Journal of Modern Mathematics and Statistics 3 (4): 69-72, 2009
ISSN: 1994-5388

A Tool for Constructing Pair-Wise Balanced Incomplete Block Designs

E.O. Effanga, J.A. Ugboh, E.I. Enang and B.E.A. Eno
Department of Mathematics/Statistics and Computer Science,
University of Calabar, Calabar, Nigeria

Abstract: This study deals with the construction of Pair-wise Balanced Incomplete Block Designs (PBIBD). We developed a non-linear non-preemptive binary integer goal programming model for the construction of PBIBD for any set of parameters (b, k, t, r, λ). The model so developed is tested using two set of parameters (b = 4, k = 4, t = 3, r = 3, λ = 2) and (b = 7, k = 7, t = 3, r = 3, λ = 1). It is shown that the PBIBD is not unique for any given set of parameters and that the design obtained by our model is D-optimal.

Key words: Goal programming, non-linear, non-preemptive, design matrix, information matrix, D-optimal

INTRODUCTION

Statistics deals with variability and how it can be controlled. In planning and conducting experiments, one of the factors used in controlling variability is the grouping of the experimental units into homogeneous sub-groups called blocks. There are many ways of blocking the experimental units in a comparative experiments with k treatments. If homogeneous blocks of size k are available to accommodate all the k treatments, a Randomized Complete Block Design (RCBD) is used.

In many situations, experimenter may not be able to run all the treatments combinations in each block, may be due to shortage of experimental materials or physical size of the block, that is block size t is less than the number of treatments k. In such situations, an incomplete block design is used by Quinoulli (1953) and Motgomery (1976).

When all treatments comparisons are equally important, the treatment combination used in each block must be selected in a balanced manner. Several methods for constructing balanced incomplete block designs are available (Federer, 1993, 1998; Khare and Federer, 1981; Patterson and Williams, 1985, 1976).

Also, there are computer software packages and toolkits, which will construct optimal or near optimal balanced incomplete block designs (Nguyen, 1993a, b; 1994, 1997; Nguyen and Williams, 1993; Federer *et al*., 1998). Tables of balanced incomplete block designs are given by Fisher and Yates (1953), Davies (1956) and Cochran and Cox (1957).

Each of the methods, toolkits and tables has its own limitations. To overcome these limitations, we developed in this study a non-linear non-preemptive goal programming model for the construction of the balanced incomplete block designs. Constructing a balanced incomplete block design is equivalent to constructing a design matrix.

PAIR-WISE BALANCED INCOMPLETE BLOCK DESIGNS

A randomized incomplete block design is said to be pair-wise balanced if treatments combinations are selected in each block so that every pair of treatments occur together in the same number of blocks. Suppose we to compare k treatments, the maximum number of blocks b required to construct a pair-wise balanced incomplete block design with only t treatments appearing in each block is b = (k/t) (Montgomery, 1976).

Let each of the k treatments appear in r of the b blocks and let every pair of treatments appear together in λ blocks, then a pair-wise balanced incomplete block design must satisfy the following conditions:

$$Kr = bt$$

$$t<k$$

$$b>r$$

$$b\geq k$$

$$\lambda = \frac{r(t-1)}{k-1} \text{ is an integer}$$

The design is called incomplete in the sense that t<k and is balanced in the sense that the parameter λ is constant. If the b = k, the design is said to be symmetric.

Corresponding Author: E.O. Effanga, Department of Mathematics/Statistics and Computer Science, University of Calabar, Calabar, Nigeria

A b by k matrix $X = (x_{ij})$ is called a design matrix for the pair-wise balanced incomplete block design with parameters, b, k, t, r, λ, if $x_{ij} = 1$, if treatment j appears in block I, $x_{ij} = 0$, otherwise

$$\sum_{j=1}^{k} x_{ij} = t,\ i = 1, 2, 3, \ldots, b \qquad (1)$$

$$\sum_{i=1}^{b} x_{ij} = r,\ j = 1, 2, 3, \ldots, k \qquad (2)$$

$$\sum_{i=1}^{b} x_{il}x_{im} = \lambda,\ m > l = 1, 2, 3, \ldots, k \qquad (3)$$

The matrix X^TX is called the information matrix or the matrix identity of the design. It has been shown that, $X^TX = (r - \lambda)I + \lambda J$, where I is a b by b identity matrix and J is b by b matrix of 1's (Federer, 1998).

A design is called a D-optimal design if the determinant of its information matrix is greater than or equal to the determinant of the information matrix of any alternative design (Hedayat and Pesotan, 2006).

GOAL PROGRAMMING

Goal programming is a mathematical tool for the analysis of problems involving multiple but conflicting objectives (Ignizio, 1978). The basic approach of goal programming is to establish a specific numeric goal for each of the objectives, formulate an objective function for each objective and eek a solution that minimizes the sum of deviations or sum of weighted deviations.

Let $f_L(x)$ denote the lth objective and G_L be the specified target value for the objective. Basically, there are three possibilities for the relation between $f_L(x)$ and G_L, namely (Eiselt *et al.*, 1987):

$$f_L(x) \le G_L$$

$$f_L(x) \ge G_L$$

$$f_L(x) = G_L$$

In the first case, the objective is to minimize the underachievement, that is negative deviation from the goal target, designated d^-_L.

In the second case, the objective is to minimize the overachievement, that is positive deviation from the goal target, designated d^+_L.

In the third case, the objective is to minimize the underachievement and overachievement simultaneously. In any of the three cases mentioned above the goal constraint is stated as $f_L(x) + d^+_L - d^-_L = G_L$ (Eiselt *et al.*, 1987).

When all the goals are of equal importance, we talk of non-preemptive goal programming. But when the goals are not all of equal importance, there is ordering of priority levels for the goals and we talk of preemptive goal programming. Goal programming can also be classified as linear or non-linear. For a non-preemptive linear goal programming the solution technique corresponds to the standard simplex methods of solution. In a preemptive case, there are two general approaches to the solution of linear goal programming, namely the sequential linear programming algorithm and the multiphase simplex method. These two methods are described by Eiselt *et al.* (1987).

DEVELOPMENT OF THE MODEL

The following indices, variables and constants are required in order to formulate the model.

Indices:

i = Block's index
j = Treatment's index
l = Treatment's index
m = Treatment's index

Variables:

r^+_j = Positive deviation from the rth goal's target for each treatment j
r^-_j = Negative deviation from the rth goal's target for each treatment j
t^+_i = Positive deviation from the tth goal's target for each block i
t^-_i = Negative deviation from the tth goal's target for each block i
λ^+_{lm} = Positive deviation from the λth goal's target for each treatment pair (l , m)
λ^-_{lm} = Negative deviation from the λth goal's target for each treatment pair (l , m)
x_{ij}: 1 = If treatment j appears in block I, 0, otherwise

Constants:

b = Number of blocks
k = Number of treatments
t = Number of treatments in each block
r = Number of blocks each treatment appears
λ = Number of blocks each pair of treatments appears together

Based on the above definitions of indices, variables and constants, the goal objectives are as given in Eq. 1-3.

The corresponding goal constraints are as follows:

$$\sum_{j=1}^{k} x_{ij} + t_i^{+} - t_i^{-} = t, \; i = 1, 2, 3, \ldots, b \qquad (4)$$

$$\sum_{j=1}^{k} x_{ij} + r_j^{+} - r_j^{-} = r, \; j = 1, 2, 3, \ldots, k \qquad (5)$$

$$\sum_{i=1}^{b} x_{il} x_{im} + \lambda_{lm}^{+} - \lambda_{lm}^{-} = \lambda \qquad (6)$$

$$m > l = 1, 2, 3, \ldots, k$$

The model is therefore,

$$\text{Min}\left\{ \sum_{j=1}^{k} (r_j^{+} - r_j^{-}) + \sum_{i=1}^{b} (t_i^{+} - t_i^{-}) + \sum_{m>1}^{k} \sum_{l=1}^{k} (\lambda_{lm}^{+} + \lambda_{lm}^{-}) \right\} \qquad (7)$$

$$x_{ij} = 0 \text{ or } 1 \qquad (8)$$

Subject to:

$$i = 1, 2, 3, \ldots, b; \; j = 1, 2, 3, \ldots, k$$

$$r_j^{+}, r_j^{-} \geq 0, \; j = 1\ 2, 3, \ldots, k \qquad (9)$$

$$t_i^{+}, t_i^{-} \geq 0, \; I = 1, 2, 3, \ldots, b \qquad (10)$$

$$\lambda_{lm}^{+}, \lambda_{lm}^{-} \geq 0, \; m > l = 1, 2, 3, \ldots, k \qquad (11)$$

TESTING THE MODEL

Inputting the parameters of the following designs into the model and solving via the available computer software packages we obtained the design matrices.

$$b = 4, k = 4, t = 3, r = 3, \lambda = 2$$

The solution is $x_{11} = 0$, $x_{12} = 1$, $x_{13} = 1$, $x_{14} = 1$, $x_{21} = 1$, $x_{22} = 1$, $x_{23} = 0$, $x_{24} = 1$, $x_{31} = 1$, $x_{32} = 1$, $x_{33} = 1$, $x_{34} = 0$, $x_{41} = 1$, $x_{42} = 0$, $x_{43} = 1$, $x_{44} = 1$.

Therefore, the design matrix is:

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix} \qquad (12)$$

The corresponding design is therefore,

	Treatments			
Block	1	2	3	4
1		*	*	*
2	*	*		*
3	*	*	*	
4	*		*	*

$$b = 7, k = 7, t = 3, r = 3, \lambda = 1$$

The solution is $x_{11} = 1$, $x_{12} = 1$, $x_{13} = 0$, $x_{14} = 1$, $x_{15} = 0$, $x_{16} = 0$, $x_{17} = 0$, $x_{21} = 0$, $x_{22} = 1$, $x_{23} = 1$, $x_{24} = 0$, $x_{25} = 1$, $X_{26} = 0$, $x_{27} = 0$, $x_{31} = 0$, $x_{32} = 0$, $x_{33} = 1$, $x_{34} = 1$, $x_{35} = 0$, $x_{36} = 1$, $x_{37} = 0$, $x_{41} = 0$, $x_{42} = 0$, $x_{43} = 0$, $x_{44} = 1$, $x_{45} = 1$, $x_{46} = 0$, $x_{47} = 1$, $x_{51} = 1$, $x_{52} = 0$, $x_{53} = 0$, $x_{54} = 0$, $x_{55} = 1$, $x_{56} = 1$, $x_{57} = 0$, $x_{61} = 0$, $x_{62} = 1$, $x_{63} = 0$, $x_{64} = 0$, $x_{65} = 0$, $x_{66} = 1$, $x_{67} = 1$, $x_{71} = 1$, $x_{72} = 0$, $x_{73} = 1$, $x_{74} = 0$, $x_{75} = 0$, $X_{76} = 0$, $x_{77} = 1$.

The design matrix is therefore,

$$X = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (13)$$

The corresponding design is:

	Treatments						
Block	1	2	3	4	5	6	7
1	*	*		*			
2		*	*		*		
3			*	*		*	
4				*	*		*
5	*				*	*	
6		*				*	*
7	*		*				*

ALTERNATIVE DESIGNS

The balanced incomplete block design is not unique for a given set of parameters, b, k, t, r, λ. Alternative designs for the same set of parameters can be obtained by interchanging rows or columns of a design matrix.

Theorem 1: If X_1 and X_2 are two design matrices for the same pair-wise balanced incomplete block design, then their information matrices are equal.

Proof: In the product $X^T X$, a diagonal entry is the inner product of X with itself.
That is,

$$\sum_{i=1}^{b} x_{ij}^2 = \sum_{i=1}^{b} x_{ij} = r,\ j = 1, 2, \ldots, k$$

Also in the product $X^T X$, an off diagonal entry is the inner product of columns of X. That is,

$$\sum_{i=1}^{b} x_{il} x_{im} = \lambda,\ m > 1 = 1, 2, \ldots, k$$

Now if any two rows or any two columns of X are interchanged, the diagonal entries and the off diagonal entries remains unchanged. Hence,

$$X_1^T X_1 = X_2^T X_2$$

Theorem 2: Every pair-wise balanced incomplete block design satisfying the set of parameters is D-optimal.

Proof: Since, all the information matrices of the a design are equal and the determinants of equal matrices are also equal, it follows that every pair-wise balanced incomplete block design is D-optimal.

CONCLUSION

In this study, some of the contributions made in this study towards the construction of pair-wise balanced incomplete block designs is summarized. First, the concepts of balanced incomplete block designs, design matrices, information matrices and D-optimality are reviewed. We developed a non-linear non-preemptive linear goal programming model for the construction of a D-optimal pair-wise balanced incomplete block designs. We observed that the design is not unique under the same set of parameters and that alternative designs can be obtained from a given design by interchanging rows or columns of the corresponding design matrix. The model developed in this study have been tested for the existence of feasible using two sets of parameters, $b = 4$, $k = 4$, $t = 3$, $r = 3$, $\lambda = 2$ and $b = 7$, $k = 7$, $t = 3$, $r = 3$, $\lambda = 1$. The solutions to the models were obtained using an optimizer in a corel quarttro pro computer software package.

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