ratio to the distances of the points from a fixed line called the directrix which does not pass through the fixed point. The constant ratio is called the eccentricity of the conic and is usually denoted by the letter e.

The eccentricity is the distance of any point on the curve from a fixed point (the focus) divided by the distance of that point from a fixed line (the directrix). A circle has an eccentricity of zero for an ellipse, it is <1 for a parabola, it is of equal 1 and for a hyperbola, it is >1. Circle is a plane curve such that each point on the curve is the same distance from a fixed point called the centre of the circle. The circle belongs to the class of curves known as conic sections because a circle can be described as the intersection of a right circular cone with a plane that is perpendicular to the axis of the cone.

Ellipse is one of the conic sections; a closed curve formed by a plane that cuts all the elements of a right circular cone. A circle which is formed by a plane perpendicular to the axis of the cone is a specialized form of ellipse. Any ellipse is symmetrical with respect to its major axis which is the straight line passing through the two foci and extended to meet the curve at each end. It is also symmetrical with respect to its minor axis, a line perpendicular to the major axis at the midpoint between the two foci. In a circle, the two foci of the ellipse coincide and the major and minor axes are equal. The eccentricity of an ellipse that is the ratio of the distance between the foci to the length of the major axis is always <1. The eccentricity of a circle is zero. The ellipse is one of the most important curves in physical science. In astronomy, the orbits of the earth and the other planets around the Sun are ellipses. It is used in engineering in the arches of some bridges and the design of gears for certain types of machinery such as punch presses.

Parabola in Mathematics, plane curve, one of the conic sections formed by the intersection of a cone with a plane parallel to a straight line on the slanting surface of the cone. Each point called the focus and a fixed straight line known as the directrix. The parabola is symmetrical about a line passing through the focus and perpendicular to the directrix. For a parabola, symmetric about the x-axis and with its vertex at the origin, the mathematical equation is $y^2 = 4ax$ where, 4a is called the latus rectum or the focal chord, the line that is parallel to y-axis and passes through the focus. A parabola is the curve that describes the trajectory of a projectile such as a bullet or a ball in the absence of air resistance.

Hyperbola, plane curve, one of the conic sections formed by a plane that cuts both nappes of a right circular cone but does not pass through the vertex of the cone. A hyperbola has two U-shaped non-intersecting branches, identical in form with the open parts facing in opposite directions, the arms of each branch separate as they recede.

A hyperbola is also defined as the locus of all points such that the difference between the distances from any point on the hyperbola to two fixed points called the foci is equal to a constant. Each branch contains one focus in its interior area; the line joining the foci intersects each branch at a point called the vertex. The line through the vertices and the foci is called the transverse axis. The line perpendicular to the transverse axis and passing through the point midway between the vertices is the conjugate axis. The two axes meet at the centre of the hyperbola. The hyperbola is symmetric with respect to each axis and the centre. A hyperbola has two asymptotes passing through the centre; an asymptote of a curve is a straight line with the property that the distance between it and the curve approaches zero as the curve recedes to infinity.

A rectangular or equilateral hyperbola has asymptotes that are perpendicular to each other. The hyperbola has useful and important properties. In particular, the angle formed at a point on the hyperbola by the lines joining the point to the foci is bisected by the tangent to the hyperbola at that point. The conic sections have numerous mathematical properties that give them important applications in mathematical physics. For example, the orbit of any astronomical object such as a planet or comet around any other object such as the Sun is always one of the conic sections.

MATERIALS AND METHODS

The logical presentation of the topic is from a low level cognitive to a high level cognitive.

Activity 1: Applications of pythagoras theorem in finding distance between two points. Recall the previous knowledge on finding the distance between two points.

Discussion 1:

Properties of eccentricity: From the given definition of conic section, the value of eccentricity determines the nature of each curve listed as:

$e = 0 \rightarrow$ Circle
$e<1 \rightarrow$ Ellipse
$e = 1 \rightarrow$ Parabola
$e>1 \rightarrow$ Hyperbola

Distinctions through diagrammatic representation of each of the curves should be properly made; PS/PM = e (Symbolic representation) and nature of the curve will depend on the value of e (Fig. 1):

$$OS = ae,\ OM = a/e$$
$$OT = x,\ OM = a/e$$
$$TM = PN = (a/e - x)$$

P is any point on the conic section. PS/PN = e (by definition):

$$(PS)^2 = e^2 (PN)^2$$
$$(x - ae)^2 + y^2 = e^2 (a/e - x)^2$$
$$x^2 - 2aex + a^2e^2 + y^2 = e^2 (a^2/e^2 - 2ax/e + x^2)$$
$$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2axe + x^2e^2$$
$$x^2 - x^2e^2 + y^2 = a^2 - a^2e^2$$
$$x^2 (1 - e^2) + y^2 = a^2 (1 - e^2)$$

Dividing through by $1 - e^2$:

$$x^2 + y^2/(1 - e^2) = a^2$$

Dividing through by a^2:

$$(x^2/a^2) + [y^2/a^2 (1 - e^2)] = 1$$

Case 1: $e<1 \rightarrow$ ellipse
$1 - e^2$ is positive
Let $a^2 (1 - e^2) = b^2$
$(x^2/a^2) + (y^2/b^2) = 1$ (Ellipse)

Case 2: $e>1 \rightarrow$ hyperbola
$1 - e^2$ is negative

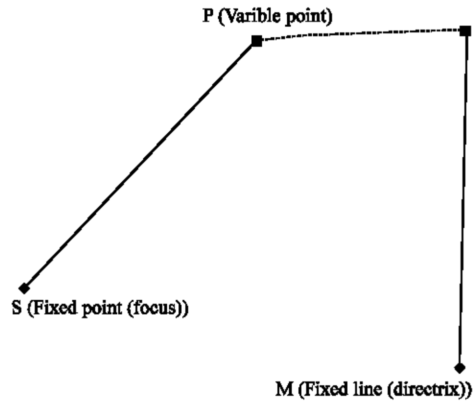

Fig. 1: Points representation

Let $a^2 (e^2 - 1) = b^2$
$(x^2/a^2) - (y^2/b^2) = 1$ (Hyperbola)

In the hyperbola:

a = Semi transverse axis
b = Semi conjugate axis

Activity 2

Problem solving techniques:
Given equation $(x^2/4) - (y^2/9) = 1$ what is the conjugate and transverse axes and what is the eccentricity.

Discussion 2

Solution method: Methods of solution should be discussed with the students' full participation:

Length of semi axes = 2, 3
Length of transverse axis = 4 units
Length of conjugate axis = 6 units

$$\text{If} \quad a^2 (e^2 - 1) = b^2$$
$$e^2 - 1 = b^2/a^2$$
$$e^2 = 1 + b^2/a^2$$
$$e = (\sqrt{1 + (b^2/a^2)})$$

Sub-stituting the values;

$$e = (\sqrt{1+(9/4)}) \rightarrow (\sqrt{13/4})$$
$$e = (\sqrt{13})/2$$

For rectangular hyperbola (i.e., a = b):

$$e = \sqrt{(1+1)} = \sqrt{2} \text{ which is} >1$$

Suppose the equation of a hyperbola:

$$(-x^2/4)+(y^2/9) = 1,\ a = 3,\ b = 2$$

Length of the semi axes = 2 and 3
Length of transverse axis = 6 units
Length of conjugate axis = 4 units
Sub-stituting the values:

$$e = \sqrt{(1+(4/9))} = \sqrt{(13/9)}$$
$$e = \sqrt{(13)}/3$$

One hyperbola is conjugate to another because they have common asymptotes (Fig. 2).

Activity 3

Derivation of parabola equation: The earlier given definition of parabola should be revised (Fig. 3).

$$PS/PM = 1 \text{ and } OS/OM = 1$$
$$PS^2 = PM^2$$
$$(x - a)^2+y^2 = (x+a)^2$$
$$x^2 - 2ax+a^2+y^2 = x^2+2ax+a^2$$
$$y^2 = 4ax$$

Focus-directrix property for e = 1; 4a = Latus rectum (length of the focal chord parallel to y-axis).

Parametric equations: Starting from ellipse:

$$(x^2/a^2)+(y^2/b^2) = 1$$

Let x = f (t), y = g (t) f, g are known functions. From these; y = f (x). In this case, sub-stitute x with new functions:

$$x = a\cos\Phi,\ y = b\sin\Phi$$
$$(a\cos\Phi)^2/a^2+(b\sin\Phi)^2/b^2 = 1$$
$$\text{Cos}^2\Phi+\text{Sin}^2\Phi = 1$$

Parametric equation of an ellipse is $x = a\cos\Phi$, $y = b\sin\Phi$
Φ = Parameter (eccentric angle)
a = Radius of the circle, OP = a

$$C: x^2+y^2 = a^2$$
$$E: (x^2/a^2)+(y^2/b^2) = 1$$

Coordinates of P; $a\cos\Phi$, $a\sin\Phi$
Q (on ellipse) corresponds to P (on circle)
Coordinates of Q; $a\cos\Phi$, y
$(a^2\cos^2\Phi/a^2)+(y^2/b^2) = 1$

$$\text{or } (y^2/b^2) = 1 - \cos^2\Phi = \sin^2\Phi$$
$$y = b\sin\Phi$$

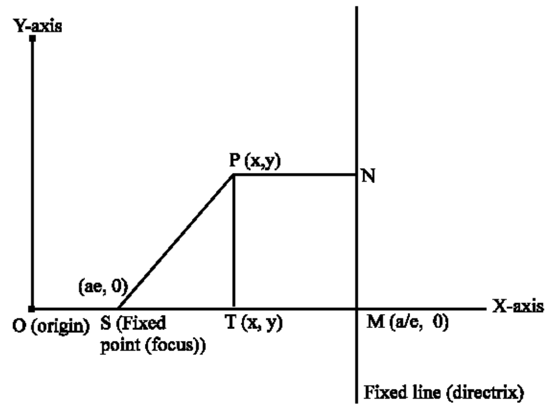

Fig. 2: Method of deriving ellipse and hyperbola equations

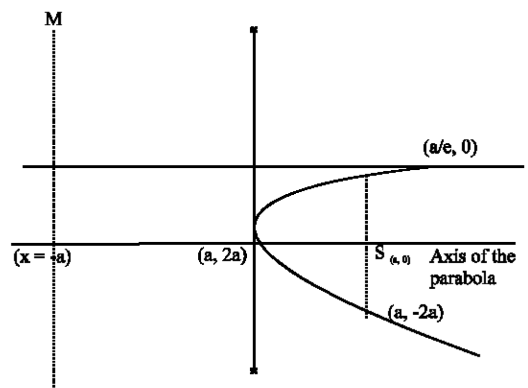

Fig. 3: Pyujnh fujbgiib km ik y5t6arabola

Ellipse: $(x^2/a^2)+(y^2/b^2) = 1$ $x = a\cos\Phi\}$ point Φ
$y = b\sin\Phi\}$ on the ellipse.

Hyperbola: $(x^2/a^2) - (y^2/b^2) = 1$ $x = a\sec\theta\}$ point θ
$y = b\tan\theta\}$ on the hyperbola.

Parabola: $y^2 = 4ax$ $x = at^2\ \}\ (at^2, 2at)$ point t
$y = 2at\}$ on the parabola (Fig. 4).

Exercise: Find the eccentricity of the ellipse $3x^2+4y^2 = 12$.
Solution:

$$(3x^2/12)+(4y/12) = 1$$
$$(x^2/4)+(y^2/3) = 1$$
$$(x^2/2^2)+(y^2/(\sqrt{3})^2) = 1$$
$$\rightarrow(x^2/a^2)+(y^2/b^2) = 1$$
$$a = 2,\ b = \sqrt{3}$$
$$b^2 = a^2(1 - e^2)$$
$$3 = 4(1 - e^2)$$
$$3/4 = 1 - e^2$$
$$e^2 = 1/4,\ ;\ e = \pm 1/2$$

Applications: Students will be motivated to learn difficult topics if only they can be exposed to the applications of

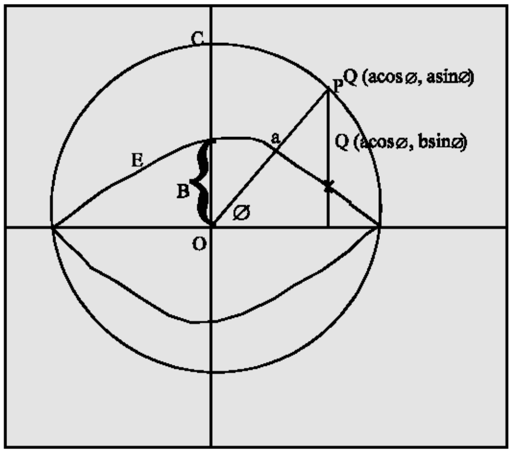

Fig. 4: Parametric circle

these topics in solving some physical sciences and engineering problems. Recall the discriminant D in the completing the square method of general quadratic equation; $ax^2+bx+c = 0$. The solution of the quadratic equation is $x = [-b\pm\sqrt{(b^2 - 4ac)}]/2a$. Investigation on the discriminant $D = b^2-4ac$ enables us to be able to determine the nature of the curve. When $D>0$, we have $b^2 - 4ac>0$, there exist two distinct real roots. When $D<0$, we have $b^2-4ac<0$, there exist two distinct complex roots. When $D = 0$, we have $b^2-4ac = 0$, there exist two coincident or repeated roots. The application is more pronounced in the linear 2nd order partial differential equations with constant coefficients. Any equation of the type:

$$a (\delta^2u/\delta x^2)+2h (\delta^2 u/\delta x\delta y)+b (\delta^2u/\delta y^2)+$$
$$2f (\delta v/\delta x)+2g (\delta v/\delta y)+e^u = 0$$

Where, a, h, b, f, g and e are constants is a linear homogeneous 2nd order partial differential equation with constant coefficient. We can see that the form of equation resembles that of a general conic section:

$$ax^2+2hxy+by^2+2fx+2gy+e = 0$$

The equation represents an ellipse, a parabola or a hyperbola when:

$ab - h^2>0$, curve is elliptic
$ab - h^2 = 0$, curve is parabolic
$ab - h^2<0$, curve is hyperbolic

We can see that the nature of differential equations depends only on the coefficients of the 2nd order derivatives whereas the nature of conic sections depends on eccentricity value. We have earlier seen that when:

$e = 0$, a circle is obtained
$e<0$, an ellipse is obtained
$e>0$, a hyperbola is obtained
$e = 1$, a parabola is obtained

Problems: Determine the nature of the following equations. Laplace equation in 2-dimensions:

$$\delta^2u/\delta x^2+\delta^2u/\delta y^2 = 0$$
$$a = 1, h = 0, b = 1$$
$$ab - h^2>0$$

The equation is of elliptic type. Wave equation:

$$(\delta^2u/\delta x^2) = 1/c^2 (\delta^2u/\delta t^2)$$
$$a = 1, h = 0, b = -1/c^2$$
$$ab - h^2<0$$

The equation is of hyperbolic type. The one-dimensional diffusion equation:

$$(\delta^2u/\delta x^2) - c^2 (\delta u/\delta t) = 0$$
$$a = 1, h = 0, b = 0$$
$$ab - h^2 = 0$$

It is of parabolic type.

CONCLUSION

Ability of students to understand difficult concepts and solve problems in Mathematics highly depends on the interest of such student's right from elementary school and teachers' general behaviour. We have attempted to show in this study that in finding the nature of any equation or sketching any curve, only the important details are required, sufficient to show the shape and position of the curve relative to the axes.

It could be generalized that for some difficult concepts or topics to be effectively taught and mastered by learners, the problem should be dismantled like peeling an onion and the teacher moves from known to unknown thereby simplifying the complex problem. That is before students learn some knowledge they should be given a problem.

The problem is posed so that the students discover that they need to learn some new knowledge before they can solve the problem. It could be seen that the form of the general solutions and the nature of the curve of the linear partial differential equation vary on the nature of the equation. Learners, therefore should be highly motivated by making teaching to be real and not abstract.

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