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Formulations of Continuous Multistep Method for Direct Solution of Third and Fourth Order ODEs Using Chebyshev Polynomials as Basis Function

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Abstract: The numerical solution of first order ordinary differential equations have been extensively discussed and well studied as reported in the literatures. Also the development of multistep schemes for solving second order ODEs there been presented by some members. In this presentation an attempt shall be made to develop a numerical scheme (multistep scheme) that can be used to solve third order and forth order ODEs directly without necessarily resolving this type of equations into system of first order ODEs.

Key words: Collocation, interpolation, zero stability, consistency chebyshev polynomial

INTRODUCTION

In the resent past many researches have developed many multistept schemes both discrete and continuous schemes in solving first order ODEs (Adeniyi, 1994; Abelman and Eyre, 1990; Alabi *et al*., 2007; Fatunla, 1988; Omolehin *et al*., 2005; Onumayin *et al*., 1993) to mention but a few because the list is inexhaustible.

Solution to higher ODEs numerically posses some problem in which on have to resolve such equation and system o first order ODEs and later solve by any known initial value solves. This problem was circumvented in the case of second order ODEs in which Lambert (1973) proposed a direct method of solving such an equation. Though the proposed method is a discrete scheme which is a-two step. In the bid to improve upon the accuracy of the existing method, Kayode and Awoyemi (2005) proposed and developed a five step method fot the solution of second order differential equations. In like manner, Awoyemi and Kayode (2003) developed an optimal order collocation method for direct solution of initial value problems of general second order ordinary differential equations, also Awoyemi and Kayod (2005) came up with an implicit collocation method for direct solution of second order ordinary differential equations, later Alabi *et al.* (2008) developed a Six-step method for solving second order ordinary differential equations.

As a result of this challenge of direct solution to differential equations of higher order and in a bid to remove the hurdle of resolving higher order ODEs into system of first order ODEs, we are presenting the duration of multistep scheme (both continuous and discrete) for solving third and fourth order ODEs directly. In order to achieve this we shall make use of Chebyshev polynomial as our basic function.

MATERIALS AND METHODS

In this research, we shall present the derivation of multistep schemes for solving third and fourth order differential equations directly.

Method for third order ordinary differential equations: Here, an attempt shall be made to derive a multistep method for solving a third order ordinary differential equations without necessarily reducing the equation to a system of first order ordinary differential equations. In this wise, we consider the solution to the differential equation:

$$y'''(x) = f(x, Y(x), Y'(x), Y''(x)) \quad (1)$$

Subject to:

$$y(a) = \xi, \quad y'(a) = \eta \text{ and } y''(a) = \varsigma \quad (2)$$

In order to carry out the desired method, we consider

$$\begin{aligned} Y'''(x) &= f\left(x,\ Y(x)\right),\ x_k \le x \le x_{k+p} \\ Y(x_k) &= Y_k \end{aligned} \quad (3)$$

in which

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$$Y(x) = \sum_{r=0}^{M} a_r T_r(x) \cong y(x) = \sum_{r=0}^{M} a_r T_r(x)\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right) \quad (4)$$

In this wise, we let M = 2n and p = n in (3) and (4), respectively that is:

$$Y```(x) = f(x,\ Y(x)),\ x_k \le x \le x_{k+n} \quad (5)$$
$$Y(x_k) = Y_k$$

and

$$Y(x) = \sum_{r=0}^{2n} a_r T_r(x)\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right) \quad (6)$$

Let n = 3 in (4.36), that is

$$Y```(x) = f(x,\ Y(x)),\ x_k \le x \le x_{k+3} \quad (7)$$
$$Y(x_k) = Y_k$$

and

$$Y(x) = \sum_{r=0}^{2n} a_r T_r(x)\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right) \quad (8)$$

With n = 3 in (6) and collocating (7) at x_{k+r}, r = 0(1)3 while interpolating (8) at x_{k+r}, r = 0(1)2 led to the set of algebraic equations

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 729 & -243 & -567 & 621 & 153 & -723 & 329 \\ 729 & 243 & -567 & -621 & 153 & 723 & 329 \\ 0 & 0 & 0 & 192 & -1536 & 6720 & -24576 \\ 0 & 0 & 0 & 5184 & -13824 & -2880 & 24576 \\ 0 & 0 & 0 & 5184 & 13824 & -2880 & -24576 \\ 0 & 0 & 0 & 192 & 1536 & 6720 & 24576 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} Y_k \\ 729Y_{k+1} \\ 729Y_{k+2} \\ 27h^3 f_k \\ 729h^3 f_{k+1} \\ 729h^3 f_{k+2} \\ 729h^3 f_{k+3} \end{pmatrix}$$

whose solution yielded

$$a_0 = \frac{1}{20480}\left(\begin{matrix} 8960Y_k - 7680Y_{k+1} + 19200Y_{k+2} + 17h^3 f_k + 2489h^3 f_{k+1} + \\ 1991h^3 f_{k+2} - 17h^3 f_{k+3} \end{matrix}\right)$$

$$a_1 = \frac{1}{20480}\left(\begin{matrix} -30720Y_{k+1} + 30720Y_{k+2} - 63h^3 f_k + 3743h^3 f_{k+1} + \\ 3743h^3 f_{k+2} - 63h^3 f_{k+3} \end{matrix}\right)$$

$$a_2 = \frac{1}{163840}\left(\begin{matrix} 92160Y_k - 184320Y_{k+1} + 92160Y_{k+2} + 89h^3 f_k + 24693h^3 f_{k+1} + \\ 21387h^3 f_{k+2} - 89h^3 f_{k+3} \end{matrix}\right)$$

$$a_3 = \frac{1}{8192}\left(9h^3 f_k + 9h^3 f_{k+1} + 567h^3 f_{k+2} + 567h^3 f_{k+3}\right)$$

$$a_4 = \frac{1}{10240}\left(-9h^3 f_k + 9h^3 f_{k+1} - 243h^3 f_{k+2} + 243h^3 f_{k+3}\right)$$

$$a_5 = \frac{1}{20480}\left(81h^3 f_k + 81h^3 f_{k+1} - 81h^3 f_{k+2} - 81h^3 f_{k+3}\right)$$

$$a_6 = \frac{1}{20480}\left(-81h^3 f_k + 81h^3 f_{k+1} + 243h^3 f_{k+2} - 243h^3 f_{k+3}\right)$$

Substituting these values into (8) with n = 3 resulted into the continuous scheme

$$Y(x) = \frac{Y_k}{163840}\left(-20480 + 184320\left(\frac{x - x_k}{3h}\right)^2\right) +$$

$$\frac{Y_{k+1}}{163840}\left(122880 - 245760\left(\frac{x - x_k}{3h}\right) - 368640\left(\frac{x - x_k}{3h}\right)^2\right) +$$

$$\frac{Y_{k+2}}{163840}\left(61440 + 245760\left(\frac{x - x_k}{3h}\right) + 184320\left(\frac{x - x_k}{3h}\right)^2\right) +$$

$$\frac{h^3 f_k}{163840}\left(\begin{array}{l} 110 - 1024\left(\frac{x - x_k}{3h}\right) - 2432\left(\frac{x - x_k}{3h}\right)^2 + 640\left(\frac{x - x_k}{3h}\right)^3 - \\ 2736\left(\frac{x - x_k}{3h}\right)^4 + 64\left(\frac{x - x_k}{3h}\right)^5 + 2592\left(\frac{x - x_k}{3h}\right)^6 \end{array}\right) +$$

$$\frac{h^3 f_{k+1}}{163840}\left(\begin{array}{l} -9187 - 4096\left(\frac{x - x_k}{3h}\right) + 76657\left(\frac{x - x_k}{3h}\right)^2 + 45440\left(\frac{x - x_k}{3h}\right)^3 - \\ 42768\left(\frac{x - x_k}{3h}\right)^4 - 64\left(\frac{x - x_k}{3h}\right)^5 + 7776\left(\frac{x - x_k}{3h}\right)^6 \end{array}\right)$$

$$\frac{h^3 f_{k+2}}{163840}\left(\begin{array}{l} -1328 - 4096\left(\frac{x - x_k}{3h}\right) + 16044\left(\frac{x - x_k}{3h}\right)^2 + 45440\left(\frac{x - x_k}{3h}\right)^3 + \\ 42768\left(\frac{x - x_k}{3h}\right)^4 - 64\left(\frac{x - x_k}{3h}\right)^5 - 7776\left(\frac{x - x_k}{3h}\right)^6 \end{array}\right) +$$

$$\frac{h^3 f_{k+3}}{163840}\left(\begin{array}{l} 16 - 1024\left(\frac{x - x_k}{3h}\right) - 2788\left(\frac{x - x_k}{3h}\right)^2 + 640\left(\frac{x - x_k}{3h}\right)^3 - \\ 2736\left(\frac{x - x_k}{3h}\right)^4 + 64\left(\frac{x - x_k}{3h}\right)^5 + 2592\left(\frac{x - x_k}{3h}\right)^6 \end{array}\right)$$

At the grid point x_{k+3} the continuous scheme yields the discrete scheme

$$Y_{k+3} - 3Y_{k+2} + 3Y_{k+1} - Y_k = \frac{h^3}{2}\left(f_{k+2} + f_{k+1}\right) \quad (9)$$

Method for fourth order ordinary differential equations: In this study, we shall present the derivation of multistep method for solving a-fourth order ordinary differential equations using Chebyshev polynomials as basis function. The main reason for this is to circumvent the hurdle of resolving the differential equations to a system of first order ordinary differential equations. In order to achieve this, we shall make use of our Chebyshev polynomials. We consider the differential equation.

$$Y^{iv}(x) = f(x, Y(x), Y'(x), Y''(x), Y'''(x)) \quad (10)$$

associated with the initial conditions

$$Y(x_0) = x_0,\ Y'(x_1) = \xi,\ Y''(x_2) = \zeta,\ Y'''(x_3) = \eta \quad (11)$$

In order to derive a multistep method to solve (10) with the associated initial conditions (11), we consider

$$Y^{iv}(x) = f(x,\ Y(x)),\ x_k \le x \le x_{k+p} \qquad (12)$$
$$Y(x_k) = Y_k$$

in which

$$Y(x) = \sum_{r=0}^{M} a_r T_r(x) \cong y(x) = \sum_{r=0}^{M} a_r T_r(x)\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right) \qquad (13)$$

in this wise, we let M = 2n and p = n in (13) and (12), respectively such that

$$Y^{iv}(x) = f(x,\ Y(x)),\ x_k \le x \le x_{k+n} \qquad (14)$$
$$Y(x_k) = Y_k$$

and

$$Y(x) = \sum_{r=0}^{2n} a_r T_r(x)\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right) \qquad (15)$$

let n = 4 in (15), that is

$$Y^{iv}(x) = f(x,\ Y(x)),\ x_k \le x \le x_{k+4} \qquad (16)$$
$$Y(x_k) = Y_k$$

and

$$Y(x) = \sum_{r=0}^{2n} a_r T_r(x)\left(\frac{2x}{nh} - \frac{2k}{n} - 1\right) \qquad (17)$$

Here, we collocate (16) at x_{k+r}, r = 0(1)r and interpolate (17) at x_{k+r}, r = 0(3) with n = 4 in (17) led to the set of algebraic equations

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 2 & -1 & -2 & 2 & -1 & -1 & 2 & -1 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 1 & -1 & -2 & -1 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 12 & -120 & 648 & -2520 & 7920 \\ 0 & 0 & 0 & 0 & 12 & -60 & 108 & 0 & -360 \\ 0 & 0 & 0 & 0 & 12 & 0 & -72 & 0 & 240 \\ 0 & 0 & 0 & 0 & 12 & 60 & 108 & 0 & -360 \\ 0 & 0 & 0 & 0 & 12 & 120 & 648 & 2520 & 7920 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} Y_k \\ 2Y_{k+1} \\ Y_{k+2} \\ 2Y_{k+3} \\ h^4 f_k \\ h^4 f_{k+1} \\ h^4 f_{k+2} \\ h^4 f_{k+3} \\ h^4 f_{k+4} \end{pmatrix}$$

in which its solution yields

$$a_0 = \frac{1}{2520}\begin{pmatrix} -h^4 f_{k+4} + 67h^4 f_{k+3} + 288h^4 f_{k+2} + 67h^4 f_{k+1} - h^4 f_k + 2520Y_{k+3} - \\ 2520Y_{k+2} + Y_{k+1} \end{pmatrix}$$

$$a_1 = \frac{1}{7560}\begin{pmatrix} -5h^4 f_{k+4} + 374h^4 f_{k+3} + 1659h^4 f_{k+2} + 494h^4 f_{k+1} - 2h^4 f_k + \\ 12600Y_{k+3} - 15120Y_{k+2} + 7560Y_{k+1} - 5040Y_k \end{pmatrix}$$

$$a_2 = \frac{1}{60480}\begin{pmatrix} -31h^4 f_{k+4} + 2476h^4 f_{k+3} + 10230h^4 f_{k+2} + 2476h^4 f_{k+1} - 31h^4 f_k + \\ 60480Y_{k+3} - 120960Y_{k+2} + 60480Y_{k+1} \end{pmatrix}$$

$$a_3 = \frac{1}{4320}\begin{pmatrix} -h^4 f_{k+4} + 124h^4 f_{k+3} + 474h^4 f_{k+2} + 124h^4 f_{k+1} - h^4 f_k + 1440Y_{k+3} - \\ 4320Y_{k+2} + 4320Y_{k+1} - 1440Y_k \end{pmatrix}$$

$$a_4 = \frac{h^4}{60}\left(f_{k+3} + 3f_{k+2} + f_{k+1}\right)$$

$$a_5 = \frac{h^4}{120}\left(f_{k+3} - f_{k+1}\right)$$

$$a_6 = \frac{h^4}{30240}\left(5f_{k+4} + 64f_{k+3} + 138f_{k+2} + 64f_{k+1} + 5f_k\right)$$

$$a_7 = \frac{h^4}{5040}\left(f_{k+4} - 2f_{k+3} + 2f_{k+1} - f_k\right)$$

$$a_8 = \frac{1}{20160}\left(f_{k+4} - 4f_{k+3} + 6f_{k+2} - 4f_{k+1} + f_k\right)$$

These values together with (17) with n = 4 leads to the continuous

$$Y(x) = \frac{Y_k}{3}\left\{\left(\frac{x-x_k}{4h}\right) - \frac{(x-x_k)^3}{16h^3}\right\} - \frac{Y_{k+1}}{4}\left\{\frac{2(x-x_k)}{h} - \frac{(x-x_k)^2}{2h^2} - \frac{(x-x_k)^3}{4h^3}\right\} +$$

$$Y_{k+2}\left\{1 + \frac{x-x_k}{4h} - \left(\frac{x-x_k}{2h}\right)^2 - \frac{(x-x_k)^3}{16h^3}\right\} + \frac{Y_{k+3}}{3}\left\{\frac{(x-x_k)}{2h} + 6\left(\frac{x-x_k}{4h}\right)^2 + \frac{(x-x_k)^3}{16h^3}\right\} +$$

$$\frac{h^4 f_k}{60480}\left\{\begin{array}{l}\frac{55(x-x_k)}{2h} + \frac{11(x-x_k)^2}{8h^2} + \frac{77(x-x_k)^3}{8h^3} - \frac{21(x-x_k)^5}{16h^5} - \frac{7(x-x_k)^6}{64h^6} + \\ \frac{3(x-x_k)^7}{64h^7} + \frac{6(x-x_k)^8}{1024h^8}\end{array}\right\} +$$

$$\frac{h^4 f_{k+1}}{15120}\left\{\begin{array}{l}\frac{-493(x-x_k)}{2h} - \frac{53(x-x_k)^2}{8h^2} + \frac{287(x-x_k)^3}{4h^3} - \frac{42(x-x_k)^5}{16h^5} + \frac{7(x-x_k)^6}{16h^6} + \\ \frac{6(x-x_k)^7}{256h^7} - \frac{6(x-x_k)^8}{1024h^8}\end{array}\right\} +$$

$$\frac{h^4 f_{k+2}}{10080}\left\{\begin{array}{l}\frac{553(x-x_k)}{2h} - \frac{1546(x-x_k)^2}{16h^2} + \frac{1106(x-x_k)^3}{16h^3} + \frac{420(x-x_k)^5}{16h^5} - \\ \frac{35(x-x_k)^6}{64h^6} + \frac{6(x-x_k)^8}{1024h^8}\end{array}\right\} +$$

$$\frac{h^4 f_{k+3}}{15120}\left\{\begin{array}{l}\frac{59(x-x_k)}{2h} - \frac{106(x-x_k)^2}{16h^2} - \frac{280(x-x_k)^3}{16h^3} + \frac{168(x-x_k)^5}{64h^5} + \frac{7(x-x_k)^6}{16h^6} - \\ \frac{6(x-x_k)^7}{256h^7} - \frac{6(x-x_k)^8}{1024h^8}\end{array}\right\} +$$

$$\frac{h^4 f_{k+4}}{60480}\left\{\begin{array}{l}\frac{-82(x-x_k)}{4h}+\frac{11(x-x_k)^2}{8h^2}+\frac{77(x-x_k)^3}{8h^3}-\frac{21(x-x_k)^5}{16h^5}-\frac{7(x-x_k)^6}{64h^6}+\\ \frac{3(x-x_k)^7}{64h^7}+\frac{6(x-x_k)^8}{1024h^8}\end{array}\right\}$$

Evaluating the continuous scheme at the grid point x_{k+4} gives the discrete formulation

$$Y_{k+4}-4Y_{k+3}+6Y_{k+2}-4Y_{k+1}+Y_k=\frac{-h^4}{720}\left(f_{k+4}-124f_{k+3}-474f_{k+2}-124f_{k+1}+f_k\right) \quad (18)$$

RESULTS AND DISCUSSION

The study represents the analysis of the result viz-a-viz the order and error constant of the methods. Here, we wish to state the following definitions:

- A linear multistep method is zero-stable if no roots of the first characteristics polynomials $\rho(\xi)$ has modulus greater than one and if every root with modulus one is simple.
- No zero-stable multistep method of step number k cannot have order exceeding k + 1 when k is odd or exceeding k + 2 when k is even.
- A linear multistep method is consistent if it has order $p \geq 1$. Note that the necessary and sufficient condition for a multistep method to be convergent are that it be both consistent and zero stable. Qualitatively, consistency controls the magnitude of the local truncation error committed at each stage of the calculation while zero-stability controls the manner in which this error is propagated as the calculation proceeds.
- The interval of absolute stability of a multistep method is the interval (α, β) of the real line if the method is absolutely stable for all $\bar{h} \in (\alpha, \beta)$.

In order to calculate the order of scheme (9), we consider

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h^3\sum_{j=0}^{k}\beta_j f_{n+j} \quad (19)$$

associated with the linear difference operator

$$\hbar\left[y(x);h\right]=\sum_{j=0}^{k}\left[\alpha_j y(x+jh)-h^3\beta_j y'''(x+jh)\right] \quad (20)$$

where, y(x) is an arbitrary function, continuously differentiable in the interval [a, b]. Assuming that y(x) has as many higher derivatives as required, then on Taylor's expansion about the point x, we obtain

$$\hbar[y(x);h]=C_0y(x)+C_1hy'(x)+\ldots+C_qh^qy^q(x)+.,$$

Where,

$$C_0=\sum_{j=0}^{k}\alpha_j,\ C_1=\sum_{j=0}^{k}j\alpha_j,$$

$$C_3=\frac{1}{3!}\sum_{j=1}^{k}j^3\alpha_j-\sum_{j=0}^{k}\beta_j$$

$$C_q=\frac{1}{q!}\sum_{j=1}^{k}j^q\alpha_j-\frac{1}{(q-3)!}\sum_{j=1}^{k}j^{q-3}\beta_j \qquad \text{where } q=4(1)k \quad (21)$$

Following Henrici argument as was presented in Lambert (1973), linear multistep method (19) is of order p if $C_0 = C_1 = \ldots C_p = C_{p+2} = 0$ but $C_{p+3} \neq 0$, where C_{p+3} is the error constant and

$$C_{p+3}h^{p+3}y^{(p+3)}(x_n)$$

is the principal local truncation error at the point x_n.

Following the steps above, it was discovered that Scheme (9) is of order 4 with error constant of 19/5040

So also to determine the order and error constant of Scheme (18), we consider

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h^4\sum_{j=0}^{k}\beta_j f_{n+j} \quad (22)$$

associated with the linear difference operator

$$\hbar\left[y(x);h\right]=\sum_{j=0}^{k}\left[\alpha_j y(x+jh)-h^4\beta_j y^{iv}(x+jh)\right] \quad (23)$$

where y(x) is an arbitrary function, continuously differentiable in the interval [a, b]. Assuming that y(x) has as many higher derivatives as required, then on Taylor's expansion about the point x, we obtain

$$\hbar[y(x);h] = C_0 y(x) + C_1 h y'(x) + \ldots + C_q h^q y^q(x) +$$

Where,

$$C_0 = \sum_{j=0}^{k} \alpha_j,\ C_1 = \sum_{j=0}^{k} j\alpha_j,\ C_2 = \sum_{j=0}^{k} j^2\alpha_j,\ C_3 = \sum_{j=0}^{k} j^3\alpha_j$$

$$C_4 = \frac{1}{4!}\sum_{j=1}^{k} j^4\alpha_j - \sum_{j=0}^{k}\beta_j$$

$$C_q = \frac{1}{q!}\sum_{j=1}^{k} j^q\alpha_j - \frac{1}{(q-4)!}\sum_{j=1}^{k} j^{q-4}\beta_j \qquad \text{where } q = 5(1)k \tag{24}$$

Following Henrici argument as presented in Lambert (1973), linear multistep method (22) is of order *p* if $C_o = C_1 = \ldots C_p = C_{p+3} = 0$ but $C_{p+4} \neq 0$, where C_{p+4} is the error constant and $C_{p+4}\, h^{p+4}\, y^{(p+4)}$ is the principal local truncation error at the point x_n.

Applying condition (24) on, multistep scheme (18), it was discovered that method (18) is of order 6, while its error constant is 1/3024.

Next, we shall consider the issue of zero stability of our methods. For scheme (9), it was discovered that all the roots of first characteristics polynomials $\rho(\xi)$ has modulus not greater that one and are simple. Also for scheme (18), all the roots of the first characteristics polynomials have modulus not greater that one and are all simple.

CONCLUSION

Here we wish to conclude that since our methods are both consistent and zero stable, we therefore recommend that the two methods can be used for solving third and fourth order differential equations directly. This removes the hurdle of resolving such equations into system of first order ordinary differential equations before solving them with any known initial value solvers.

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