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Oscillations in Systems of Neutral Impulsive Differential Equations

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Abstract: In this study, we establish new oscillation criteria for system of neutral impulsive differential equations with constant delays. Expressed as theorems, the criteria give explicit sufficient conditions for the oscillations of every solution of the said system and are readily generalized for non-autonomous cases.

Key words: Systems of neutral impulsive differential equations, oscillation criteria

INTRODUCTION

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or non-oscillation of solutions for neutral delay differential equations with or without impulses (Grammatikopoulos *et al*., 1986; Grove *et al*., 1988; Ladas and Schults, 1989; Bainov and Simeonov, 1998; Dzurina and Mihalikova, 2000). Most of these studies, particularly those pertaining to neutral impulsive equations, limit their discussions to one-dimensional linear and nonlinear neutral impulsive differential equations with single variable and/or with constant delays (El-Morshedy and Gopalsamy, 2000; Xu and Xia, 2008; Luo *et al*., 2000; Graef *et al*., 2002, 2004). In this study, we extend the concept of the oscillations in systems of neutral delay equations to systems of neutral impulsive differential equations with constant coefficients and delays.

We begin by considering the following initial value problem for systems of differential equations with impulses:

$$\begin{cases} x' = f(t,x), \ t \neq s_k \\ \Delta x(t)\big|_{t=s_k} = x(s_k+0) - x(s_k) = I_{j_k}(x(s_k)) \end{cases} \tag{1}$$

k = 1, 2, ..., with initial condition

$$x(0) = x_0 \tag{2}$$

Where:

$$f: S \to R^n, \ S = \{(t,x); t \geq 0, x \in D\}$$

D is a domain in R^n; here and further on by, s_k, k = 1, 2, ...,

$$0 < s_1 < s_2 < \dots \tag{3}$$

we denote the moments when the integral curve (t, x(t)) of problem Eq. (1) and (2) meets some of the hyper-surfaces

$$\delta_k: \ t = t_k(x), \ k = 1, 2, \cdots \tag{4}$$

jk is the number of the hypersurface met by the integral curve in the moment s_k (in general, $j_k \neq k$); I_k: D → R^n; $x(s_k) = x(s_k - 0)$, k = 1, 2, ...; $x_0 \in D$.

Definition 1: The function x = φ(t) is a solution of Eq. (1) in the interval J: = (α, β) if

- φ(t) is differentiable in J, $t \neq s_k$, $k \in N$ and satisfies the condition
$$\varphi'(t) = f(t, \varphi(t)) \text{ for all } t \in J, \ t \neq s_k \text{ and } k \in N$$
- φ(t) satisfies the relation
$$\varphi(s_k+0) - \varphi(s_k-0) = I_{j_k}(\varphi(s_k-0)), \quad s_k \in J \text{ and } k \in Z$$

Definition 2: A solution x is said to be

- Finally positive, if there exists T≥0 such that x(t) is defined for t≥T and x(t)>0 for all t≥T

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- Finally negative, if there exists $T\geq 0$ such that x(t) is defined for $t\geq T$ and $x(t)<0$ for all $t\geq T$
- Non-oscillatory, if it is either finally positive or finally negative
- Oscillatory, if it is neither finally positive nor finally negative (Lakshmikantham *et al*., 1989).

Definition 3: A solution $x(t) = [x_1(t), x_2(t), ..., x_m(t)]^T$ of a system of impulsive differential equations is said to oscillate if it is finally trivial $\forall t\geq T$ or if at least one component does not have finally constant signum and non-oscillatory.

Definition 4: A solution $x(t) = [x_1(t), x_2(t), ..., x_m(t)]^T$ of a system of impulsive differential equations is said to oscillate if every component $x_i(t)$, $1\leq i\leq m$, of the solution is neither finally positive nor finally negative and is non-oscillatory if at least one component is finally positive or finally negative.

Usually, the solution x(t) for $t\in J$, $t\notin S$ of a given impulsive differential equation or its first derivative x'(t) is a piece-wise continuous function with points of discontinuity t_k, $t_k\in J\cap S$. Therefore, in order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r, which are defined as follows:

Let, $r\in N$ and the sequence $S: = [t_k]_{k\in E}$ be fixed, where, E represents a subscript set which, can be the set of natural numbers N or the set of integers Z and satisfies the properties:

Condition 1: If $[t_k]_{k\in E}$ is defined with E = N, then

$$0 < t_1 < t_2 < \cdots \text{ and } \lim_{k\to+\infty} t_k = +\infty$$

Condition 2: If $[t_k]_{k\in E}$ is defined with E = Z, then

$$t_0 \leq 0 < t_1,\ \ t_k < t_{k+1} \text{ for } k\in Z,\ k\neq 0$$

and

$$\lim_{k\to\pm\infty} t_k = \pm\infty$$

Definition 5: PC (D, R) is the set of those functions, which are continuous for all $t\in D$, $t\notin S$, $\forall k\in N$ and have discontinuity of the first kind for $t\in S$ and $k\in N$.

Definition 6: $PC^r(D, R)$ is the set of those functions, which are r-times continuously differentiable for all $t\in D$, $t\notin S$, $\forall k\in N$ and have discontinuity of the first kind for $t\in S$ and $k\in N$ (Bainov and Simeonov, 1998; Lakshmikantham *et al*., 1989).

To specify the points of discontinuity of functions belonging to PC or PC^r, we shall sometimes use the symbols PC (D, R; S) and PC^r, (D, R; S), $r\in N$.

Now consider the impulsive delay differential Eq. (5)

$$\begin{cases} y'(t)+\sum_{i=1}^{n} q_i(t)y(t-\tau_i(t))=0,\ t\notin S \\ \Delta y(t_k)+\sum_{i=1}^{n} q_{ik}y(t_k-\tau_i(t_k))=0,\ t_k\in S \end{cases} \quad (5)$$

and the impulsive delay inequalities

$$\begin{cases} x'(t)+\sum_{i=1}^{n} p_i(t)x(t-\tau_i(t))\leq 0,\ t\notin S \\ \Delta x(t_k)+\sum_{i=1}^{n} p_{ik}x(t_k-\tau_i(t_k))\leq 0,\ t_k\in S \end{cases} \quad (6)$$

and

$$\begin{cases} z'(t)+\sum_{i=1}^{n} r_i(t)z(t-\tau_i(t))\geq 0,\ t\notin S \\ \Delta z(t_k)+\sum_{i=1}^{n} r_{ik}z(t_k-\tau_i(t_k))\geq 0,\ t_k\in S \end{cases} \quad (7)$$

We introduce the condition:

Condition 3:

$$\begin{cases} p_i,\ q_i,\ r_i\in PC(R_+,\ R_+),\ \tau_j\in C(R_+,R_+),\ i=1,\cdots,n \\ \text{and } p_{ik},\ q_{ik},\ r_{ik}\geq 0,\ k\in N,\ i=1,\cdots,n \end{cases}$$

Let, $t_0\in R_+$ and define

$$t_{-1}=\min_{1\leq i\leq n}\left\{\inf_{t\geq t_0}\{t-\tau_i(t)\}\right\} \quad (8)$$

We associate with the Eq. (5) and the inequalities Eq. (6 and 7) the initial condition

$$x(t)=\varphi(t),\ t_{-1}\leq t\leq t_0$$

where, $\varphi(t)\in PC([t_{-1}, t_0], R)$, $\varphi(t_0)>0$. The following theorem (Bainov and Simeonov, 1998) will be useful in carrying out the proofs in the main theorems.

Theorem 1: Let, condition 3 be fulfilled and let

$$\begin{cases} p_i(t)\geq q_i(t)\geq r_i(t);\ \forall t\in R_+,\ i=1,\ 2,\cdots,n \\ p_{ik}\geq q_{ik}\geq r_{ik};\ k\in N,\ i=1,\ 2,\cdots,n \end{cases} \quad (9)$$

Assume that y(t), x(t) and z(t) are solutions of Eq. (5) and inequalities Eq. (6 and 7), respectively and belong to the space $PC([t_{-1}, +\infty] R)$ and such that

$$x(t) > 0, \ t \ge t_0 \tag{10}$$

$$z(t_0^+) \ge y(t_0^+) \ge x(t_0^+) \tag{11}$$

$$\frac{x(t)}{x(t_0)} \ge \frac{y(t)}{y(t_0)} \ge \frac{z(t)}{z(t_0)} \ge 0, \ t_{-1} \le t \le t_0 \tag{12}$$

Then,

$$z(t) \ge y(t) \ge x(t), \ \forall t \ge t_0 \tag{13}$$

Consider now the impulsive differential inequality Eq. (6) together with the impulsive differential Eq. (14)

$$\begin{cases} x'(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t)) = 0, \ t \notin S \\ \Delta x(t_k) + \sum_{i=1}^{n} p_{ik} x(t_k - \tau_i(t_k)) = 0, \ t_k \in S \end{cases} \tag{14}$$

By virtue of Theorem 1, we obtain the following useful corollary.

Corollary 1: Let condition 3 be fulfilled. Then the following statements are equivalent:

- The inequality Eq. (6) has a finally positive solution
- The Eq. (14) has a finally positive solution

RESULTS

In this study, we obtain sufficient conditions for the oscillation of every solution of the system of neutral impulsive equations

$$\begin{cases} [x(t) - Px(t-\tau)]' + \sum_{\ell=1}^{N} Q_\ell x(t - \sigma_\ell) = 0, \ t \notin S \\ \Delta[x(t_k) - Px(t_k - \tau)] + \sum_{\ell=1}^{N} Q_\ell x(t_k - \sigma_1) = 0, \ t_k \in S \end{cases} \tag{15}$$

where, P is an m x m diagonal matrix with the diagonal entries p_1, p_2, ..., p_m and Q_ℓ is an m×m matrix for each $1 \le \ell \le N$ such that

$$\left.\begin{matrix} 0 \le p_i \le 1, \text{ for } 1 \le i \le m, \tau, \sigma_\ell \in [0, \infty) \\ \text{and } Q_{ij}^{(\ell)} \in R \text{ for } 1 \le \ell \le N \text{ and } 1 \le i, j \le m \end{matrix}\right\} \tag{16}$$

$Q_{\ell 0}$ is also an m x m matrix and has the entries $q_{ij}^{(\ell 0)} \in R$ for $1 \le \ell \le N$ and $1 \le i, j \le m$.

Our main result is the following theorem, which gives explicit sufficient conditions for the oscillation (component-wise) of every solution of Eq. (15). As may be verified, a similar result holds for non-autonomous impulsive systems where, oscillations are understood in the sense of definition 3.

Theorem 2: Assume that condition Eq. (16) holds. Set

$$\begin{cases} q_\ell = \min_{1 \le i \le n} \left[q_{ii}^{(\ell)} - \sum_{\substack{j=1 \\ j \ne i}}^{M} \left| q_{jl}^{(\ell)} \right| \right] \text{ for } 1 \le \ell \le N, \\ q_{\ell 0} = \min_{1 \le i \le n} \left[q_{ii}^{(\ell 0)} - \sum_{\substack{j=1 \\ j \ne i}}^{m} \left| q_{ji}^{(\ell 0)} \right| \right] \text{ for } 1 \le \ell \le N \end{cases} \tag{17}$$

Suppose that

$$q_l \ge 0 \text{ for } 1 \le \ell \le N \tag{18}$$

and that every solution of the scalar delay impulsive differential equation

$$\begin{cases} u'(t) + \sum_{\ell=1}^{N} q_\ell \, u(t - \sigma_\ell) = 0, \ t \notin S \\ \Delta u(t_k) + \sum_{\ell=1}^{N} q_{\ell 0} \, u(t_k - \sigma_\ell) = 0, \ t_k \in S \end{cases} \tag{19}$$

oscillates. Then, every solution of Eq. (15) oscillates component-wise.

Proof: Assume conversely, that Eq. (15) has a non-oscillatory solution in the sense of definition 4. Then, since Q_ℓ is an m×m matrix and $\sigma_\ell \ge 0$, Eq. (15) has a non-oscillatory solution $x(t) = [x_1(t), x_2(t), ..., x_m(t)]^T$ in the sense of Definition 3. That is, x(t) is not finally zero and for t sufficiently large, each component $x_i(t)$, $1 \le i \le m$, has finally constant signum.

For sufficiently large t, set

$$\delta_i = \operatorname{sgn} x_i(t) \text{ and } y_i(t) = \delta_i x_i(t) \text{ for } 1 \le i \le m$$

Then, for $1 \le i \le m$ and sufficiently large t, it follows from Eq. (15) that

$$\begin{cases} [y_i(t) - p_i y_i(t-\tau)]' + \sum_{\ell=1}^{N} \sum_{j=1}^{m} q_{ij}^{(\ell)} x_j(t - \sigma_\ell) \delta_i = 0, t \notin S \\ \Delta[y_i(t_k) - p_i y_i(t_k - \tau)] + \sum_{\ell=1}^{N} \sum_{j=1}^{m} q_{ij}^{(\ell 0)} x_j(t_k - \sigma_\ell) \delta_i = 0, t_k \in S \end{cases}$$

or equivalently,

$$\begin{cases} \left[y_i(t) - p_i y_i(t-\tau)\right]' + \sum_{\ell=1}^{N}\Big[q_{ii}^{(\ell)} y_i(t-\sigma_\ell) \\ + \sum_{j\neq i} q_{ij}^{(\ell)}\delta_i x_j(t-\sigma_\ell)\Big] = 0,\ t \notin S \\ \Delta\left[y_i(t_k) - p_i y_i(t_k-\tau)\right] + \sum_{\ell=1}^{N}\Big[q_{ii}^{(\ell 0)} y_i(t_k-\sigma_\ell) \\ + \sum_{j\neq i} q_{ij}^{(10)}\delta_i x_j(t_k-\sigma_\ell)\Big] = 0,\ t \in S \end{cases}$$

Hence, for $1\le i\le m$ and for sufficiently large t,

$$\begin{cases} \left[y_i(t) - p_i y_i(t-\tau)\right]' + \sum_{\ell=1}^{N}\Big[q_{ii}^{(\ell)} y_i(t-\sigma_\ell) \\ + \sum_{j\neq i} q_{ij}^{(\ell)}\delta_i x_j(t-\sigma_\ell)\Big] \le 0,\ t \notin S \\ \Delta\left[y_i(t_k) - p_i y_i(t_k-\tau)\right] + \sum_{\ell=1}^{N}\Big[q_{ii}^{(\ell 0)} y_i(t_k-\sigma_\ell) \\ + \sum_{j\neq i} q_{ij}^{(10)}\delta_i x_j(t_k-\sigma_\ell)\Big] \le 0,\ t \in S \end{cases} \quad (20)$$

Set

$$v(t) = \sum_{i=1}^{m} y_i(t) - \sum_{i=1}^{m} p_i y_i(t-\tau)$$

and

$$w(t) = \sum_{i=1}^{m} y_i(t)$$

Summing up (vertically) both sides of inequality Eq. (20) for $1\le i\le m$ and using the definitions of q_ℓ and $q_{\ell 0}$ in Eq. (17), we find that for sufficiently large t,

$$\begin{cases} v'(t) + \sum_{\ell=1}^{N} q_\ell w(t-\sigma_\ell) \le 0,\ t\notin S \\ \Delta v(t_k) + \sum_{\ell=1}^{N} q_{\ell 0}\ w(t_k-\sigma_\ell) \le 0,\ t_k \in S \end{cases} \quad (21)$$

As $w(t)>0$ and q_ℓ, $q_{\ell 0}\ge 0$, it follows that v(t) is a decreasing function. Hence, either

$$\lim_{t\to t+\infty} v(t) = -\infty \quad (22)$$

or

$$\lim_{t\to+\infty} v(t) = L \in R \quad (23)$$

First, we claim that condition Eq. (22) is impossible. Otherwise, $v(t)<0$ and at least one of the components $y_i(t)$ would be unbounded. But then finally,

$$w(t) = \sum_{i=1}^{m} y_i(t) \le \sum_{i=1}^{m} p_i y_i(t-\tau) \le \sum_{i=1}^{m} y_i(t-\tau) = w(t-\tau)$$

This implies that w(t) is bounded, which is a contradiction. Thus, condition Eq. (23) holds.

We now claim that L = 0. Indeed, by integrating inequality Eq. (22) from t_0 to t and by letting $t \to \infty$, we obtain

$$L - v(t_0) + \sum_{i=1}^{m}\left(\int_t^\infty q_\ell w(s-\sigma_\ell)ds + \sum_{t\le t_k<\infty} q_{\ell 0} w(t_k-\sigma_\ell)\right) \le 0$$

which implies that $w\in L^1(t_0,\infty)$ Then, $y_i \in L_1\ (t_0,\infty)$ for $1\le i\le m$ so $v \in L^1\ (t_0,\infty)$. But then L = 0, which proves our claim. Thus, as V(t) decreases to zero, it follows that

$$v(t)>0 \text{ and } v(t)\le w(t) \quad (24)$$

Then condition Eq. (21) implies that the finally positive function v(t) satisfies the inequality

$$\begin{cases} v'(t) + \sum_{\ell=1}^{N} q_\ell v(t-\tau_\ell) \le 0,\ t\notin S \\ \Delta v(t_k) + \sum_{\ell=1}^{N} q_{\ell 0} v(t_k-\tau_\ell) \le 0,\ t_k \in S \end{cases} \quad (25)$$

From corollary 1, it follows that Eq. (19) has a finally positive solution. This contradicts the hypothesis and thus, completes the proof of Theorem 2.

Remark 1: It can be shown that Theorem 2 holds word for word for systems of the form of Eq. (15) with the continuous Q_ℓ: $[t_0,\infty) \to R^{m\times m}$ matrix functions. In this case, the coefficients q_ℓ of Eq. (19) are the functions

$$\begin{cases} q_\ell(t) = \min_{1\le i\le m}\left[q_{ii}^{(\ell)}(t) - \sum_{\substack{j=1\\ j\neq i}}^{m}\left|q_{ji}^{(\ell)}(t)\right|\right] \text{ for } 1\le\ell\le N,\ t\notin S \\ q_{\ell k} = \min_{1\le i\le m}\left[q_{ii}^{(\ell k)} - \sum_{\substack{i=1\\ j\neq i}}^{m}\left|q_{ji}^{(\ell k)}\right|\right] \text{ for } 1\le\ell\le N,\ k\in N,\ t_k\in S \end{cases}$$

and oscillation is in the sense of definition 3.

Remark 2: In the special case where, the diagonal matrix P in Eq. 15 is a multiple of the identity matrix, that is, when

$$p_1 = p_2 = \ldots = p_n \equiv p\in[0,1] \quad (26)$$

we have

$$v(t) = w(t) - pw(t-\tau) \quad (27)$$

By substituting Eq. (27) repeatedly into inequality Eq. (21), we find after ξ steps, that finally

$$\begin{cases} v'(t) + \sum_{\ell=1}^{N} q_\ell \left[\sum_{i=0}^{\xi} p^i v(t - \sigma_\ell - i\tau) \right] \le 0, t \notin S \\ \Delta v(t_k) + \sum_{\ell=1}^{N} q_{\ell 0} \left[\sum_{i=0}^{\xi} p^{i0} v(t_k - \sigma_\ell - i\tau) \right] \le 0, t_k \in S \end{cases}$$

Hence, we obtain the following extension of Theorem 1.

Theorem 3: Assume that conditions Eq. (16 and 18) are fulfilled and that the following hypothesis holds:

Condition 4: There exists a non-negative integer ξ such that every solution of the delay impulsive equation

$$\begin{cases} u'(t) + \sum_{\ell=1}^{N} q_\ell \left(\sum_{i=0}^{\xi} p^i u(t - \sigma_\ell - i\tau) \right) = 0, t \notin S \\ \Delta u(t_k) + \sum_{\ell=1}^{N} q_{10} \left(\sum_{i=0}^{\xi} p^{i0} u(t_k - \sigma_\ell - i\tau) \right) = 0, t_k \in S \end{cases} \quad (28)$$

oscillates. Then every solution of Eq. (15) oscillates component-wise.

Hypothesis condition 2 is, for example, satisfied when for some $\xi \ge 0$,

$$\sum_{\ell=1}^{N} q_\ell \left[\sum_{i=0}^{\xi} p^i (\sigma_\ell + i\tau) \right] > e^{-1}$$

or equivalently

$$\sum_{\ell=1}^{N} \left(q_\ell \sigma_\ell \sum_{i=0}^{\xi} p^i + q_\ell \tau \sum_{i=0}^{\xi} p^i i \right) > e^{-1} \quad (29)$$

But it can be shown that inequality Eq. (29) is satisfied if and only if

$$\sum_{i=0}^{\infty} p^i \left(\sum_{\ell=1}^{N} q_\ell \sigma_\ell \right) + \left(\sum_{i=1}^{\infty} p^i i \right) \left(\sum_{\ell=1}^{N} q_\ell \right) \tau > e^{-1}$$

or equivalently,

$$\frac{1}{1-p} \sum_{\ell=1}^{N} q_\ell \sigma_\ell + \frac{\tau}{(1-p)^2} \sum_{\ell=1}^{N} q_\ell > e^{-1} \quad (30)$$

The concluding result is now an immediate consequence of the discussion.

Corollary 2: Assume that conditions Eq. (16, 18 and 30) are satisfied. Then every solution of Eq. (15) oscillates component-wise.

CONCLUSION

The oscillatory criteria stated and proved above are sufficient and are readily extendable to systems of neutral delay impulsive differential equations with variable coefficients.

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