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Application of Logistic Function to the Risk Assessment of Financial Asset Returns

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Abstract: In this study, the research considered the application of the logistic distribution function to the assessment of risk of financial assets returns. The research first derived the distribution as a compound distribution of the Gumbel and Gamma distribution functions. The risk of financial assets returns data assessed.

Key words: Logistic distribution function, financial assets, Gamma and Gumbel distribution, risk assessment, probability

INTRODUCTION

If a random variable, X has the following probability density function:

$$f_X(x) = \frac{\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)\right\}}{\sigma\left[1+\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)\right]^2}, -\infty < x < \infty \quad (1)$$

where, μ and σ>0 are the location and scale parameters, respectively.

Let,

$$y = \frac{x-\mu}{\sigma}$$

Then,

$$f_Y(y) = \frac{\exp(-y)}{\left[1+\exp(-y)\right]^2}, -\infty < y < \infty \quad (2)$$

Equation 2 is the standard logistic distribution with its moment generating function given by:

$$\begin{aligned} M_Y(t) &= E(\exp(ty)) \\ &= \int_{-\infty}^{\infty} \exp[-(1-t)y][1+\exp(-y)]^2 dy \end{aligned} \quad (3)$$

Putting $z = (1+\exp(-y))^{-1}$ in Eq. 3, The research has:

$$\begin{aligned} M_y(t) &= \int_0^1 z^{-1}(1-z)^{-1} dz \\ &= \pi t \operatorname{cosec} \pi t \end{aligned} \quad (4)$$

The corresponding cumulative distribution function of Eq. 2 is:

$$\begin{aligned} F_Y(y) &= \frac{1}{\left[1+\exp(-y)\right]}, -\infty < y < \infty \\ &= \frac{1}{2} + \frac{1}{2}\tanh(y/2) \end{aligned} \quad (5)$$

The shape of the logistic distribution and the normal distribution are very similar. This similarity makes it simpler and profitable on suitable occasions to replace the normal distribution by the logistic distribution with negligible errors in the respective theories (Olapade, 2009). The logistic distribution finds applications in a range of fields from biology to economics. Lotka (1925) used this distribution to describe how species populations grow in competition. Marchetti (1977) applied this distribution to energy and society. Modis (1992, 1998) applied the logistic function in epidemiology for the assessment of risk and the diffusion of new product sales in the market. It has been used to describe how new technologies diffuse and substitute each other (Fisher and Pry, 1971). In the estimation of bioassay and quantal response data, Berkson (1944) applied this distribution. The have been considerable studies on the subject of logistic function by many researchers among them are Wu *et al*. (2000) and Ojo and Olapade (2004). The main goal of this study is to apply this distribution into finance, in the assessment of risk incurred by financial asset returns.

MATERIALS AND METHODS

It is not enough to look at just the return one expected to earn from investing in a particular financial asset (e.g., stock) but also to consider the risk of return of this asset. Investors want to be compensated for taking risk that is the want to earn a return high enough to make them comfortable with the level of risk they are assuming.

The variance of asset returns provides a quantification of incurred risks in the approach, given the distribution of the asset returns. In order to make the approach concrete, it will be assumed that asset returns X are distributed according to the logistic distribution. Before going further, the following lemma is in order.

Lemma 1: Let:

$$f_x(x:b,\beta)=\frac{\beta^b}{\Gamma(b)}\exp\{-bx\}$$
$$\exp\{-\beta\exp(-x)\},\ -\infty<x<\infty,b,\beta>0 \tag{6}$$

be a two parameter Gumbel probability density function of a continuous random X (where α, β are the location and scale parameter, respectively). Suppose that β has a Gamma distribution with density:

$$h(\beta:\mu,b)=\frac{\mu^b}{\Gamma(b)}\beta^{b-1}\exp\{-\mu\beta\},\ b>0,\ \mu>0 \tag{7}$$

Then the compound distribution of Eq. 6 and 7 is the logistic density function.

Proof: Put,

$$f_X(x|\mu,\beta)=\int_0^\infty f_x(x|\alpha,\beta)h(\alpha|\mu,b)d\beta$$
$$=\int_0^\infty\frac{\beta^b}{\Gamma(b)}\exp\{-bx\}\exp\{-\beta\exp(-x)\}$$
$$\frac{\mu^b}{\Gamma(b)}\beta^{b-1}\exp\{-\mu\beta\}d\beta$$
$$=\frac{\mu^b\exp\{-bx\}}{[\Gamma(b)]^2}\int_0^\infty\beta^{b-1}\beta^b\exp$$
$$\{-\beta(\mu+\exp(-x))\}d\beta$$
$$=\frac{\mu^b\exp\{-bx\}}{[\Gamma(b)]^2}\int_0^\infty\beta^{2b-1}$$
$$\exp\{-\beta(\mu+\exp(-x))\}d\beta$$

Let,

$$y=\beta(\mu+\exp\{-x\}) \tag{8}$$

Then:

$$\beta=\frac{y}{(\mu+\exp\{-x\})}\text{ and }\frac{dy}{d\beta}=(\mu+\exp\{-x\})$$

So that:

$$f_X(x|\mu,b)=\frac{\mu^b\exp\{-bx\}}{[\Gamma(b)]^2}\int_0^\infty y^{2b-1}\exp(-y)dy$$
$$=\frac{\Gamma(2b)\mu^b\exp\{-bx\}}{[\Gamma(b)]^2(\mu+\exp(-x))^{2b}},-\infty<x<\infty,(\mu,\beta)>0 \tag{9}$$

as required. Equation 9 is a probability density function of a random variable that has the type 3 generalized logistic distribution (Balakrishnan and Leung, 1988).

The corresponding cumulative distribution function for this generalized logistic function is:

$$F_X(x|b)=\frac{\Gamma(2b)}{[\Gamma(b)]^2}\int_{-\infty}^{x}\frac{\exp(-bt)}{(1+\exp(-t))^{2b}}dt \tag{10}$$

Putting:

$$z=(1+\exp(-t))^{-1} \tag{11}$$

we have:

$$F_X(x|b)=\frac{\Gamma(2b)}{[\Gamma(b)]^2}\int_0^{(1+\exp(-t))^{-1}}z^{b-1}(1-z)^{b-1}dz$$
$$=\frac{\Gamma(2b)}{[\Gamma(b)]^2}\int_0^{(1+\exp(-t))^{-1}}(z-z^2)^{b-1}dz \tag{12}$$

Let $y = z-z^2$, then dz = dy/1-2z. Notice that y = 0 if z = 0 and y = 1/4 for z = 1/2. So that:

$$F_X(x|b)=\frac{\Gamma(2b)}{[\Gamma(b)]^2(1-2z)}\int_0^{1/4}y^{b-1}dy$$
$$=\frac{\left(\frac{1}{4}\right)^b\Gamma(2b)}{b[\Gamma(b)]^2(1-2z)}$$
$$=\frac{\left(\frac{1}{4}\right)^b\Gamma(2b)}{b[\Gamma(b)]^2}\sum_{i=0}^{n}2^iz^i\left(\text{for}|2z|<1\right)$$
$$\cong\frac{\left(\frac{1}{4}\right)^b\Gamma(2b)}{b[\Gamma(b)]^2}(1+2z) \tag{13}$$

Using Eq. 11:

$$F_X(x|b)=\frac{\left(\frac{1}{4}\right)^b\Gamma(2b)}{b[\Gamma(b)]^2}\left(1+\frac{2}{1+\exp(-x)}\right)$$
$$=3K(b)\left\{\frac{1+\frac{1}{3}\exp(-x)}{1+\exp(-x)}\right\} \tag{14}$$

Where:

$$K(b)=\frac{\left(\frac{1}{4}\right)^b\Gamma(2b)}{b[\Gamma(b)]^2}$$

Equation 14 is a special case of the logistic function

$$F(x)=a\frac{1+m\exp\{-x\}}{1+n\exp\{-x\}}$$

which appears in logistic regression and feed forward neural networks.

Applications: Consider an investment in which d+1 returns are received by an investor continuously on the fixed horizon [O, T], $0<t<\infty$. The portfolio of the investor is made up of risk free returns and risky returns. The risk free returns evolves according to the differential equation:

$$dr(t) = \lambda(t)r_0(t)dt, r_0(t) = i_0 \quad 0 \le t \le T \tag{15}$$

From time t to Δt, the dynamic growth of the portfolio characterized by fluctuation can be given as:

$$\frac{P_{t+\Delta T} - P_t}{P_T} = \frac{\Delta P_t}{P_t} = \alpha(t)\Delta t \tag{16}$$

due to some environmental effects (risk). Let α (t) = f (t) + noise (random error captured by normal distribution N (0, $\sigma^2 \Delta t$) with mean zero and variance in proportion to Δt). Then:

$$\frac{\Delta P_t}{P_t} = \left(f(t) + \xi_{t,\Delta t}\right)\Delta t \tag{17}$$

Suppose that $\Delta B_t \sim N(0, \Delta t)$, then $\xi_{t,\Delta t} = \beta \Delta B_t$, so that:

$$\frac{\Delta P_t}{P_t} = f(t)\Delta t + \beta\Delta B_t \tag{18}$$

Let $\Delta t \to 0$, we have Eq. 18 which becomes:

$$dP_t = f(t)P_t dt + \beta P_t dt \tag{19}$$

It is not difficult to see, using Ito's rule that:

$$d\log P_t = \left(f(t) - \frac{\beta^2}{2}\right)dt + \beta dB_t \tag{20}$$

At time t, the growth rate of the portfolio P depends on the n regulators through regulatory functions. Let x_i (t) be the returns of the ith regulator at time t. Then:

$$g\left(x:(t)\right) = 3kb\left\{\frac{1 + \frac{1}{3}\exp\left(\frac{x_i(t)-\mu}{\beta_i}\right)}{1 + \exp\left\{\frac{x_i(t)-\mu}{\beta_i}\right\}}\right\} \tag{21}$$

using Eq. 14, it is the regulatory function of a specific regulatory i described as the logistic function. μ_i and β_i are the mean and deviation of regulator i, respectively. Since the growth level is measured at n time points, there are n samples for time t = $t_1, t_2, \ldots, t_n$. μ_1 and β_1 are estimated by sample mean $\bar{X} = \sum_{i=1}^{n} X_i$ and sample standard deviation:

$$s_i = \left[\frac{\sum_{i=1}^{n}(X_i(t) - \bar{X}_i)^2}{n-1}\right]^{1/2}$$

Let,

$$f(t) = \sum_{i=1}^{n} c_i g_i\left(X_i(t)\right) \tag{22}$$

Then Eq. 20 and 22 give:

$$d\log P_i = \left[\sum_{i=1}^{n} c_i g_i\left(X_i(t)\right) - \frac{\beta^2}{2}\right]dt + \beta dB_i$$

and

$$P_i = P_0 \exp\left[\sum_{i=1}^{n} c_i g_i\left(X_i(t)\right) - \frac{\beta^2}{2}\right]t + \beta B_i \tag{23}$$

Equation 23 denotes proportion of wealth invested in by the investor.

RESULTS AND DISCUSSION

Fitting to distribution of empirical data: We now want to measure how good our model fits to empirical entreprenual financial data. The data sets obtained consist of the closing prices for the stock denoted by $(X_i)_{i \le i < n}$. The series of log returns is obtained by Vi = log $(X_i/X_{i-1})_{2 \le i < n}$. The parameters in the logistic distribution are then estimated. μ_i and β_i are the sample mean and sample standard deviation, λ is estimated by $\lambda = e^{-\mu}$ (Olapade, 2009) and the parameter b estimated by b = $1/\beta$. We do the parameter estimation for two data sets, namely Small Scale Investment (SSI) and Large Scale Investment (LSI) from December, 2008 and 13 trading months ahead. The data from the small scale business (in thousands of naira) are as follow 0.36, 0.66, 0.82, 1.30, 2.33, 3.78, 4.97, 7.07, 9.02, 12.12, 17.74, 23.93, 34.91. For these data $\mu = 9.16$, $\beta = 18.44$, b = 0.0542, Γ (b) = 17.97, Γ (2b) = 8.7399, $\lambda = 0.0001$. Next, we consider the sample data of large scale business (in million of naira) as follows: 0.14, 0.15, 0.20, 0.21, 0.37, 0.68, 1.20, 1.84, 2.08, 2.13, 2.29, 3.33, 5.62. for these $\mu = 1.56$, β = 2.79, b = 0.3584, Γ (b) = 2.4838, Γ (2b) = 1.2732, $\lambda = 0.2101$.

CONCLUSION

The logistic function is a general stochastic measurement model, it has been used here as a basis for measuring the risk incurred by a financial assets returns. It is observed that initial stage of growth of the worth of a business enterprise is approximately exponential. At a time the growth slows. This could be due to diversification(invetsting in more than one stock), since the returns on different stocks do not move in exactly the same way all the time. Ultimately, the growth of the firm is stable. Likely, it may not be affected by risk since diversification reduces risk.

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