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A New Derivation of Continuous Collocation Multistep Methods Using Power Series as Basis Function

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Abstract: Some derivations of continuous linear multistep methods are given in this study. The study provides, the use of both collocation and interpolation techniques to obtain the schemes. Rather than using Chebyshev polynomials as basis function as it was always done in the past, we introduced the use of direct form of power series as an alternative to the derivation of these schemes. Multistep methods have over the years been one of the most popular and acceptable methods for generating solutions to initial value problems of ordinary differential equations.

Key words: Chebyshev polynomials, optimal order, continuous collocation, scheme, alternative

INTRODUCTION

Over the years, several researchers have considered the collocation methods as ways of generating numerical solutions to Ordinary Differential Equations (ODE). The collocation method is dated as far back as 1956 in the research carried out by Lanczos (1956) and Brunner (1996).

Lanczos (1956) introduced the standard collocation method with some selected points. However, Fox and Parker (1968) introduced the use of Chebyshev polynomials in the collocating the existing method, which was captioned as the Lanczos-Tau method. Also, Ortiz (1969) went on to discuss the general Tau method, which was later extended by Onumanyi and Ortiz (1984) to a method known as the collocation Tau method. The standard collocation method with method of selected points provides a direct extension of the Tau method to linear ODEs with non polynomial coefficients. The collocation Tau method however, uses the Chebyshev perturbation terms to select the collocation points. Okunuga and Onumanyi (1985, 1986) gave the generalized Tau method, which permits exact fractional values in the computation with >1 τ-term as perturbation on the right hand side of the linear differential equation. This was later extended to non-linear differential equations with some linearization being introduced on the Tau method by Okunuga and Sofoluwe (1990).

Other researchers such as Onumanyi *et al*. (1993), Adeniyi and Alabi (2006) and Fatokun (2007) have however, introduced some other variants of the collocation methods, which recently led to some continuous collocation approach. The introduction of the continuous collocation schemes as against the discrete schemes includes the fact that better global error can be estimated and approximations can be equally obtained at all interior points. Furthermore, the introduction of the continuous collocation method has been able to bridge the gap between the discrete collocation methods and the conventional multistep methods. Thus, it is possible to write the Linear Multistep Methods (LMM) in form of some continuous collocation schemes.

Various techniques have been suggested for the derivation of linear multistep methods. In this study, we propose the use of generalized power series as a basis function on the collocation method, which will lead to some continuous collocation schemes and are easily linked to the Linear Multistep Methods (LMM).

GENERAL COLLOCATION METHOD

It is a known fact that the Linear Multistep Methods (LMM) have over the years being very useful in generating solutions to IVP in ODEs.

Consider the Initial Value Problem (IVP) Eq. 1:

$$y'(x) = f(x, y(x)),\ y(x_0) = y_0 \tag{1}$$

The Linear Multistep Methods (LMM) for solving the IVP Eq. 1 can be developed by some collocation and interpolation techniques.

The general k-step method or LMM of step number k given by Lambert (1973) is written as:

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$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j},\ \alpha_k \neq 0 \qquad (2)$$

where:

α_j and β_j = Uniquely determined
h = The step length

Such that

$$x_{k+n} - x_k = nh$$

The Linear Multistep Methods (LMM) generate discrete multistep schemes Eq. 2, which are used for solving the IVP Eq. 1.

There have been various forms of the LMM, which were Henrici (1962), Lambert (1973), Fatunla (1988) and Butcher (2003).

Many of the linear multistep schemes given by Eq. 2 have been proved to have satisfied some stability conditions. Due to the nature of various problems, other variants of the LMM do exist in the study. Some of these include the hybrid LMM, second derivative LMM and the general multiderivative LMM (Okunuga, 1999; Lambert, 1991; Butcher, 2003). These are variously developed to improve the accuracy of the results being obtained when solving the IVP Eq. 1 and other higher order linear ODEs.

Thus, the continuous collocation approach, which require collocating at some points x_k of the linear k-step method Eq. 2 is rewritten in continuous form as:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j(x) f_{n+j} \qquad (3)$$

where, $\beta_j(x)$ is now defined as a function of x and it is continuously differentiable at least once. In this study, we developed some continuous multistep collocation methods with some collocation points taken at the grid points using some form of series or polynomials $(x - x_j)^k$ as the basis function.

Power series collocation: The Taylor polynomial is the ultimate in osculation. For a single argument x_0, the values of the polynomial and its first n derivatives are required to match those of a given function y(x) that is:

$$p^{(r)}(x_0) = y^{(r)}(x_0),\ i = 0, 1, 2, \ldots, n$$

The existence and uniqueness of such a polynomial is well known and they are classical results of analysis.

The Taylor formula settles the existence issue directly by exhibiting such polynomial in the form Eq. 4:

$$p(x) = \sum_{r=0}^{n}\frac{y^{(r)}(x_0)}{r!}(x - x_0)^r \qquad (4)$$

Analytic function has the property that for $n \to \infty$, the approximate function p (x) reduces to y (x).

Based on this argument, we propose a polynomial series in form of Eq. 4 as the basis function for deriving the LMM.

In the research, Onumanyi *et al.* (1993) derived some finite difference methods that lead to some linear multistep methods for the solution of initial value problems in ordinary differential equations of the form Eq. 1. By appropriate selection of points for both interpolation and collocation, many important classes of finite difference methods were recovered and new methods were generated. These authors also used a collocating function of the form:

$$y(x) = \sum_{k=0}^{M}\alpha_k x^k$$

Adeniyi and Alabi (2006) derived the continuous LMM by using some Chebyshev polynomial function as a basis function. The researchers proposed a collocating function of the form:

$$Y(x) = \sum_{j=0}^{M} a_j T_j(x)\left(\frac{x - x_k}{h}\right),\ x_k \le x \le x_{k+p}$$

where, $T_j(x)$ are some Chebyshev functions, which are used as basis function.

We however, propose in this study a basis function of the form in Eq. 5

$$y(x) = \sum_{r=0}^{M} a_j (x - x_k)^r \qquad (5)$$

which is in form of Eq. 4 and will be shown to have identical results and methods with the research of other previous researchers. We are able to generate more methods by our new approach, which makes this different from other previous reseaches. Equation 5 proposed here shall be used for both collocation and interpolation techniques that the methods may require.

The use of this basis function will permit us to derive some continuous LMM of various orders and consequently the discrete formulas are also obtained. We shall make comparison of our methods with those generated by using the Chebyshev polynomials. The power series (Eq. 5) permit smooth functions in which a_i's are suitably determined by collocation techniques, so as to generate some LMM in continuous form.

LINEAR MULTISTEP METHODS

The LMM have over the years been very useful in generating solutions to IVP in ODE. There has been various form of the LMM, which were derived by Lambert (1973, 1991), Fatunla (1988) and Butcher (2003). All of these schemes given in the form of Eq. 2 are in discrete form. Among the existing methods of deriving the LMM in discrete form include the interpolation approach, numerical integration, Taylor series expansion and through the determination of the order of the LMM. These various techniques were developed over the years because no single approach can really produce all existing multistep schemes. There are still some schemes that can be written in the form of Eq. 2 by fixing certain values for the coefficients of y(x) and f(x,y) which may not be easily obtained by the techniques shown above. Hence, the need to seek more approaches of deriving these all important schemes.

It is also useful to note that many of these schemes have been proved to have satisfied some stability conditions. Due to the nature of various problems, other variant of the LMM exist also in study. Some of these include the second derivative LMM. These are equally developed to improve the accuracy of the numerical results being obtained when solving the IVP.

In this study, we shall develop the continuous form of the LMM, which permits collocating at various points rather than the usual discrete formulas.

The derivation given in this study is quite different from the usual techniques given by Lambert (1991, 1973) and Butcher (2003), but will end up to yield the same LMM, which in this study could be written both in discrete and continuous form.

Definition 1: Consider the IVP,

$$y'(x) = f(x,y(x)),\ y(x_0) = y_0,$$
$$x \in (x_0, y_0),\ y(x),\ f(x,y) \in R^m$$

Where, we assume that there exist some Lipschitz constant L such that:

$$\|f(x,y) - f(x,z)\| \le L\|y - z\|,$$
$$\forall (x,y),\ (x,z) \in (x_0, y_0) \times R^m$$

This implies that the IVP has a unique solution.

Definition 2: The first characteristics polynomial of the LMM Eq. 2 is given by:

$$\rho(\xi) = \sum_{r=0}^{\infty} \alpha_r \xi^r$$

The methods for which $\rho(\xi) = \xi^k - \xi^{k-1}$ are called Adams methods, while those that have $\rho(\xi) = \xi^k - \xi^{k-2}$ are the Nystrom methods.

Derivation of continuous LMM: Consider the polynomial function:

$$y(x) = \sum_{j=0}^{M} a_j (x - x_k)^j \cong y(x),\quad x_k \le x \le x_{k+p}$$

over each of the sub-interval (x_k, x_{k+p}) of (a, b) where, M is appropriately chosen. This shall be used as basis function to derive some LMM in the continuous form.

The technique, which is being employed is using the trial or basis function:

$$Y(x) = \sum_{j=0}^{n+1} a_j (x - x_k)^j \cong y(x) \qquad (6)$$
$$x_k \le x \le x_{k+p}$$

This satisfies the unperturbed ODE:

$$Y'(x) = f(x, y(x)),\quad x_k \le x \le x_{k+p} \qquad (7)$$
$$Y(x_k) = Y_k$$

Collocating Eq. 7 at (n + 1) points x_{k+j}, j = 0, 1, 2,...n, and interpolating the trial polynomial (Eq. 6) at x_k to give the required (n + 2) equations for the unique determination of α_j.

Doing this, we write

$$f(x_{k+j}) = f_{k+j},\ j = 0,1,2,.... \qquad (8)$$
$$Y(x_k) = Y_k$$

To derive a one step LMM, we set n = 1, in Eq. (6), so that

$$Y(x) = a_0 + a_1(x - x_k) + a_2(x - x_k)^2 \qquad (9)$$

From Eq. 8, we have:

$$Y'(x_k) = f_k$$
$$Y'(x_{k+1}) = f_{k+1} \qquad (10)$$
$$Y(x_k) = Y_k$$

Using Eq. 9 in 10, we obtain the 3 equations:

$$Y(x_k) = a_0 = Y_k$$
$$Y'(x_k) = a_1 = f_k$$
$$Y'(x_{k+1}) = a_1 + 2a_2(x_{k+1} - x_k) = f_{k+1}$$

Representing this in the matrix form to get:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & h & 0 \\ 0 & h & 2h^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} Y_k \\ hf_k \\ hf_{k+1} \end{pmatrix}$$

On solving, α_2 is determined as:

$$a_2 = \frac{f_{k+1} - f_k}{2(x_{k+1} - x_k)}$$

Substituting in Eq. 9, we obtain:

$$Y(x) = Y_k + f_k(x - x_k) + \frac{f_{k+1} - f_k}{2(x_{k+1} - x_k)}(x - x_k)^2 \quad (11)$$

Equation 11 is the continuous formulation of a one step method.

To obtain its discrete form, we evaluate at $x = x_{k+1}$:

$$Y(x_{k+1}) = Y_k + f_k(x_{k+1} - x_k) + \frac{f_{k+1} - f_k}{2(x_{k+1} - x_k)}(x_{k+1} - x_k)^2$$

Which, reduces to:

$$Y_{k+1} - Y_k = \frac{h}{2}(f_{k+1} + f_k) \quad (12)$$

Equation 12 is the well-known trapezoidal method of order 2 and it is an implicit one-step scheme.

On the other hand, if we put $x = x_{k+2}$ in Eq. 11, we shall obtain the scheme:

$$Y_{k+2} - Y_k = 2hf_{k+1}$$

This is a two-step linear scheme and which is called the mid-point rule.

We can also derive some other two step methods by setting n = 2 in Eq. 6 that is:

$$Y(x) = \sum_{j=0}^{3} a_j (x - x_k)^j \quad (13)$$

This leads to:

$$Y'(x_k) = f_k$$
$$Y'(x_{k+1}) = f_{k+1}$$
$$Y'(x_{k+2}) = f_{k+2}$$
$$Y(x_k) = Y_k$$

This in turn is rewritten to include a_i as:

$$Y(x_k) = a_0 = Y_k$$
$$Y'(x_k) = a_1 = f_k$$
$$Y'(x_{k+1}) = 3a_3(x_{k+1} - x_k)^2 + 2a_2(x_{k+1} - x_k) + a_1$$
$$Y'(x_{k+2}) = 3a_3(x_{k+2} - x_k)^2 + 2a_2(x_{k+2} - x_k) + a_1$$

Representation on a matrix, yields the system:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & h & 2h^2 & 3h^3 \\ 0 & h & 4h^2 & 12h^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} Y_k \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \end{pmatrix}$$

Solving for a_2 and a_3, we obtain:

$$a_2 = \frac{1}{2(x_{k+1} - x_k)}\left[-\tfrac{1}{2}f_{k+2} + 2f_{k+1} - \tfrac{3}{2}f_k\right]$$

$$a_3 = \frac{1}{6(x_{k+1} - x_k)^2}(f_{k+2} - 2f_{k+1} + f_k)$$

Substituting for α_0, α_1, α_2, α_3 in Eq. 13, we obtain:

$$Y(x) = Y_k + f_k(x - x_k) + \frac{1}{2h}\left[-\frac{1}{2}f_{k+2} + 2f_{k+1} - \frac{3}{2}f_k\right](x - x_k)^2 + \frac{1}{6h^2}\left[f_{k+2} - 2f_{k+1} + f_k\right](x - x_k)^3 \quad (14)$$

Evaluating at $x = x_{k+2}$, we obtain the discrete form of Eq. 14 after simplification as:

$$Y_{k+2} - Y_k = \tfrac{1}{3}h\left[f_{k+2} + 4f_{k+1} + f_k\right] \quad (15)$$

Which is the Simpson's method, while Eq. 14 is the continuous formulation of the discrete scheme Eq. 15 and it is known to be of order 4 (Henrici, 1962).

The N-step optimal order: We shall at this point consider in a general form a LMM of optimal order with n steps. We consider our trial polynomial Eq. (6) that is:

$$Y(x) = \sum_{j=0}^{n+1} a_j (x - x_k)^j$$

On substituting into the IVP (Eq. 1) and collocating at n + 1 points x_{k+j}, j = 0, 1, 2,..., n and interpolating at x_k to give a (n + 2) systems of equations for the unique determination of:

$$a_j\text{'s},\ j=0,1,2,\ldots,n+1$$

We shall obtain:

$$\begin{aligned}Y(x) &= a_{n+1}(x-x_k)^{n+1}+a_n(x-x_k)^n+a_{n-1}(x-x_k)^{n-1}+\ldots\\ &\quad +a_3(x-x_k)^3+a_2(x-x_k)^2+a_1(x-x_k)+a_0\\ Y'(x) &= (n+1)a_{n+1}(x-x_k)^n+na_n(x-x_k)^{n-1}+(n-1)a_{n-1}(x-x_k)^{n-2}+\ldots\\ &\quad +3a_3(x-x_k)^2+2a_2(x-x_k)+a_1=f(x,y)\end{aligned}$$

Interpolating at x_k and collocating at x_k, x_{k-1}, $x_{k,2}$, ..., x_{k+n}, we obtain:

$$\begin{aligned}Y(x_k) &= a_0 = Y_k\\ Y'(x_k) &= a_1 = f_k\\ Y'(x_{k+1}) &= (n+1)a_{n+1}(x_{k+1}-x_k)^n+na_n(x_{k+1}-x_k)^{n-1}+\ldots\\ &\quad +3a_3(x_{k+1}-x_k)^2+2a_2(x_{k+1}-x_k)+a_1=f_{k+1}\\ Y'(x_{k+2}) &= (n+1)a_{n+1}(x_{k+2}-x_k)^n+na_n(x_{k+2}-x_k)^{n-1}+\ldots\\ &\quad +3a_3(x_{k+2}-x_k)^2+2a_2(x_{k+2}-x_k)+a_1=f_{k+2}\\ Y'(x_{k+3}) &= (n+1)a_{n+1}(x_{k+3}-x_k)^n+na_n(x_{k+3}-x_k)^{n-1}+\ldots\\ &\quad +3a_3(x_{k+3}-x_k)^2+2a_2(x_{k+3}-x_k)+a_1\\ &\ldots\ldots\ldots\ldots.\\ Y'(x_{k+n}) &= (n+1)a_{n+1}(x_{k+n}-x_k)^n+na_n(x_{k+n}-x_k)^{n-1}+\ldots\\ &\quad +3a_3(x_{k+n}-x_k)^2+2a_2(x_{k+n}-x_k)+a_1\ f_{k+n+1}\end{aligned}$$

This leads to:

$$\begin{aligned}&a_0=Y_k\\ &ha_1=hf_k\\ &(n+1)a_{n+1}h^{n+1}+na_nh^n+\ldots+3a_3h^3+2a_2h^2+a_1h=hf_{k+1}\\ &2^n(n+1)a_{n+1}h^{n+1}+2^{n-1}na_nh^n+\ldots+2^2.3a_3h^3+2.2a_2h^2+a_1(h)=hf_{k+2}\\ &3^n\cdot(n+1)a_{n+1}h^{n+1}+3^{n-1}\cdot na_nh^n+\ldots+3^2\cdot 3a_3h^3+3\cdot 2a_2h^2+a_1h=hf_{k+3}\\ &\ldots\ldots\ldots.\\ &(n+1)\cdot n^n\cdot a_{n+1}h^{n+1}+n^n\cdot a_nh^n+\ldots+n^2\cdot 3a_3h^3+n\cdot 2a_2h^2+a_1h=hf_{k+n+1}\end{aligned}$$

Representing this in a matrix form, the following is deduced:

$$\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & h & 0 & 0 & 0 & 0 & 0\\ 0 & h & 2h^2 & 3h^3 & \cdots & nh^n & (n+1)h^{n+1}\\ 0 & h & 2\cdot 2h^2 & 2^2.3h^3 & \cdots & 2^{n-1}\cdot nh^n & 2^{n}\cdot(n+1)h^{n+1}\\ 0 & h & 3\cdot 2h^2 & 3^2.3h^3 & \cdots & 3^{n-1}.nh^n & 3^n\cdot(n+1)h^{n+1}\\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & h & n\cdot 2h^2 & n^2.3h^3 & \cdots & n^{n-1}.nh^n & .(n+1)\cdot n^nh^{n+1}\end{pmatrix}$$

$$\begin{pmatrix}a_0\\ a_1\\ a_2\\ a_3\\ a_4\\ \\ a_{n+1}\end{pmatrix}=\begin{pmatrix}Y_k\\ hf_k\\ hf_{k+1}\\ hf_{k+2}\\ hf_{k+3}\\ \vdots\\ hf_{k+n+1}\end{pmatrix}$$

This can be solved using numerical solvers for the independent solutions of α_0, α_1, ...α_{n+1}. When, the values of α_0, α_1, ...α_{n+1} are substituted into the basis function, the result obtained gives the continuous formulation of the linear multistep method, while specific evaluation at point x_k gives the discrete equivalent.

Derivation of classes of Adams methods: We shall further construct some continuous schemes which yield classes of Adams Methods. The Adams methods are broadly classified into two, namely Adam-Bashforth (explicit) schemes and Adam-Moulton (implicit) schemes. Thus, for the IVP Eq. 1, the technique involves seeking the trial or basis function in the form.

$$Y(x)=\sum_{j=0}^{n}a_j(x-x_k)^j \cong y(x),\ x_k \le x \le x_{k+p} \qquad (17)$$

This satisfies the unperturbed equations:

$$\begin{aligned}&Y'(x)=f(x,y(x)),\quad x_k \le x \le x_{k+p}\\ &Y(x_k)=Y_k\end{aligned} \qquad (18)$$

Collocating Eq. 18 at n points x_{k+j}, j = 0, 1, 2, ..., (n-1), and interpolating the trial polynomial Eq. 17 at the x_{k+n-1} to give the required (n + 1) equations for the unique determination of α_j, j = 0, 1, 2, ..., n.

To derive a one step Adam-Bashforth scheme, we set n = 1 in Eq. 17 and using Eq. 19, we have:

$$\begin{aligned}Y'(x_k)&=f_k\\ Y(x_k)&=Y_k\end{aligned}$$

Using the basis function Eq. 17, we obtain:

$$\begin{aligned}Y'(x_k)&=a_1=f_k\\ Y(x_k)&=a_0=Y_k\end{aligned}$$

Representation on the matrix yields:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} Y_k \\ f_k \end{pmatrix}$$

On substituting α_0 and α_1 in the basis function (Eq. 17), we have the continuous formulation as:

$$Y(x) = Y_k + f_k(x - x_k) \tag{19}$$

Evaluating at x_{k+1}, we obtain the Euler explicit method:

$$Y_{k+1} = Y_k + hf_k \tag{20}$$

Similarly, for a 2-step Adam-Bashforth method, set n = 2 in Eq. 17. Interpolating Eq. 17 at $x = x_{k+1}$ and collocating the derivative of Eq. (17) at $x = x_k$, we obtain the following equations.

$$\begin{aligned} &Y(x_{k+1}) = a_0 + a_1(x_{k+1} - x_k) + a_2(x_{k+1} - x_k)^2 \\ &Y'(x) = a_1 + 2\cdot a_2(x - x_k) \\ &Y'(x_k) = a_1 = f_k \\ &Y'(x_{k+1}) = a_1 + 2\cdot a_2(x_{k+1} - x_k) \end{aligned}$$

Solving, we get:

$$a_0 = Y_{k+1} - \frac{h}{2}[f_{k+1} + f_k]$$

$$a_2 = \frac{1}{2h}(f_{k+1} - f_k)$$

This gives the continuous method as:

$$\begin{aligned} Y(x) = Y_{k+1} - \frac{h}{2}[f_{k+1} + f_k] + f_k(x - x_k) \\ + \frac{1}{2h}[f_{k+1} - f_k](x - x_k)^2 \end{aligned} \tag{21}$$

Evaluating at $x = x_{k+2}$, we obtain the discrete form as:

$$Y_{k+2} = Y_{k+1} + \frac{h}{2}(3f_{k+1} - f_k) \tag{22}$$

Equation 21 is the continuous formulation of the two step Adams Bashforth scheme and the discrete form is given by Eq. 22.

N-step Adams Bashforth scheme: Using the trial polynomial Eq. 17 and substituting into the ordinary differential Eq. 1, collocating at n points x_{k+j}, j = 0, 1, 2, ..., n - 1 and interpolating at x_{k+n-1} to get an (n + 1) systems of equations for unique determination of α_j' j = 0, 1, 2, ..., n.

Doing this, we obtain:

$$\begin{aligned} Y(x) &= a_n(x - x_k)^n + a_{n-1}(x - x_k)^{n-1} + \dots \\ &\quad + a_3(x - x_k)^3 + a_2(x - x_k)^2 + a_1(x - x_k) + a_0 \\ Y'(x) &= na_n(x - x_k)^{n-1} + (n-1)a_{n-1}(x - x_k)^{n-2} + \dots \\ &\quad + 3a_3(x - x_k)^2 + 2a_2(x - x_k) + a_1 = f(x, y) \end{aligned}$$

Interpolating at x_{k+n-1} and collocating at x_k, x_{k+1}, x_{k+2}, ..., x_{k+n}, we obtain.

Interpolation:

$$\begin{aligned} Y(x_{n+k-1}) = Y_{n+k-1} &= a_n(x_{k+n-1} - x_k)^n + a_{n-1}(x_{k+n-1} - x_k)^{n-1} + \dots \\ &\quad + a_3(x_{k+n-1} - x_k)^3 + a_2(x_{k+n-1} - x_k)^2 + a_1(x_{k+n-1} - x_k) + a_0 \end{aligned}$$

Collocation:

$$\begin{aligned} &Y'(x_k) = a_1 = f_k \\ &Y'(x_{k+1}) = na_n(x_{k+1} - x_k)^{n-1} + \dots + 3a_3(x_{k+1} - x_k)^2 \\ &\qquad + 2a_2(x_{k+1} - x_k) + a_1 = f_{k+1} \\ &Y'(x_{k+2}) = na_n(x_{k+2} - x_k)^{n-1} + \dots + 3a_3(x_{k+2} - x_k)^2 \\ &\qquad + 2a_2(x_{k+2} - x_k) + a_1 = f_{k+2} \\ &Y'(x_{k+3}) = na_n(x_{k+3} - x_k)^{n-1} + \dots + 3a_3(x_{k+3} - x_k)^2 \\ &\qquad + 2a_2(x_{k+3} - x_k) + a_1 = f_{k+3} \\ &\dots\dots\dots \\ &Y'(x_{k+n-1}) = na_n(x_{k+n-1} - x_k)^{n-1} + \dots + 3a_3(x_{k+n-1} - x_k)^2 \\ &\qquad + 2a_2(x_{k+n-1} - x_k) + a_1 = f_{k+n-1} \\ &(n-1)^n h^n a_n + (n-1)^{n-1} h^{n-1} a_{n-1} + \dots + (n-1)^3 h^3 a_3 \\ &\qquad + (n-1)^2 h^2 a_2 + (n-1)ha_1 + a_0 = Y_{k+n-1} \end{aligned}$$

Multiplying the collocations by h, we obtain:

$$\begin{aligned} &ha_1 = hf_k \\ &na_n h^n + \dots + 3a_3 h^3 + 2a_2 h^2 + a_1 h = hf_{k+1} \\ &2^{n-1} na_n h^n + \dots + 2^2.3a_3 h^3 + 2.2a_2 h^2 + a_1(h) = hf_{k+2} \\ &3^{n-1}\cdot na_n h^n + \dots + 3^2\cdot 3a_3 h^3 + 3\cdot 2a_2 h^2 + a_1 h = hf_{k+3} \\ &\dots\dots\dots \\ &n(n-1)^{n-1} a_n h^n + \dots + (n-1)^2\cdot 3a_3 h^3 \\ &\qquad + (n-1)\cdot 2a_2 h^2 + a_1 h = hf_{k+n-1} \end{aligned}$$

Representing this on a matrix, the following is derived:

$$\begin{pmatrix} 1 & (n-1)h & (n-1)^2h^2 & (n-1)^3h^3 & \cdots & (n-1)^nh^n \\ 0 & h & 0 & 0 & \cdots & 0 \\ 0 & h & 2h^2 & 3h^3 & \cdots & nh^n \\ 0 & h & 2\cdot 2h^2 & 2^2\cdot 3h^3 & \cdots & 2^{n-1}\cdot nh^n \\ 0 & h & 3\cdot 2h^2 & 3^2\cdot 3h^3 & \cdots & 3^{n-1}\cdot nh^n \\ & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & h & (n-1)\cdot 2h^2 & (n-1)^2\cdot 3h^3 & \cdots & .n(n-1)^{n-1}h^n \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} Y_{k+n-1} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \\ hf_{k+3} \\ \vdots \\ hf_{k+n-1} \end{pmatrix}$$

This can similarly be solved using some numerical solvers for the independent solutions of $\alpha_0, \alpha_1..., \alpha_n$. When, the values of $\alpha_0, \alpha_1..., \alpha_n$ are substituted in the basis function, the result obtained is called the continuous formulation of the linear multistep method for its discrete equivalent.

In a similar manner, we can formulate the classes of Adams-Moulton schemes in continuous form by using the series as the basis function. The Adams-Moulton schemes are LMM that implicit in nature and are often used as the corrector to the Adams-Bashforth schemes. As such the derivation of this is equally important. Since the Adams-Moulton methods are implicit, the appropriate technique is to use the trial or basis function of the form.

$$Y(x) = \sum_{j=0}^{n+1} a_j (x - x_k)^j \cong y(x),\ x_k \le x \le x_{k+p} \quad (24)$$

If we set n = 2, we can obtain a 2-step implicit method as follows:

$$Y'(x_k) = f_k$$
$$Y'(x_{k+1}) = f_{k+1}$$
$$Y'(x_{k+2}) = f_{k+2}$$
$$Y(x_{k+1}) = Y_{k+1}$$

Using the basis function Eq. 24 for n = 2, we get:

$$Y(x_{k+1}) = a_0 + ha_1 + h^2a_2 + h^3a_3 = Y_{k+1}$$
$$Y'(x_k) = Y'(x_k) = a_1 = f_k$$
$$Y'(x_{k+1}) = a_1 + 2ha_2 + 3h^2a_3 = f_{k+1}$$
$$Y'(x_{k+2}) = a_1 + 4ha_2 + 12h^2a_3 = f_{k+2}$$

Solving, we obtain:

$$a_3 = -\frac{1}{3h^2}\left[f_{k+1} - \frac{1}{2}f_k - \frac{1}{2}f_{k+2}\right]$$
$$a_2 = \frac{1}{4h}\left[4f_{k+1} - 3f_k - f_{k+2}\right]$$
$$a_0 = Y_{k+1} - \frac{h}{12}\left[5f_k + 8f_{k+1} - f_{k+2}\right]$$

Substituting for α_0, α_1, α_2 and α_3 in Eq. 24, we obtain:

$$\begin{aligned} Y(x) = {} & Y_{k+1} - \frac{h}{12}\left[5f_k + 8f_{k+1} - f_{k+2}\right] + f_k(x - x_k) \\ & + \frac{1}{4h}\left[4f_{k+1} - 3f_k - f_{k+2}\right](x - x_k)^2 \\ & - \frac{1}{3h^2}\left[f_{k+1} - \frac{f_k}{2} - \frac{f_{k+2}}{2}\right](x - x_k)^3 \end{aligned} \quad (25)$$

Evaluating at $x = x_{k+2}$ and simplify we obtain:

$$Y_{k+2} = Y_{k+1} + \frac{h}{12}\left[5f_{k+2} + 8f_{k+1} - f_k\right] \quad (26)$$

Equation 25 is the continuous formulation of the discrete formulation which is the Adams-Moulton scheme Eq. 26 of order 3. Several other methods can be obtained by the same technique.

NUMERICAL EXAMPLE

Consider the system of ODE:

$$\frac{d}{dx}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -0.5y \\ 4 - 0.3z - 0.1y \end{pmatrix}, \begin{pmatrix} y(0) = 4 \\ z(0) = 6 \end{pmatrix} \quad (27)$$

The differential equation is known to have an analytical solution:

$$y = 4e^{-0.5x}$$

and

$$z = \frac{-4e^{-\frac{x}{2}} + 40}{3} - 6$$

Using the numerical schemes, we obtain, $Y(x) \cong y(x)$ and $z(x) \cong z(x)$.

The approximate solution of the system of ordinary differential Eq. 27 is solved using the following schemes derived above

- 2-step optimal order scheme (PTD2)
- 2-step Adams-Bashforth Scheme (ABS2)
- 2-step Adams-Moulton scheme (AMM2)

The numerical results obtained by each Scheme with their errors are shown in Table 1-4.

Table 1: The numerical solution of Y (x)

X	PTD2 (y)	ABS2 (y)	AMM2 (y)	Exact y (x)
0.1	3.804918	3.804918	3.804918	3.804918
0.2	3.619354	3.619355	3.619355	3.619355
0.3	3.442822	3.442823	3.442831	3.442832
0.4	3.274916	3.274917	3.274923	3.274924
0.5	3.115197	3.115198	3.115202	3.115203
0.6	2.963268	2.963269	2.963272	2.963273
0.7	2.818748	2.818750	2.818752	2.818752
0.8	2.681278	2.681279	2.681281	2.681281
0.9	2.550511	2.550511	2.550513	2.550513
1.0	2.426121	2.426121	2.426123	2.426123

Table 2: The error of numerical solution of Y(x)

	Error of y_i		
X	PTD2 (y)	ABS2 (y)	AMM2 (y)
0.1	6.14E-07	5.61E-07	3.71E-08
0.2	3.73E-07	3.37E-07	2.34E-08
0.3	9.40E-06	8.89E-06	9.19E-07
0.4	7.92E-06	6.79E-06	8.87E-07
0.5	5.78E-06	4.86E-06	7.49E-07
0.6	4.41E-06	3.74E-06	6.04E-07
0.7	4.05E-06	2.74E-06	4.93E-07
0.8	3.24E-06	2.32E-06	4.12E-07
0.9	2.59E-06	2.23E-06	3.73E-07
1.0	2.24E-06	1.92E-06	2.19E-07

Table 3: The numerical solution of z (x)

X	PTD2 (z)	ABS2 (z)	AMM2 (z)	Exact z (x)
0.1	6.064983	6.064964	6.06502	6.065027
0.2	6.126846	6.126829	6.126876	6.126883
0.3	6.185693	6.185679	6.185715	6.185722
0.4	6.241674	6.241653	6.241686	6.241692
0.5	6.293974	6.294903	6.294926	6.294932
0.6	6.344732	6.345551	6.345571	6.345576
0.7	6.393009	6.393732	6.393746	6.393749
0.8	6.438913	6.438607	6.439545	6.439573
0.9	6.482669	6.482369	6.48314	6.483162
1	6.524402	6.524004	6.524607	6.524626

Table 4: The error of numerical solution of z (x)

	Error of z_i		
X	PTD2 (z)	ABS2 (z)	AMM2 (z)
0.1	4.40E-05	6.34E-05	7.34E-06
0.2	3.73E-05	5.37E-05	7.05E-06
0.3	2.94E-05	4.29E-05	6.84E-06
0.4	1.79E-05	3.92E-05	6.24E-06
0.5	9.58E-04	2.88E-05	5.52E-06
0.6	8.44E-04	2.48E-05	4.76E-06
0.7	7.41E-04	1.74E-05	3.32E-06
0.8	6.60E-04	9.66E-04	2.78E-05
0.9	4.93E-04	7.93E-04	2.17E-05
1	2.24E-04	6.22E-04	1.89E-05

CONCLUSION

It has been shown that continuous collocation methods for solving ordinary differential equations can equally be derived through the approach in this study. It is not compulsory to use the special function as a basis function to derive these schemes. A simple power series used in this study is suffice for such derivations. It should be noted that the optimal order produces a better result than the Adams-Bashforth schemes of the same step, but the Adams Moulton scheme is most accurate. The schemes generated are stable and consistent.

The results generated in this study could be compared with the continuous collocation schemes generated by other researchers cited in this research. All the derivations agreed with known discrete formulas.

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