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Solving Second Kind Fredholm Integral Equations with Special Oscillatory Kernel

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Abstract: In this study, we find numerical solution of

$$u(x) = f(x) + \lambda \int_a^b \underbrace{\cos(\omega(x \pm t))}_{k(x,t)} u(t)dt$$

$$\omega >> 1,$$

$$u(x) = f(x) + \lambda \int_a^b \underbrace{\sin(\omega(x \pm t))}_{k(x,t)} u(t)dt$$

by Newton Interpolation. We determined coefficients $\{a_i\}_{i=0}^{n+1}$ such that

$$\sum_{j=0}^{k} aj\ n_j(x)$$

to be an approximation for u (x). This method give an approximate solution for integral equation and also it is powerful in solving both Fredholm and Volterra integral equations, specially for the first kind. In this study, we use special interpolation and quadrature rule for numerical integration. Effectiveness and accuracy of new method are presented with numerical examples.

Key words: Integral equation, newton interpolation, numerical examples, quadrater method

INTRODUCTION

The theory of differential equations is an essential ingredient in mathematics and the majority of numerical analysis.

For some reason the theory and, perhaps more so, the numerical solution, of integral equations are deferred to a later stage: in some sense integral equations must be felt to be either more advanced or of less practical interest than differential equations. This reflects the situation in practical calculations and probably t helps to perpetuate it, we turn more readily to a differential formulation of a problem than to an integral formalism.

In mathematics, an integral equation is an equation in which an unknown function appears under an integral sign. There is a close connection between differential and integral equations and some problems may be formulated either way. See, for example, Maxwell's equations The most basic type of integral equation is a Fredholm equation of the first type:

$$f(x) = \lambda \int_a^b k(x,t)\ \phi(t)dt$$

The notation follows Arfken. ϕ is a unknown function, f is an known function k is another known function of 2 variables, often called the kernel function. Note that the limits of integration are constant; this is what characterizes a Fredholm equation. If the unknown function occurs both inside and outside of the integral, it is known as: a Fredholm equation of the second type:

$$u(x) = f(x) + \lambda \int_a^b k(x,t)\ u(t)dt$$

The parameter λ is an unknown factor, which plays the same role as the eigenvalue in linear algebra. In this study, we worked at Fredholm equation of the second type.

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Newton polynomial: In the mathematical field of numerical analysis, a Newton polynomial (Meinardus, N.D), named after its inventor Isaac Newton, is the interpolation polynomial for a given set of data points in the Newton form. The Newton polynomial is sometimes called Newton's divided differences interpolation polynomial because the coefficients of the polynomial are calculated using divided differences.

As there is only one interpolation polynomial for a given set of data points it is a bit misleading to call the polynomial Newton interpolation polynomial. The more precise name is interpolation polynomial in the Newton form. Given a set of k + 1 data points $(x_0, y_0),..., (x_k, y_k)$ where no 2 x_j are the same, the interpolation polynomial in the Newton form is a linear combination of Newton basis polynomials

$$N(x) = \sum_{i=1}^{k} a_j n_j(x)$$

with the Newton basis polynomials defined as:

$$n_j(x) = \prod_{i=0}^{j-1}(x - x_i)$$

and the coefficients defined as:

$$a_j = [y_0,..., y_j]$$

where,

$$[y_0,..., y_j]$$

is the notation for divided differences. Thus, the newton polynomial can be written as:

$$N(x) = [y_0] + [y_0, y_1]\,(x - x_0) +...+ [y_0,..., y_k]\,(x - x_0)\,(x - x_1)...\,(x - x_{k-1})$$

Solving an interpolation problem leads to a problem in linear algebra where we have to solve a system of linear equations. Using a standard monomial basis for our interpolation polynomial we get the very complicated Vandermonde matrix. By choosing another basis, the Newton basis, we get a system of linear equations with a much simpler lower triangular matrix which can be solved faster. For k + 1 data points we construct the Newton basis as:

$$n_j(x) := \prod_{i=0}^{j-1}(x - x_i)\; j = 0,...,k.$$

Using these polynomials as a basis for Π_k we have to solve:

$$\begin{bmatrix} 1 & & & & \\ 1 & x_1 - x_0 & (x_1 - x_0)(x_2 - x_0) & & \\ 1 & x_2 - x_0 & & & 0 \\ \vdots & \vdots & \cdots & \ddots & \prod_{j=0}^{k-1}(x_k - xj) \\ 1 & x_k - x_0 & & \cdots & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_k \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ \vdots \\ y_k \end{bmatrix}$$

to solve the polynomial interpolation problem. This system of equations can be solved recursively by solving

$$\sum_{i=0}^{j} a_j n_j(x_j) = y_j \qquad j = 0,..., k$$

The new method for solving second kind fredholm integral equation with Special oscillatory Kernel using asymptotic method consider the linear fredholm integral equation:

$$u(x) = f(x) + \lambda\int_a^b \cos(\omega(x \pm t))u(t)dt$$

$$\omega >> 1, \qquad (1)$$

$$u(x) = f(x) + \lambda\int_a^b \sin(\omega(x \pm t))u(t)dt$$

Note that

$$\int f(x)e^{i\omega g(x)}dx = \int \cos(\omega g(x))dx + i\int \sin(\omega g(x))dx \qquad (2)$$

Therefore, from (2), for solving (1) can be solve:

$$u(x) = f(x) + \int_a^b u(t)e^{i\omega g(x \pm t)}dt, \qquad (3)$$

For solve the integral Eq. 3, let

$$u(x) \cong u_N(x) = \sum_{i=0}^{N+1} \alpha_i n_i(x), \quad i = 0,1,..., N+1, \qquad (4)$$

That α_i is a unknown parameter that be definited.
From (3) and (4):

$$\sum_{i=0}^{N+1} \alpha_i n_i(x) = f(x) + \lambda\int_a^b e^{i\omega(x \pm t)} \sum_{i=0}^{N+1} \alpha_i n_i(x)\,dt \qquad (5)$$

By simplify and arrange relation (5) at α_i, have

$$\sum_{i=0}^{N+1}(n_i(x)-\lambda\int_a^b e^{i\omega(x\pm t)}n_i(t)\,dt)\alpha_i = f(x) \quad (6)$$

Let

$$x = x_i = a + i\frac{b-a}{N+1}, \quad i=0, 1, \ldots, N+1$$

Therefore, from relation (6) can be derived the algebraic system:

$$((N - \lambda K) = f)_{N+1\times N+1} \quad (7)$$

That

$$N = \begin{pmatrix} n_0(x_0) & n_1(x_0) & \cdots & n_{N+1}(x_0) \\ n_0(x_1) & n_1(x_1) & \cdots & n_{N+1}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ n_0(x_{N+1}) & n_1(x_{N+1}) & \cdots & n_{N+1}(x_{N+1}) \end{pmatrix},$$

$$K = \begin{pmatrix} \int_a^b e^{i\omega(x_0\pm t)}n_0(t) & \int_a^b e^{i\omega(x_0\pm t)}n_1(t) & \cdots & \int_a^b e^{i\omega(x_0\pm t)}n_{N+1}(t) \\ \int_a^b e^{i\omega(x_1\pm t)}n_0(t) & \int_a^b e^{i\omega(x_1\pm t)}n_1(t) & \cdots & \int_a^b e^{i\omega(x_1\pm t)}n_{N+1}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^b e^{i\omega(x_{N+1}\pm t)}n_0(t) & \int_a^b e^{i\omega(x_{N+1}\pm t)}n_1(t) & \cdots & \int_a^b e^{i\omega(x_{N+1}\pm t)}n_{N+1}(t) \end{pmatrix}$$

And

$$F = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$$

By solved the system (7) can be denote the coefficient,

$$\{\alpha_i\}_{i=0}^{n+1}$$

therefore,

$$u(x) \cong u_N(x) = \sum_{i=0}^{N+1}\alpha_i n_i(x), \quad i = 0, 1, \ldots, N+1.$$

Is approximated at u (x) that solution of the linear Fredholm integral equation.

Computing of integrals: We approximate above integrals in (1), (3), (5), (6) and (7) by asymptotic method (Dahmardeh *et al*., 2008):

$$K_{ij} = \int_a^b e^{i?(x_i\pm t)}(t)\,dt$$

$$K_{ij} \approx -\sum_{l=0}^{N+1}{}'' \frac{1}{(-i?)^l}\begin{pmatrix} e^{i\omega(x_i\pm b)}n_j^{(l-1)}(b) \\ -e^{i\omega(x_i\pm i)}\ n_j^{(l-1)}(a) \end{pmatrix}$$

where here the double prime on the above summation sign implies that the first and latest terms are halved.

After solving (N+2)_ Eq. (5-7). We get an appropriate set of

$$\{a_i\}_{i=0}^{n+1}$$

coefficients. Then we approximate the solution of integral equations. By

$$\sum_{j=0}^{N+1} a_j\ n_j(x)$$

Numerical examples: In this study we give three examples for integral equations second kind (Delves and Mohamed, 1985). For following examples we assumed that N = 10. All computations were carried out using MATLAB 7.

Example 1: Fredholm integral equation of the second kind $\lambda = 1$ (Fig. 1):

$$u(x) = f(x) + \lambda\int_0^{\pi}\sin(10(x+t))u(t)dt$$

$$f(x) = 1 - 3.3630\cos(x) - 0.6815\pi\sin(x)$$

$$u(x) = 1 + \frac{2\lambda\cos(x) + \lambda^2\ p\sin(x)}{1 - \frac{1}{4}\lambda^2\pi^2},$$

Example 2: Fredholm integral equation of the second kind $\lambda = 20$ (Fig. 2):

$$u(x) = f(x) + \lambda\int_0^{\frac{\pi}{3}}\sin(100(x+t))u(t)dt,$$

$$f(x) = e^{-x^2} + \frac{x(1-e^{-x^2})}{2} + \frac{1}{5}\cos(100x + \frac{\pi}{6}) - \frac{1}{5}\sin(100x)$$

$$u(x) = e^{-x^2} + \frac{x(1-e^{-x^2})}{2}$$

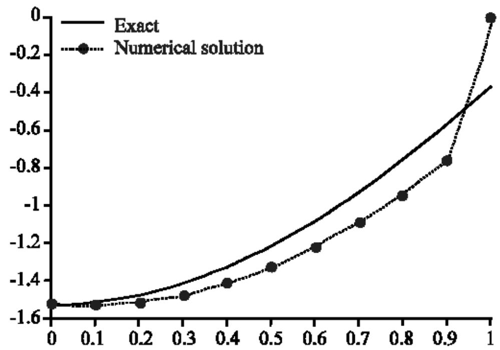

Fig. 1: Results for example 1 with. Source (Maleknejad and Rahbar, 2000; Kanwal, 1971)

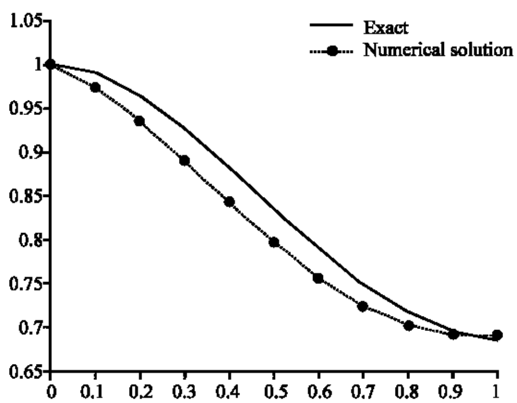

Fig. 2: Results for example 2 with. Source (Rahbar, 2000; Baker, 1969)

CONCLUSION

This method is applicable for both the second and first kind Fredholm integral equation. In addition this method instead of finding values of the function in approximate function by expansion of Newton Interpolation.

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$$\int_0^1 f(x)e^{i\omega g(x)}dx$$

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