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Oscillations in Neutral Impulsive Logistic Differential Equations

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Abstract: In this study, the neutral delay impulsive differential equation model of a single-species dynamical system is considered. Some sufficient conditions for the oscillation of the solutions are also provided.

Key words: Neutral delay impulsive differential equation, single-species population dynamics

INTRODUCTION

Until now, most dynamical models were constructed mainly with the help of delay differential equations (Xia and Cao, 2007; Xia *et al*., 2008; Wong, 2000; Ahmed, 2000, 2001; Akca *et al*., 2002). Neutral delay equations also made their way through and their importance are by no means insignificant (Peics and Karsai, 2002; Saker and Manojlovic, 2004). However, there still occur situations that suggest the inadequacy of our existing means. In this study, we are considering the oscillatory implication of a single-species population dynamical model obtained from a neutral impulsive differential equation with constant delays. A neutral impulsive differential equation with constant delays is a differential Eq. 1.1:

$$\begin{cases} \left[x(t)+p(t)x(t-\tau)\right]' + q(t)x(t-\sigma) = 0, \ t \neq t_k \\ \Delta\left[x(t_k)+p(t_k)x(t_k-\tau)\right] + q_k x(t_k-\sigma) = 0, \ t = t_k \end{cases} \tag{1.1}$$

that is, a system consisting of a differential equation together with an impulsive condition in which the first order derivative of the unknown function appears in the equation both with and without delay.

The above definition becomes more meaningful if, we define other related terms and concepts that will continue to be useful as we progress.

Let $\Omega \subset R^n$ be an open set and let $D = R + x\ \Omega$, where, x defines a Cartesian product. Let us assume that for each $k = 1, ..., \tau_k \in C(\Omega(0, \infty))$, $\tau_k(x) < \tau_{k+1}(x)$ and $\lim_{k\to\infty} \tau_k(x) = \infty$ for $x \in \Omega$. For convenience of notation, we shall assume that $\tau_0(x) \equiv 0$. Except stated otherwise, we will assume that the elements of the sequence S: = $\{t_k\}_{k\in E}$ are moments of impulse effect, where, E represents a subscript set, which can be the set of natural numbers N or the set of integers Z and satisfy the properties:

C 1.1: If the sequence $\{t_k\}$ is defined for all $k \in N$, then

$$0<t_1<t_2<\dots \text{ and } \lim_{k\to+\infty} t_k = +\infty$$

C 1.2: If the sequence $\{t_k\}$ is defined for all $k \in Z$, then

$$t_0 \le 0 < t_1,\ t_k < t_{k+1} \text{ for all } k \in Z,\ k \neq 0 \text{ and } \lim_{k\to\pm\infty} t_k = \pm\infty$$

Consider the initial value problem of the impulsive differential system

$$\begin{cases} x' = f(t,x),\ t \neq \tau_k(x), \\ \Delta x = I_k(x),\ t = \tau_k(x) \\ x(t_0^+) = x_0,\ t_0 \ge 0 \end{cases} \tag{1.2}$$

where,

$f \;=\; D \to R^n$
$I_k \;=\; \Omega \to R^n$

Definition 1.1: A function x: $(t_0, t_0 + a) \to R^n$, $t_0 \ge 0$, $a > 0$, is said to be the solution of system (1.2) if

- $x(t_0^+) = x_0$ and $(t, x(t)) \in D$ for all $t \in [t_0, t_0 + a)$
- x (t) is continuously differentiable and satisfies $x'(t) = f(t, x(t))$ for all $t \in [t_0, t_0 + a)$ and $t \neq \tau_k(x(t))$
- If $t \in [t_0, t_0 + a)$ and $t = \tau_k(x(t))$, then $x(t^+) = x(t) + I_k(x(t))$ and for such t's, we always assume that x (t) is left continuous and $s \neq \tau_j(x(s))$ for any j, $t<s<\delta$, for some $\delta>0$

Definition 1.2: A solution x is said to be

- Finally positive, if there exists $T \ge 0$ such that x (t) is defined for $t \ge T$ and $x(t)>0$ for all $t \ge T$
- Finally negative, if there exists $T \ge 0$ such that x (t) is defined for $t \ge T$ and $x(t)<0$ for all $t \ge T$

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- Non-oscillatory, if it is either finally positive or finally negative
- Oscillatory, if it is neither finally positive nor finally negative (Lakshmikantham *et al*., 1989)

Usually, the solution x (t) for $t \in j$, $t \notin S$ of the impulsive differential equation or its first derivative x' (t) is a piece-wise continuous function with points of discontinuity t_k, $t_k \in j \cap S$, where, $J \subset R$ is a given interval. Therefore, in order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r, which are defined as follows:

Let $r \in N$, D: = $[T, \infty) \subset R$ and let the set S be fixed. We denote by PC (D, R) the set of all functions φ: D→R, which is continuous for all $t \in D$, $t \notin S$. They are continuous from the left and have discontinuity of the first kind at the points for which $t \in S$.

By PC^r (D, R), we denote the set of functions φ: D→R, having derivative $d^j_\varphi / d^j_t \in$ PC (D, R), $0 \le j \le r$ (Bainov and Simeonov, 1998; Lakshmikantham *et al.*, 1989).

To specify the points of discontinuity of functions belonging to PC or PC^r, we shall sometimes use the symbols PC (D, R; S) and PC^r (D, R; S), $r \in N$.

STATEMENT OF THE PROBLEM

Before, we formulate our results, we state some lemmas and theorems that will assist us in carrying out the investigation.

Lemma 2.1: Let f, g: $[t_0, \infty]$ → R be such that

$$f(t) = g(t) + pg(t-\tau),\ t \ge t_0 + \max\{0, \tau\} \tag{2.1}$$

where, p, $\tau \in R$ and $p \ne 1$. Assume further that

$$\lim_{t\to\infty} f(t) = L \in R$$

exists. Then the following statements hold:

- If $\lim_{t\to\infty} \inf g(t) \equiv a \in R$, then L = (1 + p)a
- If $\lim_{t\to\infty} \sup g(t) \equiv b \in R$, then L = (1 + p)b
- If g(t) is bounded and $p \ne 1$, then $\lim_{t\to\infty} g(t) = L/1 + p$

Lemma 2.2: Let F, G, P: $[T_0, \infty)$ → R and $c \in R$ be such that

$$F(t) = G(t) + P(t)G(t-c),\ t \ge t_0 + \max\{0, c\} \tag{2.2}$$

Assume that there exist numbers P_1, P_2, P_3, $P_4 \in R$ such that P(t) is in one of the following ranges:

- $P_1 \le P(t) \le 0$
- $0 \le P(t) \le P_2 < 1$
- $1 < P_3 \le P(t) \le P_4$

Suppose that G (t)>0 for $t \ge t_0$, $\lim_{t\to\infty} \inf G(t) = 0$ and that $\lim_{t\to\infty} F(t) \equiv L \in R$ exists. Then, L = 0 (Gyori and Ladas, 1991).

Consider the linear impulsive differential equation with delay

$$\begin{cases} x'(t) + p(t)x(t-\tau) = 0,\ t \notin S \\ \Delta x(t_k) + p_k x(t_k - \tau) = 0,\ \forall t_k \in S \end{cases} \tag{2.3}$$

together with the corresponding inequalities

$$\begin{cases} x'(t) + p(t)x(t-\tau) \le 0,\ t \notin S \\ \Delta x(t_k) + p_k x(t_k - \tau) \le 0,\ \forall t_k \in S \end{cases} \tag{2.4}$$

and

$$\begin{cases} x'(t) + p(t)x(t-\tau) \ge 0,\ t \notin S \\ \Delta x(t_k) + p_k x(t_k - \tau) \ge 0,\ \forall t_k \in S \end{cases} \tag{2.5}$$

Let the following condition be fulfilled:

C 2.1: $p \in PC(R_+, R)$ and $\tau \ge 0$.

Theorem 2.1: Assume that condition C 2.1 is satisfied and let there exist a sequence of disjoint intervals $J_n = [\zeta_n, \eta_n)$ with $\eta_n - \zeta_n = 2\tau$, such that:

- For each $n \in N$, $t \in J_n$ and $t_k \in J_n$

$$p(t) \ge 0,\ p_k \ge 0 \tag{2.6}$$

- There exits $v_1 \in N$ such that for $n \ge v_1$

$$\int_{\eta_n - \tau}^{\eta_n} p(s)ds + \sum_{\eta_n - \tau \le t_k < \eta_n} p_k \ge 1 \tag{2.7}$$

Then:

- The inequality Eq. 2.4 has no finally positive solution
- The inequality Eq. 2.5 has no finally negative solution
- Each regular solution of Eq. 2.3 is oscillatory

Next, consider the linear impulsive differential equation with advanced argument

$$\begin{cases} x'(t) - p(t)x(t+\tau) = 0,\ t \notin S \\ \Delta x(t_k) - p_k x(t_k + \tau) = 0,\ t_k \in S \end{cases} \tag{2.8}$$

together with the corresponding inequalities

$$\begin{cases} x'(t) - p(t)x(t+\tau) \geq 0, \ t \notin S \\ \Delta x(t_k) - p_k x(t_k + \tau) \geq 0, \ t_k \in S \end{cases} \quad (2.9)$$

and

$$\begin{cases} x'(t) - p(t)x(t+\tau) \leq 0, \ t \notin S \\ \Delta x(t_k) - p_k x(t_k + \tau) \leq 0, \ t_k \in S \end{cases} \quad (2.10)$$

The following result is valid:

Theorem 2.2: Let condition C 2.1 be fulfilled and let there exist a sequence of disjoint intervals $J_n = [\zeta_n, \eta_n)$ with $\eta_n - \zeta_n = 2\tau$, such that:

- For each $n \in N$, $t \in J_n$ and $t_k \in J_n$

$$p(t) \geq 0, \ p_k \geq 0 \quad (2.11)$$

- There exits $v_1 \in N$ such that for $n \geq v_1$

$$\int_{\zeta_n}^{\zeta_n+\tau} p(s)ds + \sum_{\zeta_n < t_k \leq \zeta_n + \tau} p_k \geq 1 \quad (2.12)$$

Then:

- The inequality Eq. 2.9 has no finally positive solution
- The inequality Eq. 2.10 has no finally negative solution
- Each regular solution of Eq. 2.8 is oscillatory

RESULTS

Let us modify the classical delay logistic equation

$$r(t) = \frac{\dot{N}(t)}{N(t)} = r\left[1 - \frac{N(t-\sigma)}{K}\right]$$

by introducing additional term r_τ (t) to accommodate our present needs. Consequently, we obtain a modified delay logistic equation in the form

$$r(t) = r_\sigma(t) + r_\tau(t) \quad (3.1)$$

where:

$$r_\sigma(t) = r\left[1 - \frac{N(t-\sigma)}{K}\right]$$

is the growth rate associated with density dependence and

$$r_\tau(t) = c\frac{\dot{N}(t-\tau)}{K}$$

is the growth rate associated with the growth rate at time t-τ.

The expansion of Eq. 3.1 leads to a neutral delay differential Eq. 3.2:

$$\dot{N}(t) = N(t)\left\{r\left[1 - \frac{N(t-\sigma)}{K}\right] + c\frac{\dot{N}(t-\tau)}{K}\right\}, \quad t \geq 0 \quad (3.2)$$

where, r, K ∈ (0, ∞), τ, σ ∈ [0, ∞) and c may assume any value in the interval (-∞, ∞). Here, the different parameters in the equation represent different physical quantities. Precisely, N (t) represents the population density at time t, r_σ (t) denotes the feedback mechanism, which takes σ units to respond to changes in the size of the population and the constant K is the carrying capacity of the environment (Gyori and Laddas, 1991).

We introduce the change of variable

$$x(t) = \ln\frac{N(t)}{K}$$

and hence, transform Eq. 3.2 to the form

$$\left[x(t) - c\left(e^{x(t-\tau)} - 1\right)\right]' + r\left(e^{x(t-\sigma)} - 1\right) = 0, \quad t \geq 0 \quad (3.3)$$

Suppose x (t)>0 for all t≥t-max {τ, σ} and set

$$P(t) = -c\frac{e^{x(t-\tau)} - 1}{x(t-\tau)}, \quad Q(t) = r\frac{e^{x(t-\sigma)} - 1}{x(t-\sigma)} \quad \forall t \geq t_0 \quad (3.4)$$

assuming that $t_0 \geq 0$ exist, we obtain a linear neutral delay differential Eq. 3.5:

$$[x(t) + P(t)x(t-\tau)]' + Q(t)x(t-\sigma) = 0, \quad t \geq t_0 \quad (3.5)$$

Let us assume that Eq. 3.2 models a single-species population system and that the population is experiencing a periodic increase perhaps, due to heavy immigration. Suppose further that the moments $t_1, t_2, \ldots, t_k$ $1 \leq k < \infty$, where, $t_1 < t_2 < \ldots < t_k$ and $\lim_{k\to\infty} t_k = +\infty$ represent rapid changes in the population density, we can build a neutral delay impulsive differential Eq. 3.6:

$$\begin{cases} [x(t) + P(t)x(t-\tau)]' + Q(t)x(t-\sigma) = 0, \ t_k \in S, t \geq t_0 \\ \Delta[x(t_k) + P(t_k)x(t_k-\tau)] + Q_k x(t_k - \sigma) = 0, \ \forall t_k \in S \end{cases} \quad (3.6)$$

where, C 3.1, P (t)∈ $PC^1(R_+, R)$, Q (t) ∈ $PC(R_+, R_+)$, $Q_k \geq 0$, τ≥0 and σ≥0.

Literarily speaking, since the impulsive condition is based on heavy immigration, the population density is expected to go up that is, Δx > 0 (Δx (t_k)>0).

If, with Eq. 3.6, we associate the condition

$$x(t) = \phi(t), \; -\gamma \le t \le 0, \; \gamma = \max\{\tau, \sigma\} \tag{3.7}$$

where, the function ϕ satisfies the following condition:

$$\left.\begin{array}{l} \phi \in PC\left[[-\gamma, 0], R_k\right], \quad \phi(t) > 0 \text{ for } -\gamma \le t \le 0 \text{ and} \\ \phi(t) \text{ is absolutely continuous with locally bounded} \\ \text{derivative on } -\tau \le t \le 0 \end{array}\right\}$$

then the initial value problem Eq. 3.6 and 3.7 has a unique solution, which exists and remains positive on $[0, \infty)$ (Gyori and Laddas, 1991).

The task of establishing the oscillatory status of the solution to the original differential Eq. 3.2 about the steady state K appears to be extremely involving especially, in view of its impulsive requirement. However, this is hardly a problem if we are conscious of the fact that every positive solution of Eq. 3.2 along with its impulsive conditions oscillates if and only if every solution of Eq. 3.6 oscillates. Against this backdrop, we will shift our emphasis to Eq. 3.6 believing that whatever conclusions we arrive at, will remain binding to Eq. 3.2 about K. This is accomplished through the following lemmas and theorems.

We return to the neutral delay impulsive differential Eq. 3.6 together with the conditions for its coefficients and delays.

Lemma 3.1: Assume that C 3.2

$$\int_{t_0>0}^{\infty} x(s)ds \to \infty \Rightarrow \int_{t_0>0}^{\infty} Q(s)x(s-\sigma)ds = \infty$$

for any $x \in PC(R_+, R_+)$ and $\forall \sigma \ge 0$. Let x (t) be a finally positive solution of Eq. 3.6 and set:

$$z(t) = x(t) + P(t)x(t-\tau) \tag{3.8}$$

Then the following statements are true:

- z (t) is a finally non-increasing function
- If P $(t) \le 1$, then z (t) is finally negative
- If $-1 \le P(t) \le 0$, then z $(t) > 0$ and $\lim_{k\to\infty}$ z (t) = 0

Definition 3.1: The solution x (t) is said to be

- Finally non-increasing if $t_1 < t_2$ implies x $(t_1) \ge$ x (t_2) for $t_1, t_2 > T$ and $T > 0$
- Finally non-decreasing if $t_1 < t_2$ implies x $(t_1) \le$ x (t_2) for $t_1, t_2 > T$ and $T > 0$

Proof: We have

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma) \le 0, \; t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma) \le 0, \; \forall t_k \in S \end{cases} \tag{3.9}$$

and so, z (t) is a finally non-increasing function.

Assume, on the contrary that z $(t) > 0$, $\forall t \ge T_0$. But then,

$$\neg\text{"}z(t) < 0\text{"} \Rightarrow z(t) \ge 0$$

If however, z (t) = 0, then by condition C3.2, Eq. 3.9 and the fact that x $(t) > 0$, $\forall t \ge T_0 \Rightarrow z(t) < 0 \; \forall t \ge T_1$. Hence,

$$\neg\text{"}z(t) < 0\text{"} \Leftrightarrow z(t) > 0, \; \forall t \ge T_0$$

Let us start with the statement

$$x(t) \ge -P(t)x(t-\tau) \ge x(t-\tau) \tag{3.10}$$

We show that x $(t) \ge \beta > 0$ for $([t_k - \tau_k, t_k)$. Also, we show that the statement holds for (t_l, t_{l+1}). Since, x $(t) > 0$ for all continuity points (t_l, t_{l+1}), only $\lim_{t\to t_\ell+0} x(t) = 0$ can contradict our statement. Actually, if $\lim_{t\to t_\ell+0} x(t) = 0$, then by Eq. 3.10, $\lim_{t\to t_\ell+0} x(t-\tau) = 0$ also. Then, $\lim_{t\to t_\ell+0} z(t) = 0$ follows and from Eq. 3.9, z fulfils the initial condition in (t_l, t_{l+1}) that is:

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma) \\ z(t_\ell) = 0 \end{cases}$$

hence,

$$0 = z(t_\ell) \ge z(s), \; s \in (t_\ell, \; t_{\ell+1}]$$

which contradicts the hypothesis that z $(t) > 0$, $t_0 \le t < \infty$.

Therefore,

$$\lim_{t\to t_\ell+0} x(t) > 0$$

Consequently,

$$\min_{t_\ell < t \le t_{\ell+1}} x(t) > \beta_\ell > 0$$

Hence,

$$\min_{t_k-\tau\le t\le t_k} x(t) \min_{t_k-\tau\le t_\ell\le t_k} \min_{t_\ell\le t\le t_{\ell+1}} x(t) = \min_{t_k-\tau\le t_\ell\le t_k} \beta_\ell = \beta > 0$$

Thus, x (t) is bounded from below by a positive constant on the sequence $t + k\tau$, $0 \le k < \infty$. Therefore, from Eq. 3.9, we see that

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma) \le -Q(t)\beta, \\ \Delta z(t_k) = -Q_k x(t_k - \sigma) \le -Q_k \beta \end{cases} \tag{3.11}$$

which, in view of condition C3.2, implies that

$$\lim_{t\to+\infty} z(t) = -\beta\left(\int_{t_0}^{+\infty} Q(t)dt + \sum_{k=1}^{\infty} Q_k\right) = -\infty$$

This is a contradiction and so completes the proof of Lemma 3.1 (b). Notice, in the Eq. 3.11 that the condition

$$\int_{t_0}^{+\infty} Q(t)dt + \sum_{k=1}^{\infty} Q_k = \infty$$

constitutes a special case of condition C3.2.

Let us claim inversely that z (t)<0. We recall that

$$\neg"z(t) > 0" \Leftrightarrow z(t) \le 0$$

Hence, reasoning like in b (i) above, z (t)<0 for $t>T_0$. Thus, x (t)≤x (t-τ), hence, x (t) is a bounded function and so also is z (t). Since z (t)<0,

$$\lim_{t\to+\infty} z(t) = L < 0$$

Hence,

$$\int_{t_0}^{\infty} z(s)ds = -\infty$$

On the other hand,

$$\begin{aligned}
&\int_{t_0}^{\infty}\left(x(s)+P(s)x(s-\tau)\right)ds = \lim_{T\to\infty}\int_{t_0}^{T}\begin{pmatrix} x(s)+P(s) \\ x(s-\tau)\end{pmatrix}ds \\
&= \lim_{T\to\infty}\left(\int_{t_0}^{T} x(s)ds + \int_{t_0-\tau}^{T-\tau} P(s+\tau)x(s)ds\right) \\
&= \lim_{T\to\infty}\left(\int_{t_0}^{T-\tau} x(s)ds + \int_{t_0}^{T-\tau} P(s+\tau)x(s)ds + \right. \\
&\left. + \int_{T-\tau}^{T} x(s)ds + \int_{t_0-\tau}^{t_0} P(s+\tau)x(s)ds\right) \\
&= \int_{t_0}^{\infty} x(s)\left(1+P(s+\tau)\right)ds + \\
&+ \lim_{T\to\infty}\int_{T-\tau}^{T} x(s)ds + \int_{t_0-\tau}^{t_0} P(s+\tau)x(s)ds
\end{aligned} \tag{3.12}$$

But the component

$$\int_{t_0}^{\infty} x(s)\left(1+P(s+\tau)\right)ds + \lim_{T\to\infty}\int_{T-\tau}^{T} x(s)ds \ge 0$$

and

$$\int_{t_0-\tau}^{t_0} P(s+\tau)x(s)ds \le 0$$

meaning that Eq. 3.12 cannot tend to -∞. This is a contradiction, therefore, $z(t) \nrightarrow L<0$, which implies that $z(t) \nless 0$.

Hence, we have established that z (t)>0, $t \ge T_0$ and that z (t)→L≥0. Clearly, if L>0, then

$$\int_{t_0}^{\infty} z(s)ds = \infty$$

hence,

$$\int_{t_0}^{\infty} x(s)ds = \infty$$

Thus, by condition C3.2, z (t)→0 and this completes the proof of Lemma 3.1.

Now consider the neutral Eq. 3.13:

$$\begin{cases} \left[x(t)+px(t-\tau)\right]' + Q(t)x(t-\sigma) = 0, \ t \notin S \\ \Delta\left[x(t_k)+px(t-\tau)\right] + Q_k x(t_k-\sigma) = 0, \ \forall t_k \in S \end{cases} \tag{3.13}$$

where $t_k \in R$. We introduce the following condition:

Condition 3.3: There exist nonnegative integers m_1 and m_2 such that

$$t_{k+m_1} = t_k + |\tau|, \ t_{k+m_2} = t_k + |\sigma|, \ k \in Z$$

Lemma 3.2: Let us assume that conditions C3.2 and C3.3 hold. We further assume that p≠+1 in Eq. 3.13 and that

$$Q \in PC\left(R_+, R_+\right), Q_k \ge 0, \tau \ge 0, \sigma \ge 0 \tag{3.14}$$

Let x (t) be a finally positive solution of Eq. 3.13 and set z (t) = z (t) + px (t - τ). Then

- z (t) is a finally non-increasing function and either

$$\lim_{t\to+\infty} z(t) = -\infty \tag{3.15}$$

or

$$\lim_{t\to+\infty} z(t) = 0^+ \tag{3.16}$$

The following statements are equivalent:

- Equation 3.15 holds
- p≤1
- $\lim_{t\to+\infty} x(t) = +\infty$

The following statements are equivalent:

- Equation 3.16 holds
- p≥1
- $\lim_{t\to+\infty} x(t) = 0^+$

Proof: From Eq. 3.13, bearing inequalities Eq. 3.14 in mind, we obtain

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma),\ t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma),\ \forall t_k \in S \end{cases} \quad (3.17)$$

which implies z is a non-increasing function. It converges therefore, either to $-\infty$ or to a number L, where $-\infty<L<+\infty$, for $t\to\infty$.

If z converges to $-\infty$, then the proof of (a) is complete. Otherwise, if z (t) converges to L as $t\to\infty$, then

$$\int_{t_0}^{\infty} z(t)\,dt = \pm\infty$$

(depending on whether $L > 0$ or $L \le 0$).

Again, if L = 0, the proof of (a) is complete. Otherwise, we integrate both sides of Eq. 3.17 from t to ∞ for sufficiently large t, to obtain

$$L - z(t_0) = -\int_{t_0}^{+\infty} Q(t)x(t-\sigma)\,dt - \sum_{t_k > t_0} Q_k x(t_k - \sigma) \quad (3.18)$$

Equation (3.18) is clearly, finite and this implies

$$\int_{t_0}^{\infty} x(s)\,ds < \infty$$

by condition 3.2.

Notice the modification

$$\int_{t_0}^{+\infty} x(s)\,ds + \sum_{t_k \ge t_0} x(t_k) = \infty$$

$$\Rightarrow \int_{t_0}^{+\infty} Q(s)x(s-\sigma)\,ds + \sum_{t_k \ge t_0} Q_k x(t_k - \sigma) = \infty,\ \forall \sigma \ge 0$$

of condition 3.2 or equivalently,

$$\int_{t_0}^{+\infty} Q(s)x(s-\sigma)\,ds + \sum_{t_k \ge t_0} Q_k x(t_k - \sigma) < \infty$$

$$\Rightarrow \int_{t_0}^{+\infty} x(s)\,ds + \sum_{t_k \ge t_0} x(t_k) < \infty$$

where in this case, $\sigma\ge0$ is assumed to exist. Statement 3.18 contradicts the hypothesis that

$$\int_{t_0}^{\infty} z(t)\,dt = \infty$$

Hence, L = 0 and this completes the proof of (a).

Let (i) hold that is, condition (3.15) is fulfilled. We are to prove that

$$\text{(i)} \Rightarrow \text{(ii)}$$

By definition,

$$z(t) = x(t) + px(t-\tau)$$

Both x (t) and x (t - τ) are positive functions, meaning that the above expression can be negative only if $p<0$. Consequently, z (t)$\to-\infty$ only if x (t) is unbounded.

We show that there exists $T_0 \in R$ such that $z(T_0^+)<0$ and

$$x(T_0^+) \ge \sup_{t\le T_0} x(t)$$

Let us assume conversely that such T_0 does not exist. Then,

$$x(T_0^+) < \sup_{t\le T_0} x(t)\ \forall T_0 \in R$$

Consequently, $\exists\ \varepsilon>0$ such that $\forall s,\ T_0 < s < T_0 + \varepsilon$, $x(s) < \sup_{t\le T_0} x(t)$. Hence,

$$\sup\left\{ s : x(s) < \sup_{t\le T_0} x(t) \right\} = T \in R$$

must exist, otherwise x (s) is bounded contrary to our earlier assertion. But then for T,

$$\sup_{t\le T_0} x(t) \le x(T)$$

holds. With this T_0: = T, we obtain the inequality

$$0 > z(T_0^+) = x(T_0^+) + px(T_0 - \tau^+) \ge x(T_0^+)(1+p)$$

This is only possible if $p\le1$, since x (T_0^+)> 0.

$$\text{(ii)} \Rightarrow \text{(iii)}$$

Let $p\le0$. Also, let us assume that z is finally positive. Then z is decreasing and $z\to0$, by Lemma 3.1. If

$$0 < z(t) = x(t) + px(t-\tau)$$

then

$$x(t) > (-p)x(t-\tau) \quad (3.19)$$

On the other hand, by z (t)$\to$0 and

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma),\ t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma),\ \forall t_k \in S \end{cases}$$

$$0 - z(t) = -\int_t^{\infty} Q(s)x(s-\sigma)ds - \sum_{t \le t_k} Q_k x(t_k - \sigma) > -\infty$$

Hence, by condition 3.2,

$$\int_{t-\sigma}^{\infty} x(s)ds < \infty, \sum_{t \le t_k} x(t_k - \sigma) < \infty \qquad (3.20)$$

Consequently, inequality Eq. 3.19 brings contradiction since

$$x(t_k + i\sigma) > (-p)^i x(t_k - \sigma), 1 \le i < \infty$$

would have led to infinity in Eq. 3.20. Hence, z cannot be finally positive. Thus, by Lemma 3.1, $z \to -\infty$ if $t \to \infty$. Therefore, there exists T_0 such that $z(s) < 0$ if $s > T_0$.
Since,

$$z(t) = x(t) + px(t-\tau)$$

and $z(t) \to -\infty$,

$$0 > z(t) > px(t-\tau)$$

which implies

$$0 < \frac{z(t)}{p} < x(t-\tau) \to +\infty$$

(iii) ⇒ (i)

Assume that $x(t) \to -\infty$ for $t \to \infty$. We show that if $z(t) \to 0$, it implies that $x(t) \not\to \infty$. Really,

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma), t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma), \forall t_k \in S \end{cases}$$

Hence,

$$0 - z(t) = -\int_t^{\infty} Q(s)x(s-\sigma)ds - \sum_{t \le t_k} Q_k x(t_k - \sigma) < \infty$$

which, by condition 3.2, implies

$$\int_t^{\infty} x(s-\sigma)ds < \infty \text{ and } \sum_{t \le t_k} x(t_k - \sigma) < \infty$$

This contradicts the statement that $x(t) \to \infty$. Hence, $z(t) \to 0 \Rightarrow x(t) \not\to \infty$. Therefore, $x(t) \to \infty \Rightarrow z(t) \to \infty$, by Lemma 3.1. This completes the proof of (b).

Applying contraposition to the statements of Lemma 3.2 (a), we obtain ¬ (j) ⇒ ¬ (jj) ⇒ ¬ (jjj)
Thus,

¬ (j) ⇒ ¬ (jj) means $z(t) \to \Rightarrow p \ge -1$

¬ (j) ⇒ ¬ (jjj) means $z(t) \to 0 \Rightarrow x(t) \not\to \infty$

(j) ⇒ (jj)

We know that $z(t) \to 0 \Rightarrow p \ge -1$. Let us assume that $p = -1$. If z, being a decreasing function, takes on negative values, then z (t) finally tends to $-\infty$ by Lemma 3.1. Hence, $z(t) \to 0$ implies that z is finally positive. Thus,

$$0 < z(t) = x(t) - x(t-\tau), \forall t > T_0$$

Hence,

$$x(t-\tau) < x(t), \forall t > T_0$$

Iterating the above inequality, we obtain

$$x(t + i\tau) > x(t-\tau) > 0 \qquad (3.21)$$

On the other hand,

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma) \\ \Delta z(t_k) = -Q_k x(t_k - \sigma) \end{cases}$$

where, t_k belongs to the set of points of impulse effect. Hence,

$$0 - z(t) = -\int_t^{\infty} Q(s)x(s-\sigma)ds - \sum_{t \le t_k} Q_k x(t_k - \sigma) < \infty$$

This follows, from condition 3.2 that

$$\sum_{t \le t_k} x(t_k - \sigma) < \infty$$

which contradicts condition 3.21. Hence, the assumption that $z(t) \to 0$ when $p = -1$ leads to a contradiction. Therefore, $p \ge 1$ is admissible only.

(jj) ⇒ (jjj)

Now, we are familiar with the fact when $p \ge 1$, $x(t) \not\to \infty$, Let us check what happens when $p \ge 0$. Since, whenever $x(t) \not\to \infty$ implies $z(t) \not\to -\infty$, it follows, by Lemma 3.1 that $z(t) \to 0$. Therefore,

$$z(t) = x(t) + px(t-\tau) > x(t) > 0, \quad \forall t > T_0$$

Hence, $x(t) \to 0$.

Let, $-1 < p < 0$. Then

$$x(t) = (-p)x(t-\tau) + z(t) < (-p)x(t-\tau) + z(T_0 - \tau)$$

where, we have used the fact that z is a strictly decreasing function and $t \in (t_0, t_0 + \tau)$. We rewrite the above inequality in the form:

$$x(t) < (-p)x(t-\tau) + z(T_0 - \tau)$$

and replace the function x (t - τ) with its supremum

$$x(t-\tau) \le \sup_{s\in[T_0-\tau,\, T_0]} x(s)$$

Then,

$$x(t) < (-p) \sup_{s\in[T_0-\tau,\, T_0]} x(s) + z(T_0 - \tau)$$

hence,

$$\sup_{s\in[T_0,\, T_0+\tau]} x(s) < (-p) \sup_{s\in[T_0-\tau,\, T_0]} x(s) + z(T_0 - \tau)$$

Let,

$$\theta_k := t_0 + k\tau,\ M_k := \sup_{s\in[\theta_k-\tau,\, \theta_k]} x(s)\ \forall -1 \le k < \infty$$

Then, we get

$$M_{k+1} < (-p)M_k + z(\theta_k - \tau)$$

Applying this iteratively, we obtain, for $\ell > k$:

$$M_\ell \le (-p)^{\ell-k} M_k + z(\theta_{k-1})\sum_{i=k}^{\ell}(-1)^i$$

$$< (-p)^{\ell-k} M_k + z(\theta_{k-1})\frac{1}{1+p}$$

Hence, for $\ell \to -\infty$,

$$\limsup M_\ell \le z(\theta_{k-1})\frac{1}{1+p} \to 0$$

therefore,

$$M_\ell \to 0 \Rightarrow x(t) \to 0$$

This means (jj) ⇒ (jjj) and thus, completes the proof of Lemma 3.2.

Theorem 3.1: Assume that conditions C3.1, C3.2 and (3.14) are satisfied. Then every solution of the Eq. 3.22

$$\begin{cases} [x(t) - x(t-\tau)]' + Q(t)\,x(t-\sigma) = 0,\ t \notin S \\ \Delta[x(t_k) - x(t_k - \tau)] + Q_k x(t_k - \sigma) = 0,\ \forall t_k \in S \end{cases} \quad (3.22)$$

is oscillatory.

Proof: By the definition of z (t), the expression

$$z(t) = x(t) - x(t-\tau)$$

immediately implies p = -1. Hence, by the implication (j) ⇒ (jj) of Lemma 3.2 (c), x (t) is neither finally positive nor finally negative. Consequently, the solution of Eq. 3.22 oscillates. This completes the proof of Theorem 3.1.

Theorem 3.2: Assume that conditions C3.1 and C3.2 hold, $-1 < P(t) \le 0$ and every solution of the equation

$$\begin{cases} z'(t) + Q(t)\,z(t-\sigma) = 0,\ t \ge t_0,\ t \notin S \\ \Delta z(t_k) + Q_k z(t_k - \sigma) = 0,\ \forall t_k \in S \end{cases}$$

is oscillatory. Then every solution of Eq. 3.6 is oscillatory.

Proof: Assume conversely that Eq. 3.6 has a finally positive solution x (t). We set

$$z(t) = x(t) + P(t)\,x(t-\tau)$$

Then by Lemma 3.1 (c),

$$z(t) > 0,\ t \ge t_0 \quad (3.23)$$

Since, $z(t) \le x(t)$ $(t \ge t_0)$, it follows from equation

$$\begin{cases} z'(t) = -Q(t)x(t-\sigma),\ t \notin S \\ \Delta z(t_k) = -Q_k x(t_k - \sigma),\ \forall t_k \in S \end{cases}$$

that

$$\begin{cases} z'(t) + Q(t)z(t-\sigma) \le 0,\ t \ge t_0 \\ \Delta z(t_k) + Q_k z(t_k - \sigma) \le 0 \end{cases} \quad (3.24)$$

In view of condition 3.2 and Theorem 2.1 (i), the delay impulsive differential inequality Eq. 3.24 cannot have a finally positive solution and this contradicts condition 3.23. Thus, the proof of Theorem 3.2 is complete.

CONCLUSION

The beauty and effectiveness of the above results are indications that it is now possible to inject adequate mathematical components into those frequently encountered natural disasters such as earthquakes, tsunamis, etc. In addition, the effect of periodical increase (decrease) in a given population can also be given a more accurate mathematical treatment.

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