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Mathematical Model of Blood Flow Through a Tapered Artery with Mild Stenosis and Hematocrit

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Abstract: High wall shear stress cause the innermost membrane of an artery or a vein thickening but may also activate platelets cause platelet aggregation and finally result in the formation of a thrombus and there is no suspicion that height of the stenosis is a more important factor influencing blood flow than tapering.

Key words: Wall shear stress, resistance parameter, Hematocrit, tapered artery, mild stenosis, India

INTRODUCTION

Severe stenosis may cause critical flow conditions, related to artery collapse which leads directly to heart attack and stroke. It also causes pressure changes at the throat of stenosis and shear stress changes direction in the region just distal to stenosis under unsteady conditions (Carrocio *et al*., 2002; Frank *et al*., 2002). Different mathematical models have been studied by some researchers to explore the various aspects of blood flow in stenosed artery (Smith *et al*., 2002; Shukla *et al*., 1980a, b). Tu *et al* . (1992) have studied the Pulsatile flow of blood in stenosed artery. Verma and Parihar (2009) have considered the behavior of blood in stenosed artery in the presence of magnetic effect and Hematocrit.

Blood is made up of a suspension of particles in a solution of proteins and electrolytes called plasma. Erythrocytes, leukocytes and platelets are the main constituents of blood. The erythrocytes or Red Blood Cells (RBCs) are more than a thousand times more numerous than the leukocytes or White Blood Cells (WBCs) and much larger than platelets. For this reason, the flow properties of blood mainly involve the RBCs. The Hematocrit (percentage of the blood volume that is made up of red blood cells) is the major determinant of blood viscosity. Initially red blood cells have round shape (Fig. 1a). But in people who have sickle-cell anemia, the blood cells break down. They lose their round shape and take the form of a sickle, a farmer's implement (Fig. 1b), (Bonn, 1999). These sickled blood cells get stuck in the small veins of the body, blocking other red blood cells from passing. When the flow of blood to a part of the body is reduced, the oxygen supply to that part of the body is cut off and cells begin to die. The wall shear stress distribution is an important diagnostic factor for examining the flow characteristics of the blood through the arteries. There is no question that tapering is a significant aspect of mammalian arterial system. Keeping in these views in this study, an attempt has been made to find the effects of the stenosis and Hematocrit on the flow rate, wall shear stress and resistance parameter for Newtonian fluid through the tapered artery.

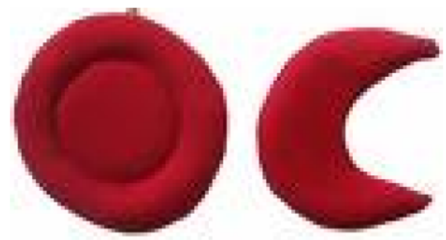

Fig. 1: a) Round red blood cell and b) Obstructed red blood cell

Development of the mathematical model: For the development of mathematical model in the present analysis, following assumptions are made:

- Blood is assumed to be a Newtonian fluid
- Motion of the fluid is laminar and steady
- The inertia term is neglected as the motion is slow
- No body force acts on the fluid
- There is no slip velocity at the wall
- Cylindrical polar co-ordinates are used
- The axis of symmetry of the artery taken as z-axis
- Stenosis is mild
- Height of the stenosis is much less than the length of the stenosis
- Tapered artery is taken
- Tapered angle is very very small

However, it is also assumed that there is no flow of blood in θ-direction in the artery, so system is reduced to

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(r, z) co-ordinates. Therefore, the equations of motion in r and z directions and equation of continuity are written as follows (Bird *et al*., 1960):

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right) \quad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right) \quad (2)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z}=0 \quad (3)$$

Where:

u and w = Radial velocity component and axial velocity component, respectively
p = The pressure of fluid
n = The constant density of blood
r = Radial co-ordinate
z = Axial co-ordinate
μ = Viscosity of blood

Here, the magnitude of radial velocity (u) is very less in comparison to magnitude of axial velocity (w) (i.e., u<<w), variation of velocity gradient in z direction is very less in comparison to velocity gradient in r direction:

$$\left(\frac{\partial w}{\partial z} \prec\prec \frac{\partial w}{\partial r}\right) \text{ and } \frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial z^2}$$

also neglected, therefore Eq. 1 and 2 are reduced as:

$$\frac{\partial p}{\partial z}=\frac{\mu}{r}\left(r\frac{\partial^2 w}{\partial r^2}+\frac{\partial w}{\partial r}\right) \quad (4)$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial r}=0 \quad (5)$$

It is seen from Eq. 5, the pressure gradient is the function of z (p = p (z)) only which caused the motion of flow, therefore:

$$\frac{\partial p}{\partial z}=\frac{dp}{dz}$$

The boundary conditions are shown as:

$$\frac{\partial w}{\partial r}=0 \qquad r=0 \quad 6\,(a)$$

$$w=0 \qquad r=R(z) \quad 6\,(b)$$

By applying the boundary conditions, maximum velocity at the centerline (6 (a)) and no-slip velocity at the wall (for finite velocity, 6 (b)) on Eq. 4, the velocity profile of the fluid through the artery is shown as:

$$w=\left(\frac{dp/dz}{4\mu}\right)\left(R^2-r^2\right) \quad (7)$$

Guo-Tao *et al.* (2004) discussed the physical model of tapered artery with stenosis. In such a case the geometry of the artery (Fig. 2) is modeled mathematically as follows:

$$R(z)=\begin{cases} R_1-m(z+L)\ ; & 0\le z\le d_0 \\ R_1-m(z+L)-\frac{h\cos\varphi}{2}\left[1+\cos\left(\frac{\pi z}{L_0}\right)\right] & d_0\le z\le L_0+d_0 \\ R_1-m(z+L) & L_0+d_0\le z\le L \end{cases} \quad (8)$$

Where:

R(z) = Effective radius of the tapered artery
R_1 = The radius of the untapered artery
φ = The angle of tapering
H = h cosΦ = The height of stenosis in tapered artery
L_0 = The length of stenosis
m = tanφ = Slope of tapered vessel

To find the effect of Hematocrit on the flow rate, wall shear stress and resistance parameter, the relation of blood viscosity and the Hematocrit which is quite effective is shown as Einstein (1906):

$$\mu=\mu_p(1+2.5H) \quad (9)$$

Where:

H = The hematocrit
μ_p = The viscosity of plasma

Flow rate: Flow rate for Newtonian fluid is defined as:

$$Q=\int_0^R 2\pi w r\,dr \quad (10)$$

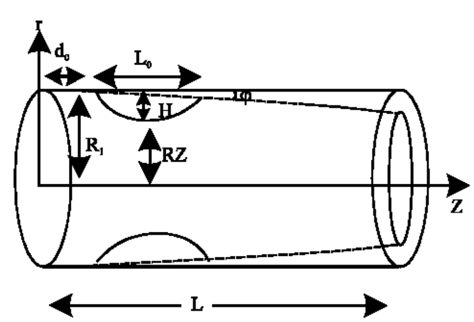

Fig. 2: Physical model and co-ordinate system

Using Eq. 7 into Eq. 10, flow rate is shown as:

$$Q = \frac{\pi G R^4}{8\mu} \tag{11}$$

Where, G = dp/dz. Thus, the expression of total volumetric flow flux for the tapered artery is shown as:

$$Q_L = \frac{\pi G}{8\mu}\int_0^L R^4 dz \tag{12}$$

By using the geometry of artery (Fig. 2) as shown in Eq. 8 into Eq. 12, the expression of non-dimensional flow rate (n.d. Q_L, $\bar{Q}_L$) is shown as:

$$\bar{Q}_L = \frac{40 m\mu Q_L}{\pi G} = [A + B + C] \tag{13}$$

Where:

$$A = \left[(R_1 - mL)^5 - (R_1 - m(d_0 + L))^5\right]$$

$$B = \left[(R_1 - h\cos\varphi - m(d_0 + L))^5 - (R_1 - m(d_0 + L_0 + L))^5\right]$$

$$C = \left[(R_1 - m(d_0 + L_0 + L))^5 - (R_1 - 2mL)^5\right] \tag{14}$$

By using Eq. 9 and 13, the expression of non-dimensional $\bar{Q}_L$ in terms of Hematocrit is shown as:

$$\bar{Q}_L = \frac{1}{1 + 2.5H}[A + B + C] \tag{15}$$

Wall shear stress: The constitutive relationship for the Newtonian fluid is given as:

$$\tau = \mu\left(-\frac{\partial w}{\partial r}\right) \tag{16}$$

From Eq. 4 and 11, the wall shear stress, τ_L for the tapered artery is given as:

$$\tau_L = \frac{8\mu Q}{\pi}\int_0^L R^{-3} dz \tag{17}$$

Using Eq. 8 in Eq. 17, the expression of non dimensional wall shear stress (n.d.τ_L, $\bar{\tau}_L$):

$$\bar{\tau}_L = \frac{m\pi\tau_L}{2Q\mu} = [A_1 + B_1 + C_1] \tag{18}$$

Where:

$$A_1 = \left[\frac{1}{(R_1 - m(d_0 + L))^2} - \frac{1}{(R_1 - mL)^2}\right]$$

$$B_1 = \left[\frac{1}{(R_1 - h\cos\varphi - m(d_0 + L_0 + L))^2} - \frac{1}{(R_1 - h\cos\varphi - m(d_0 + L))^2}\right]$$

$$C_1 = \left[\frac{1}{(R_1 - 2mL)^2} - \frac{1}{(R_1 - m(d_0 + L_0 + L))^2}\right] \tag{19}$$

Similarly, the expression of non-dimensional τ_L (n.d.τ_L) from Eq. 9 and 18 in terms of Hematocrit is shown as:

$$\bar{\tau}_L = (1 + 2.5H)[A_1 + B_1 + C_1] \tag{20}$$

Resistance parameter: The resistance to flow λ (resistance parameter) is defined as follows:

$$\lambda = \frac{p_1 - p_0}{Q} \tag{21}$$

From Eq. 11 and using the conditions that the inlet pressure $p = p_1$ at $z = 0$ and outlet pressure $p = p_0$ at $z = L$, the resistance parameter is shown as:

$$\lambda = \frac{8\mu}{\pi}\int_0^L \frac{dz}{R^4} \tag{22}$$

Thus, the non dimensional resistance parameter, $\bar{\lambda}$ for the tapered artery is given as:

$$\bar{\lambda} = \frac{R_1^4}{3mL}[A_2 + B_2 + C_2] \tag{23}$$

Where:

$$A_2 = \left[\frac{1}{(R_1 - m(d_0 + L))^3} - \frac{1}{(R_1 - mL)^3}\right]$$

$$B_2 = \left[\frac{1}{(R_1 - h\cos\varphi - m(d_0 + L_0 + L))^3} - \frac{1}{(R_1 - h\cos\varphi - m(d_0 + L))^3}\right]$$

$$C_2 = \left[\frac{1}{(R_1 - 2mL)^3} - \frac{1}{(R_1 - m(d_0 + L_0 + L))^3}\right]$$

Thus, using Eq. 9 in Eq. 23, the non-dimensional resistance parameter $\bar{\lambda}$ for the tapered artery in the term of Hematocrit is shown as:

$$\bar{\lambda} = \frac{(1 + 2.5H)R_1^4}{L}[A_2 + B_2 + C_{21}] \quad (24)$$

Wall shear stress is an important factor in the study of blood flow. Accurate predictions of the distribution of the wall shear stress are particularly useful for the understanding of the effect of blood flow on endothelial cells. Since low shear stress has been related to the initiation and localization of arteriosclerosis, hence the post-stenotic zone could be prone to the propagation of the disease.

From the study, it is observed that there is a negative correlation between height of stenosis, length of stenosis, slope of tapered vessel, Hematocrit and the flow rate but there is a positive correlation between height of stenosis, length of stenosis, slope of tapered vessel, Hematocrit and the wall shear stress and resistance parameter in tapered vessel.

Thus, the investigation represents that as the height of stenosis, length of stenosis, slope of tapered vessel and Hematocrit increase, the flow rate decreases (Fig. 3a-c) but the wall shear stress (Fig. 4a-c) and resistance parameter in the tapered vessel increases (Fig. 5a-c).

Numerical solution of $\bar{Q}_L$ (Fig. 3d), $\bar{\tau}_L$ (Fig. 4d) and $\bar{\lambda}$ (Fig. 5d) for tapered artery is also calculated to compare with the analytic solution with respect to shape of stenosis and Hematocrit and it is found that numerical solutions and analytical solutions are close to each other.

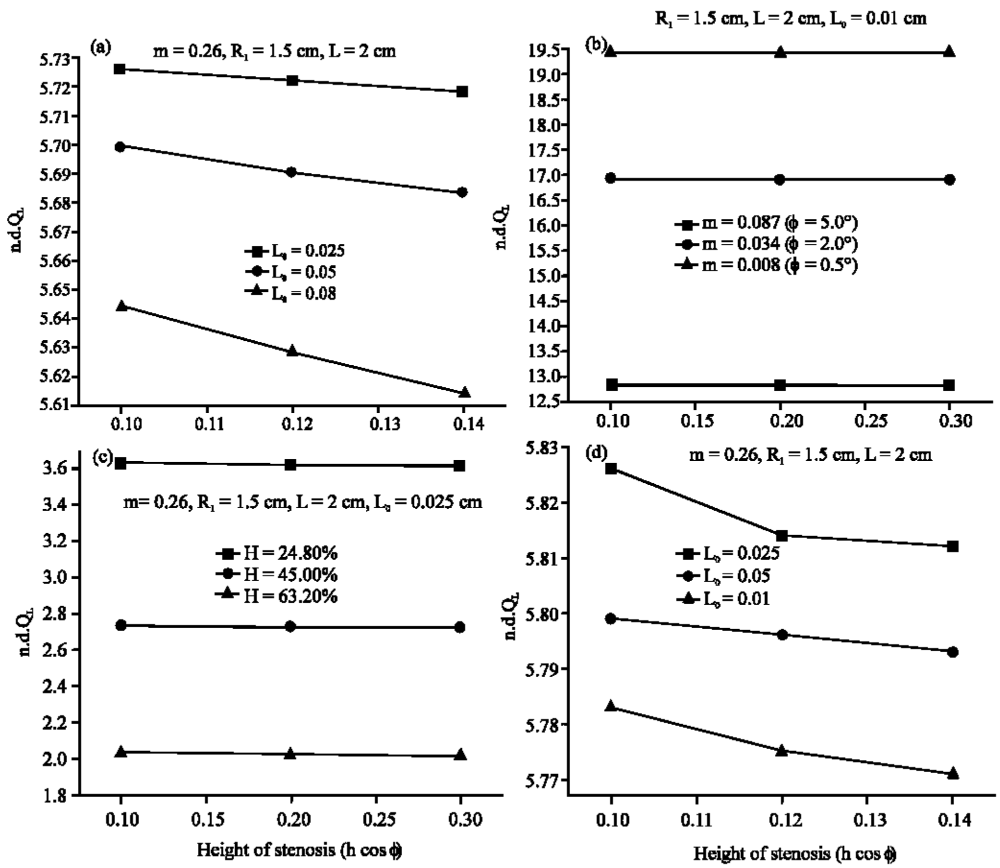

Fig. 3: Represents (a, b) the variation of $\bar{Q}_L$ against height of stenosis for different length of stenosis and slope of tapered vessel, respectively, while other parameters are fixed, represents (c, d) the variation of $\bar{Q}_L$ against height of stenosis for different Hematocrit and numerical solution for different length of stenosis, respectively while other parameters are fixed

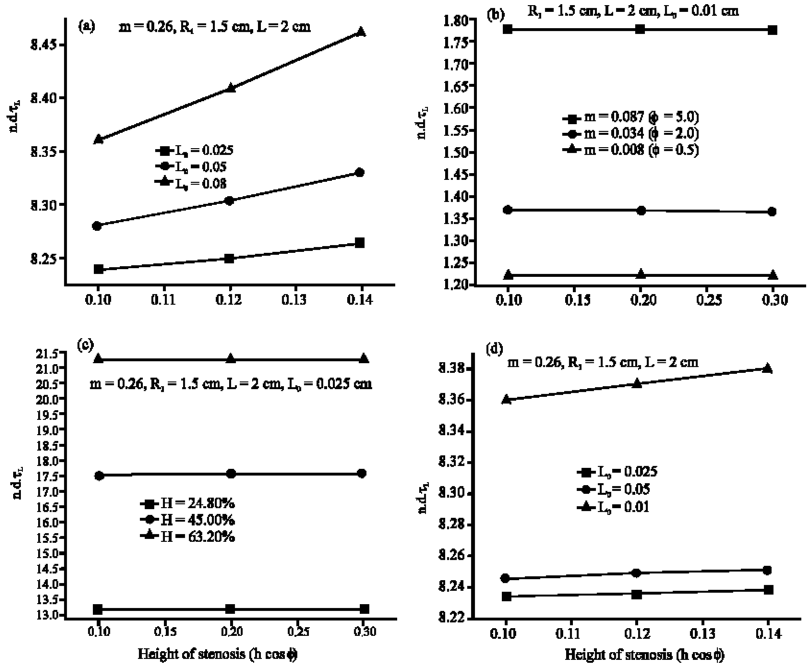

Fig. 4: Represents (a, b) the variation of $\bar{\tau}_L$ against height of stenosis for different length of stenosis and slope of tapered vessel, respectively, while other parameters are fixed, represents (c, d) the variation of $\bar{\tau}_L$ against height of stenosis for different Hematocrit and numerical solution for different length of stenosis, respectively while other parameters are fixed

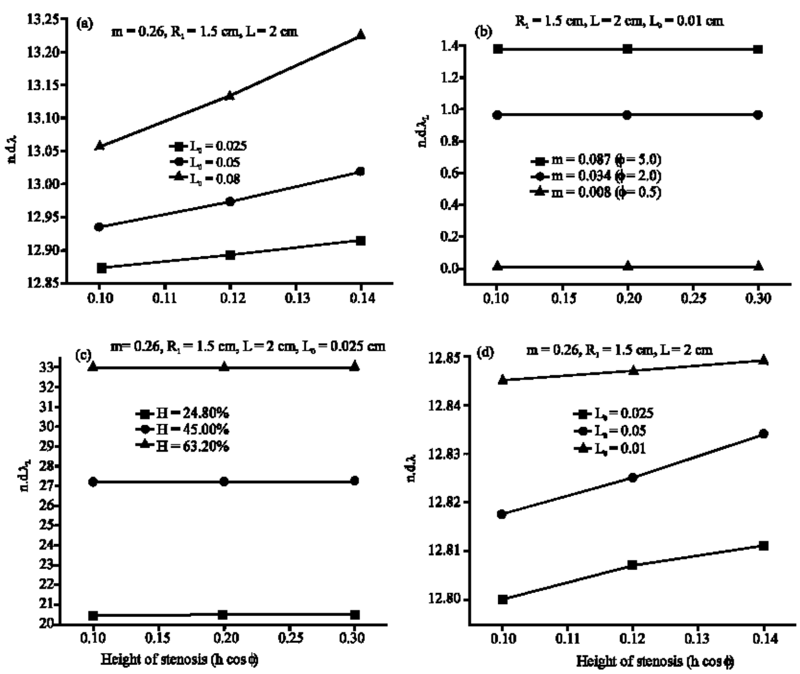

Fig. 5: Represents (a, b) the variation of $\bar{\lambda}$ against height of stenosis for different length of stenosis and slope of tapered vessel, respectively, while other parameters are fixed, represents (c, d) the variation of $\bar{\lambda}$ against height of stenosis for different Hematocrit and numerical solution for different length of stenosis, respectively while other parameters are fixed

CONCLUSION

In this study, a mathematical model has been developed to study the effect of stenosis and Hematocrit on flow rate, wall shear stress and resistance parameter through tapered artery under stenotic conditions by considering laminar flow, rigid walls and Newtonian fluid. The numerical results for these expressions also have been carried out.

REFERENCES

Frank, A.O., P.W. Walsh and J.E. Jr. Moore, 2002. Computational fluid dynamics and stent design. Artif. Organs, 26: 614-630.

Bird, R.B., W.E. Stewart and E.N. Lightfoot, 1960. Transport Phenomena. John Wiley and Sons, New York.

Bonn, D., 1999. Plaque detection: The key to tackling atherosclerosis. Lancet, 354: 656-656.

Carrocio, A., P.L. Faries and N.J. Morrissey, 2002. Predicting iliac limb occlusions after bifurcated aortic stent grafting: Anatomic and device-related causes. J. Vasc Surg., 36: 679-684.

Einstein, A., 1906. Eine neue bestimmung der molekuldimensionen. Ann. Phys., 19: 289-306.

Guo-Tao, L., W. Xian-Ju, A. Bao-Quan and L. Liang-Gang, 2004. Numerical study of pulsating flow through a tapered artery with stenosis. Chinese J. Phys., 42: 401-409.

Shukla, J.B., R.S. Parihar and S.P. Gupta, 1980a. Effects of peripheral layer viscosity on blood flow through the artery with mild stenosis. Bull. Math. Biol., 42: 797-805.

Shukla, J.B., R.S. Parihar, B.R.P. Rao and S.P. Gupta, 1980b. Effects of peripheral layer viscosity on peristaltic transport of a bio-fluid. J. Fluid Mech., 97: 225-235.

Smith, N.P., A.J. Pullan and P.J. Hunter, 2002. An anatomically based model of transient coronary blood flow in the heart. SIAM J. Applied Mathematics, 62: 990-1018.

Tu, C., M. Deville, L. Dheur and L. Vanderschuren, 1992. Finite element simulation of pulsatile flow through arterial stenosis. J. Biomechanics, 25: 1141-1152.

Verma, N. and R.S. Parihar, 2009. Effects of magneto-hydrodynamic and hematocrit on blood flow in an artery with multiple mild stenosis. J. Applied Math. Comput., 1: 30-46.