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Dynamic Bhaviour Under Moving Concentrated Masses of Elastically Supported Finite Bernoulli-Euler Beam on Winkler Foundation

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Abstract: The dynamic response of an elastically supported Bernoulli-Euler beam carrying moving masses and resting on a constant elastic foundation is investigated in this study. This problem, involving non-classical boundary conditions is solved and illustrated with 2 commonest examples often encountered in Engineering practice. Analysis of the closed form solutions shows that, for the same natural frequency the response amplitude for the moving mass problem is greater than that of the moving force problem for fixed axial force and foundation modulli, The critical speed for the moving mass problem is smaller than that for the moving force problem and so resonance is reached earlier in the former. Similarly, an increase in the value of foundation modulli and axial force reduces the critical speed for both illustrative examples. The response amplitudes of both moving force and moving mass problems was also found to decrease when both the foundation moduli K and the axial force N are increased.

Key words: Dynamic behaviour, concentrated masses, elastically, Bernoulli-Euler

INTRODUCTION

The analyses of elastic structures (beams, plates and shells), resting on a subgrade, such as railway tracks, highway pavements, navigation locks and structural foundations, constitute an important part of the Civil Engineering and applied Mathematics literatures. In general, such analyses are mathematically complex due to the difficulty in modeling the mechanical response of the subgrade which is governed by many factors. When these structures are acted upon by moving loads, the dynamic analyses of the system become more cumbersome (Fryba, 1972). The crudest approximation known to the literatures to this problem is the so called moving force problem, in which the vehicle-track interaction is completely neglected and the action of the vehicle is described as a concentrated force moving along the beam (Muscolino and Palmeri, 2007). Several researchers have considered vehicle-track interaction in their analyses. These they commonly termed moving mass problems. These researchers include Stanisic *et al*. (1974), Milornir *et al*. (1969), Clastornic *et al*. (1986), Sadiku and Leipholz (1981) and Gbadeyan and Oni (1995). More recently, Douglas *et al*. (2002) solved the problem of plate strip of varying thickness and the center of shear. In their work, they considered a free-vibrating strip with classical boundary conditions, precisely, they assumed the plate strip clamped at one end and free at the other end. Pesterev *et al*. (2001) came up with a series expansion method for calculating bending moment and shear force in the problem of vibration of a damped beam subject to an arbitrary number of moving loads. This kind of solution, though could be accurate, cannot account for vital information such as the phenomenon of resonance in the dynamical system.

However, in all these, the problem of determining the dynamic response of beams under the action of moving concentrated masses has been almost exclusively reserved for elastic beams having the normal ideal boundary conditions. Such ideal boundary conditions include among others, Clamped edge, Free edge, Simply supported edge and Sliding edge boundary conditions. For practical applications in many cases, it is more realistic to consider non-classical boundary conditions because the ideal boundary conditions can seldom be realized. A common example is the elastically supported end conditions. As a problem of this kind, Wilson (1974) studied the response of a cantilever plate strip restrained elastically against rotation and subjected to a moving normal line load. In a later development, Saito *et al*. (1980) presented a theoretical analysis of the steady state response of a plate strip constrained elastically along its edges against rotation and translation under the action of a moving transverse line load. The first 5 speeds of the applied load for which a resonance effect occurs in the system are plotted as functions of the edge constraint

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parameters. The profiles of the displacement and the moment of the plate are also shown graphically for several values of the load speed and the edge constraint.

It is remarked at this juncture, that the results of these works on non-classical boundary conditions could seriously be misleading as only the force effect of the moving load is taken into consideration in their calculations while the inertia effect is neglected. Thus, in this study, the dynamical analysis of an elastically supported finite Bernoulli-Euler beam resting on a Winkler elastic foundation and under moving concentrated masses is considered. The object is to classify the effects of the elastic constraints, foundation modulli and prestress on the response of the beam.

Governing equation: The equation governing the response of elastically supported Bernoulli-Euler beam on a constant foundation and traversed by an arbitrary number of concentrated masses is given by the 4th order partial differential equation:

$$EI\frac{\partial^4 V(x,t)}{\partial x^4}+\mu\frac{\partial^2 V(x,t)}{\partial t^2}-N\frac{\partial^2 V(x,t)}{\partial x^2} + KV(x,t)=P(x,t) \quad (1)$$

Where,

x = The special coordinate.
t = The time.
V(x,t) = The transverse displacement.
E = The Young's modulus.
I = The moment of inertia.
μ = The mass per unit length of the beam.
N = The axial force
K = The elastic foundation.

The moving load on the beam under consideration has mass commensurable with the mass of the beam. Thus, the load P(x,t) takes the form (Gbadeyan and Oni, 1995):

$$P(x,t)=P_{mf}[1-\frac{\eta}{g}(V(x,t))] \quad (2)$$

The operator η is defined as follows:

$$\eta=\frac{\partial^2}{\partial t^2}+2c_i\frac{\partial^2}{\partial x\partial t}+c_i^2\frac{\partial^2}{\partial x^2} \quad (3)$$

and the continuous moving force P_{mf} acting on the beam model is given by:

$$P_{mf}=\sum_{i=1}^{m}M_i g\delta(x-c_i t) \quad (4)$$

The time t is assumed to be limited to that interval of time within which the mass μ is on the beam that is:

$$0\le c_i t\le L, \quad (5)$$

Where,

L = The length of the beam.
δ(x-ct) = The Dirac-delta function defined as:

$$\delta(x-ct)=\begin{cases}0, x\ne ct\\ \infty, x=ct\end{cases} \quad (6)$$

with the properties

$$(-x) = (x)$$

and

$$\int_a^b\delta(x-k)f(x)dx=\begin{cases}0, & k<a<b\\ f(k), & a<k<b\\ 0, & a<b<k\end{cases} \quad (7)$$

In mechanics, the Dirac-delta function $\delta(x)$ may be thought of as a unit concentrated force acting at point x = 0 (Gbadeyan and Oni, 1995). Substituting Eq. 2-4 into Eq. 1, we have:

$$EI\frac{\partial^4 V(x,t)}{\partial x^4}+\mu\frac{\partial^2 V(x,t)}{\partial t^2}-N\frac{\partial^2 V(x,t)}{\partial x^2}+KV(x,t) =\sum_{i=1}^{m}M_i g\delta(x-c_i t)\left[1-\frac{1}{g}\left(\frac{\partial^2}{\partial t^2}+2c_i\frac{\partial^2}{\partial x\partial t}+c_i^2\frac{\partial^2}{\partial x^2}\right)V(x,t)\right] \quad (8)$$

In this study, in the first instance, a Bernoulli-Euler Beam with classical boundary conditions at the end x = 0 and elastically supported at the end x = L is considered. For example, the associated boundary conditions at x=0 can be any of

$V(0, t) = 0 = V^{1}(0, t)$ for the Clamped end
$V(0, t) = 0 = V^{11}(0, t)$ for the Simply supported end
$V^{111}(0,t)=0=V^{11}(0,t)$ for the Free end
$V^{1}(0,t) = 0 = V^{111}(0, t)$ for the Sliding end (9a)

while for the other end x = L, we have

$$V^{11}(L, t)-K_1V^{1}(L, t) = 0$$
$$V^{111}(L, t) + K_2V(L, t) = 0 \quad (9b)$$

Where,

K_1 = The stiffness against rotation.

K_2 = The stiffness against translation.

It is clearly understood from (9b) that $K_1 = 0$ and $K_2 = \infty$ implies the Simply supported end, $K_1 = \infty$ and $K_2 = \infty$ implies the Clamped end, $K_1 = 0$ and $K_2 = 0$ implies the Free end while $K_1 = \infty$ and $K_2 = 0$ implies the Sliding end.

In what follows, we consider a Bernoulli-Euler beam elastically supported at both ends, that is, with the boundary conditions:

$$\begin{aligned} &V^{11}(0,t)-K_1V^1(0,t)=0\\ &V^{111}(0,t)+K_2V(0,t)=0 \end{aligned} \tag{10a}$$

at the end x = 0 and

$$\begin{aligned} &V^{11}(L,t)-K_1V^1(L,t)=0\\ &V^{111}(L,t)+K_2V(L,t)=0 \end{aligned} \tag{10b}$$

at the other end x = L.

In both cases, the initial conditions, without any loss of generality are taken as:

$$V(x,0)=0=V_t(x,0) \tag{11}$$

METHOD OF SOLUTION

In an attempt to solve Eq. (8), the method of separation of variables is evidently inapplicable because it becomes difficult to get separate equations where functions are functions of a single variable. In fact, a closed form solution of the above singular differential Eq. (8) does not exist. As a result of the above difficulty, an approximate solution is sought. Here, we employ the generalized finite integral transform defined as follows:

$$Z(n,t)=\int_0^L V(x,t)V_n(x)dx \tag{12}$$

Where,

$$V(x,t)=\sum_{n=1}^{\infty}\frac{\mu}{V_n}Z(n,t)V_n(x) \tag{13}$$

$V_n(x)$ is the general kernel chosen so that the pertinent boundary conditions are satisfied and V_n is defined as:

$$V_n=\int_0^L \mu V_n^2(x)dx \tag{14}$$

$V_n(x)$ is the normal mode of vibrations of the beam and is given as:

$$V_n(x)=\sin\frac{\theta_n x}{L}+A_n\cos\frac{\theta_n x}{L}+B_n\sinh\frac{\theta_n x}{L}+C_n\cosh\frac{\theta_n x}{L} \tag{15}$$

where,

A_n, B_n and C_n are constants that can be determined using the boundary conditions

θ_n is the mode frequency.

ANALYTICAL APPROXIMATE SOLUTION

By applying the generalized finite integral transform (12) (8) can be written as:

$$\begin{aligned} &r_1T(0,L,t)+r_1T_A(t)+Z_{tt}(n,t)-r_2T_B(t)+r_3Z(n,t)\\ &+T_C(t)+T_D(t)+T_E(t)=\sum_{i=1}^{m}\frac{m_ig}{\mu}T_i^*(t) \end{aligned} \tag{16}$$

where

$$r_1=\frac{EI}{\mu},\ r_2=\frac{N}{\mu},\ r_3=\frac{K}{\mu} \tag{17}$$

$$T(0,L,t)=\left[\begin{aligned} &\frac{\partial^3V(x,t)}{\partial x^3}V_n(x)-\frac{\partial^2V(x,t)}{\partial x^2}\frac{dV_n(x)}{dx}\\ &+\frac{\partial V(x,t)}{\partial x}\frac{d^2V_n(x)}{dx^2}-V(x,t)\frac{d^3V_n(x)}{dx^3} \end{aligned}\right]_0^L \tag{18}$$

$$\begin{aligned} &T_A(t)=\int_0^L V(x,t)\frac{d^4V_n(x)}{dx^4}dx,\\ &T_B(t)=\int_0^L \frac{\partial^2V(x,t)}{\partial x^2}V_n(x)dx \end{aligned} \tag{19}$$

$$\begin{aligned} &T_C(t)=\sum_{i=1}^{m}\frac{m_i}{\mu}\int_0^L\delta(x-c_it)\frac{\partial^2V(x,t)}{\partial t^2}V_n(x)dx,\\ &T_D(t)=\sum_{i=1}^{m}\frac{2c_im_i}{\mu}\int_0^L\delta(x-c_it)\frac{\partial^2V(x,t)}{\partial x\partial t}V_n(x)dx, \end{aligned} \tag{20}$$

and

$$\begin{aligned} &T_E(t)=\sum_{i=1}^{m}\frac{c_i^2m_i}{\mu}\int_0^L\delta(x-c_it)\frac{\partial^2V(x,t)}{\partial x^2}V_n(x)dx\\ &\text{and } T_i^*(t)=\int_0^L\delta(x-c_it)V_n(x)dx \end{aligned} \tag{21}$$

It is clear that the natural modes (15) satisfy the homogeneous differential equation:

$$EI\frac{d^4V_n(x)}{dx^4} - \mu\omega_N^2 V_n(x) = 0 \quad (22)$$

for the Euler's beam. The parameter ω_N is the natural circular frequency defined by:

$$\omega_N^2 = \frac{\theta_n^4 EI}{L^4\mu} \quad (23)$$

Equation (22) implies:

$$\int_0^L V(x,t)\frac{d^4V_n(x)}{dx^4}dx = \frac{\mu\omega_N^2}{EI}\int_0^L V(x,t)V_n(x)dx \quad (24)$$

Thus, taking Eq. (12) into account,

$$T_A(t) = \frac{EI\theta_n^4}{L^4\mu}Z(n,t) \quad (25)$$

Since, Z (k,t) is just the coefficient of the generalized finite integral transforms,

$$V(x,t) = \sum_{k=1}^{\infty}\frac{\mu}{V_k}Z(k,t)V_k(x) \quad (26)$$

Therefore,

$$\frac{\partial^2 V(x,t)}{\partial x^2} = \sum_{k=1}^{\infty}\frac{\mu}{V_k}Z(k,t)\frac{d^2V_k(x)}{dx^2} \quad (27)$$

Then

$$T_B(t) = \sum_{k=1}^{\infty}Z(k,t)\int_0^L \frac{d^2V_k(x)}{dx^2}V_n(x)dx \quad (28)$$

Using the property of the Dirac-delta function as an even function, it can be shown that:

$$\delta(x-ct) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\cos\frac{n\pi x}{L} \quad (29)$$

when use is made of Eq. (29) and (26), we have that:

$$T_C(t) = \frac{1}{L}\sum_{k=1}^{\infty}\frac{m_i}{\mu\tau_k}Z_{tt}(k,t)$$
$$\left[\int_0^L V_k(x)V_n(x)dx + 2\sum_{n=1}^{\infty}\cos\frac{n\pi c_i t}{L}\int_0^L \cos\frac{n\pi x}{L}V_k(x)V_n(x)dx\right] \quad (30)$$

Where,

$$\tau_k = \frac{V_k}{\mu}$$

Using similar arguments in Eq. (29) and (26), it is straight forward to show that

$$T_D(t) = \frac{2}{L}\sum_{k=1}^{\infty}\frac{m_i c_i}{\mu\tau_k}Z_t(k,t)$$
$$\left[\int_0^L \frac{dV_k(x)}{dx}V_n(x)dx + 2\sum_{n=1}^{\infty}\cos\frac{n\pi c_i t}{L} \int_0^L \cos\frac{n\pi x}{L}\frac{dV_k(x)}{dx}V_n(x)dx\right] \quad (31)$$

and

$$T_E(t) = \frac{1}{L}\sum_{k=1}^{\infty}\frac{m_i c_i^2}{\mu\tau_k}Z(k,t)$$
$$\left[\int_0^L \frac{d^2V_k(x)}{dx^2}V_n(x)dx + 2\sum_{n=1}^{\infty}\cos\frac{n\pi c_i t}{L} \int_0^L \cos\frac{n\pi x}{L}\frac{d^2V_k(x)}{dx^2}V_n(x)dx\right] \quad (32)$$

By the property of Dirac-delta function in Eq. (7), it is clear that:

$$T_i^*(t) = V_n(c_i t) \quad (33)$$

Substituting Eq. (25) (28) (30) (31) (32) and (33) into Eq. (16), after some simplifications and rearrangements, one obtains:

$$Z_{tt}(n,t) + \omega_n^2 Z(n,t) - \frac{N}{\mu}\sum_{k=1}^{\infty}R_1(k,n)Z(k,t)$$
$$+\sum_{i=1}^{m}\frac{M_i}{L\mu}\left[\sum_{k=1}^{\infty}R_2(k,n)Z_{tt}(k,t)\right.$$
$$+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{n\pi c_i t}{L}R_4(k,n)Z_{tt}(k,t)+$$
$$2c_i\sum_{k=1}^{\infty}R_3(k,n)Z_t(k,t) + 4c_i\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{n\pi c_i t}{L}R_5(k,n) \quad (34)$$
$$Z_t(k,t) + c_i^2\sum_{k=1}^{\infty}R_1(k,n)Z(k,t)+$$
$$\left.2c_i^2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{n\pi c_i t}{L}R_6(k,n)Z(k,t)\right]$$
$$=\sum_{i=1}^{m}\frac{M_i g}{\mu}V_n(c_i t)$$

Where,

$$\omega_n^2 = \frac{EI\theta_n^4}{\mu L^4} + \frac{K}{\mu}$$

$$R_1(k,n) = \frac{1}{\tau_k}\int_0^L \frac{d^2V_k(x)}{dx^2}V_n(x)dx$$

$$R_2(k,n) = \frac{1}{\tau_k}\int_0^L V_k(x)V_n(x)dx$$

$$R_3(k,n)=\frac{1}{\tau_k}\int_0^L \frac{dV_k(x)}{dx}V_n(x)dx$$

$$R_4(k,n)=\frac{1}{\tau_k}\int_0^L V_k(x)V_n(x)\cos\frac{n\pi x}{L}dx$$

$$R_5(k,n)=\frac{1}{\tau_k}\int_0^L \frac{dV_k(x)}{dx}V_n(x)\cos\frac{n\pi x}{L}dx$$

$$R_6(k,n)=\frac{1}{\tau_k}\int_0^L \frac{d^2V_k(x)}{dx^2}V_n(x)\cos\frac{n\pi x}{L}dx$$

Considering only a mass moving with c and using Eq. (15), thus (34) reduces to:

$$\begin{aligned}&Z_{tt}(n,t)+\omega_n^2 Z(n,t)-\frac{N}{\mu}\sum_{k=1}^{\infty}R_1(k,n)Z(k,t)+\\&\Gamma\Bigg[\sum_{k=1}^{\infty}R_2(k,n)Z_{tt}(k,t)+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(k,n)Z_{tt}(k,t)\\&+2c\sum_{k=1}^{\infty}R_3(k,n)Z_t(k,t)+4c\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{n\pi ct}{L}R_5(k,n)\\&Z_t(k,t)+c^2\sum_{k=1}^{\infty}R_1(k,n)Z(k,t)\Bigg]\\&+2c^2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{n\pi ct}{L}R_6(k,n)Z(k,t)\\&=\frac{Mg}{\mu}\begin{bmatrix}\sin\frac{\theta_n ct}{L}+A_n\cos\frac{\theta_n ct}{L}+B_n\\ \sinh\frac{\theta_n ct}{L}+C_n\cosh\frac{\theta_n ct}{L}\end{bmatrix}\end{aligned}\tag{35}$$

Where,

$$\Gamma=\frac{M}{L\mu}\tag{36}$$

Equation (35) is the transformed equation governing the problem of a Bernoulli-Euler beam on a constant elastic foundation. This coupled non-homogeneous second order ordinary differential equation holds for all variants of both classical and non-classical (elastically supported) boundary conditions.

SOLUTION OF THE TRANSFORMED EQUATION

If we neglect the inertial effect of the moving mass in Eq. (35), that is, setting Γ equals to zero, we have:

$$\begin{aligned}&Z_{tt}(n,t)+\omega_n^2 Z(n,t)-\frac{N}{\mu}\sum_{k=1}^{\infty}R_1(k,n)Z(k,t)\\&=\frac{Mg}{\mu}\begin{bmatrix}\sin\frac{\theta_n ct}{L}+A_n\cos\frac{\theta_n ct}{L}+\\ B_n\sinh\frac{\theta_n ct}{L}+C_n\cosh\frac{\theta_n ct}{L}\end{bmatrix}\end{aligned}\tag{37}$$

Equation (37) represents the classical case of a moving force problem associated with our system. Evidently, an exact analytical solution to the Eq. (37) is not possible. Though, the equation yields readily to numerical technique, an analytical approximate method is desirable as solutions so obtained often shed light on vital information about the vibrating system. To this end, we are going to use a modification of the asymptotic method due to Struble's (Gbadeyan and Oni, 1995) often used in treating weakly homogeneous and non-homogeneous nonlinear oscillatory systems. For this purpose, Eq. (37) is arranged to take the form

$$\begin{aligned}&Z_{tt}(n,t)+\left[\omega_n^2-\varepsilon^*R_1(n,n)\right]Z(n,t)-\varepsilon^*\sum_{\substack{k=1\\k\neq n}}^{\infty}R_1(k,n)Z(k,t)\\&=\frac{Mg}{\mu}\begin{bmatrix}\sin\frac{\theta_n ct}{L}+A_n\cos\frac{\theta_n ct}{L}+\\ B_n\sinh\frac{\theta_n ct}{L}+C_n\cosh\frac{\theta_n ct}{L}\end{bmatrix}\end{aligned}\tag{38}$$

Where

$$\varepsilon^*=\frac{N}{\mu}$$

By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of axial force N. An equivalent free system operator defined by the modified frequency then replaces Eq. (38). Thus, we set the right-hand-side of Eq. (38) to zero and consider a parameter λ-1 for any arbitrary ratio ε* defined as:

$$\lambda=\frac{\varepsilon^*}{1+\varepsilon^*}\tag{39}$$

So that

$$\varepsilon^*=\lambda+0(\lambda^2)+\ldots\tag{40}$$

Hence, the homogeneous part of Eq. (38) becomes

$$\begin{aligned}&Z_{tt}(n,t)+\left[\omega_n^2-\lambda R_1(n,n)\right]\\&Z(n,t)-\lambda\sum_{\substack{k=1\\k\neq n}}^{\infty}R_1(k,n)Z(k,t)=0\end{aligned}\tag{41}$$

to order of λ only. When λ is set to zero in Eq. (41), a situation corresponding to the case in which the axial force effect is regarded as negligible is obtained, then the solution of Eq. (41) becomes:

$$Z_r(n,t)=C_1\cos(\omega_n t-\phi_n)\tag{42}$$

Where,

$$C_1,\ \omega_n \text{ and } \phi_n \text{ are constants} \tag{43}$$

Furthermore, as $\lambda<1$, Struble's technique requires that the asymptotic solutions of the homogeneous part of the Eq. (38) be of the form

$$Z(n,t) = \phi(n,t)\cos[\omega_n t - \Omega(n,t)] + \lambda Z_1(n,t) + o(\lambda^2) \tag{44}$$

When Eq. (44) is substituted into the homogenous part of Eq. (38), one arrives at

$$\begin{aligned}&-2\dot{\phi}(n,t)\omega_n \sin[\omega_n t - \Omega(n,t)] + 2\dot{\Omega}(n,t)\\ &\phi(n,t)\omega_n\cos[\omega_n t - \Omega(n,t)] + \lambda\omega_n^2 Z_1\\ &-\lambda R_1(n,n)\phi(n,t)\cos[\omega_n t - \Omega(n,t)]\\ &-\lambda\sum_{\substack{k=1\\k\neq n}}^{\infty} R_1(k,n)\begin{bmatrix}\phi(k,t)\cos[\omega_k t - \Omega\\ (k,t)] + \lambda Z_1(k,t)\end{bmatrix} = 0\end{aligned} \tag{45}$$

retaining term to $o(\lambda)$ only. The variational equations are obtained by equating the coefficients of $\sin(\omega_n t - \Omega(n,t))$ and $\cos(\omega_n t - \Omega(n,t))$ terms on both sides of the equation. Thus, neglecting those terms that do not contribute to the variational equations, Eq. (45) reduces to:

$$\begin{aligned}&\left[2\omega_n\dot{\Omega}(n,t)\phi(n,t) - \lambda R_1(n,n)\phi(n,t)\right]\\ &\cos[\omega_n t - \Omega(n,t)] - 2\omega_n\dot{\phi}(n,t)\sin[\omega_n t - \Omega(n,t)] = 0\end{aligned} \tag{46}$$

From Eq. (46), the variational equations are obtained, respectively as:

$$-2\dot{\phi}(n,t)\omega_n = 0 \tag{47}$$

and

$$2\omega_n\dot{\Omega}(n,t)\phi(n,t) - \lambda R_1(n,n)\phi(n,t) = 0 \tag{48}$$

Solving Eq. (47) and (48), respectively, we have:

$$\phi(n,t) = \phi_0, \tag{49}$$

where, ϕ_0 is a constant.

$$\Omega(n,t) = \frac{\lambda R_1(n,n)}{2\omega_n}t + \Omega_0 \tag{50}$$

and where, Ω_0 is a constant.

Therefore, when the effect of the axial force is considered, the first approximation to the homogenous system is:

$$Z(n,t) = \phi_0[Y_{FN}t - \Omega_0] \tag{51}$$

Where,

$$Y_{FN} = \omega_n\left[1 - \frac{\lambda R_1(n,n)}{2\omega_n^2}\right] \tag{52}$$

is called the modified natural frequency representing the frequency of the free system.

Using Eq. (51), the homogenous part of the Eq. (38) can be written as:

$$Z_{tt}(n,t) + Y_{FN}^2 Z(n,t) = 0 \tag{53}$$

Hence, the entire Eq. (38) takes the form:

$$Z_{tt}(n,t) + Y_{FN}^2 Z(n,t) = P^0\begin{bmatrix}\sin\frac{\theta_n ct}{L} + A_n\cos\frac{\theta_n ct}{L} + \\ B_n\sinh\frac{\theta_n ct}{L} + C_n\cosh\frac{\theta_n ct}{L}\end{bmatrix} \tag{54}$$

Where

$$P^0 = \frac{Mg}{\mu} \tag{55}$$

Now, if the moving load has mass commensurable with that of the structure, the inertial effect of the moving mass is not negligible. Thus, $\Gamma \neq 0$ and we are required to solve the entire Eq. (35). This is called the moving mass problem. We take note that, neglecting the terms representing the inertia effect of the moving mass, we obtain Eq. (38). The homogeneous part of this equation can be replaced by a free system operator defined by the modified frequency Y_{FN}, due to the presence of the effect of the axial force N. Thus, Eq. (35) can be written in the from

$$\begin{aligned}&Z_{tt}(n,t) + Y_{FN}^2 Z(n,t) + \Gamma\Bigg[\sum_{k=1}^{\infty} R_2(k,n)Z_{tt}(k,t) +\\ &2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(k,n)Z_{tt}(k,t) +\\ &2c\sum_{k=1}^{\infty}R_3(k,n)Z_t(k,t) + 4c\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{n\pi ct}{L}R_5(k,n)\\ &Z_t(k,t) + c^2\sum_{k=1}^{\infty}R_1(k,n)Z(k,t)\Bigg]\\ &+ 2c^2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{n\pi ct}{L}R_6(k,n)\\ &Z(k,t) = \frac{Mg}{\mu}\begin{bmatrix}\sin\frac{\theta_n ct}{L} + A_n\cos\frac{\theta_n ct}{L} + \\ B_n\sinh\frac{\theta_n ct}{L} + C_n\cosh\frac{\theta_n ct}{L}\end{bmatrix}\end{aligned} \tag{56}$$

Rearranging Eq. (56), we have

$$Z_{tt}(n,t)+\frac{\Gamma\left[2cR_3(n,n)+4c\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_5(n,n)\right]Z_t(n,t)}{\left[1+\Gamma\left(R_2(n,n)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(n,n)\right)\right]}$$

$$+\frac{\left[Y_{FN}^2+\Gamma\left(c^2R_1(n,n)+2c^2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_6(n,n)\right)\right]Z(n,t)}{\left[1+\Gamma\left(R_2(n,n)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(n,n)\right)\right]}$$

$$+\sum_{\substack{k=1\\k\neq n}}^{\infty}\Gamma\left\{\frac{\left[R_2(k,n)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(k,n)\right]}{\left[1+\Gamma\left(R_2(n,n)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(n,n)\right)\right]}Z_{tt}(k,t)\right.$$

$$\frac{\left[2cR_3(k,n)+4c\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_5(k,n)\right]}{\left[1+\Gamma\left(R_2(n,n)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(n,n)\right)\right]}Z_t(k,t)$$

$$\left.+\frac{\left[c^2R_1(k,n)+2c^2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_6(k,n)\right]}{\left[1+\Gamma\left(R_2(n,n)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(n,n)\right)\right]}Z(k,t)\right\} \quad (57)$$

As above, in the first instance, we shall consider the homogenous part of Eq. (57) and obtain a modified frequency corresponding to the frequency of the free system due to the presence of the axial force and the moving mass. An equivalent free system operator defined by the modified frequency then replaces the Eq. (57). Considering a parameter $\epsilon_0 \angle 1$ for any arbitrary mass ratio Γ defined as:

$$\varepsilon_0=\frac{\Gamma}{1+\Gamma} \quad (58)$$

It is then clear that:

$$\Gamma=\varepsilon_0[1+o(\varepsilon_0)+o(\varepsilon_0^2)+\dots] \quad (59)$$

But, all the various time dependent coefficients of the differential operator which acts on Z(n,t) in Eq. (57) can be written in terms of ε_0 when we notice that to $o(\varepsilon_0)$, $\Gamma=\varepsilon_0$ and

$$\frac{1}{1+\varepsilon_0\left[R_2(n,n)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(n,n)\right]}$$

$$=\left[1-\varepsilon_0\left(R_2(n,n)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(n,n)\right)+o\left(\varepsilon_0^2\right)+\dots\right]$$

where,

$$\left|\varepsilon_0\left(R_2(n,n)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}R_4(n,n)\right)\right|<1 \quad (60)$$

Following the same argument as above one obtains:

$$\beta_n=\frac{Y_{FN}}{2}\left[2-\varepsilon_0\left(R_2(n,n)-\frac{c^2}{Y_{FN}^2}R_1(n,n)\right)\right] \quad (61)$$

as the modified natural frequency representing the frequency of the free system due to the presence of both the axial force and the moving mass.

Using Eq. (61), the homogenous part of the Eq. (57) can be written as:

$$Z_{tt}(n,t)+\beta_n^2Z(n,t)=0 \quad (62)$$

Hence, the entire Eq. (57) takes the form:

$$Z_{tt}(n,t)+\beta_n^2Z(n,t)=\varepsilon^0\left[\begin{array}{l}\sin\frac{\theta_nct}{L}+A_n\cos\frac{\theta_nct}{L}+\\B_n\sinh\frac{\theta_nct}{L}+C_n\cosh\frac{\theta_nct}{L}\end{array}\right] \quad (63)$$

where, $\varepsilon^0=\varepsilon_0\,Lg$

Using the Laplace transformation technique and the convolution theory, expression for Z(n,t) is obtained. Thus, in view of Eq. (13), one obtains:

$$V(x,t)=\sum_{n=1}^{\infty}\frac{\mu}{V_n}\left\{\frac{\varepsilon^0}{\beta_n\left[\beta_n^4-\gamma_0^4\right]}\left\{\left[\beta_n^2+\gamma_0^2\right]\right.\right.$$

$$\left[A_n\beta_n\left(\cos(\gamma_0t)-\cos(\beta_nt)\right)-\left(\gamma_0\sin(\beta_nt)-\beta_n\sin(\gamma_0t)\right)\right]$$

$$\left.\left.+\left[\beta_n^2-\gamma_0^2\right]\left[\begin{array}{l}C_n\beta_n\left(\cosh(\gamma_0t)-\cos(\beta_nt)\right)+\\B_n\left(\beta_n\sinh(\gamma_0t)-\gamma_0\sin(\beta_nt)\right)\end{array}\right]\right\}\right\}$$

$$\left[\sin\frac{\theta_nx}{L}+A_n\cos\frac{\theta_nx}{L}+B_n\sinh\frac{\theta_nx}{L}+C_n\cosh\frac{\theta_nx}{L}\right] \quad (64)$$

where,

$$\gamma_0 = \frac{\theta_n c}{L}$$

Equation (64) represents the transverse-displacement response to a moving mass of a Bernoulli-Euler beam resting on a constant elastic foundation for all variants of both classical and non-classical (elastically supported) boundary conditions. The corresponding moving force solution is:

$$V(x,t) = \sum_{n=1}^{\infty} \frac{\mu}{V_n} \left\{ \frac{P^0}{Y_{FN}\left[Y_{FN}^4 - \gamma_0^4\right]} \left\{ \left[Y_{FN}^2 + \gamma_0^2\right] \begin{bmatrix} A_n Y_{FN}\left(\cos(\gamma_0 t) - \cos(Y_{FN} t)\right) \\ -\left(\gamma_0 \sin(Y_{FN} t) - Y_{FN} \sin(\gamma_0 t)\right) \end{bmatrix} + \left[Y_{FN}^2 - \gamma_0^2\right] \begin{bmatrix} C_n Y_{FN}\left(\cosh(\gamma_0 t) - \cos(Y_{FN} t)\right) + \\ B_n \begin{pmatrix} Y_{FN} \sinh(\gamma_0 t) - \\ \gamma_0 \sin(Y_{FN} t) \end{pmatrix} \end{bmatrix} \right\} \right\} \begin{bmatrix} \sin\left(\frac{\theta_n x}{L}\right) + A_n \cos\left(\frac{\theta_n x}{L}\right) + \\ B_n \sinh\left(\frac{\theta_n x}{L}\right) + C_n \cosh\left(\frac{\theta_n x}{L}\right) \end{bmatrix} \quad (65)$$

ILLUSTRATIVE EXAMPLES

In this study, the foregoing analysis is illustrated by various practical examples. In particular, 2 cases are considered, namely, Bernoulli-Euler beam simply supported at the end x = 0 and elastically supported at the other end x = L.

Bernoulli-Euler beam elastically supported at both ends.

Example (i): The conditions are expressed as:

$$(0,t) = 0 = V^{11}(0,t) \text{ and } V^{11}(L,t)-k_1 V^1(L,t) = 0 = V^{111}(L,t) + k_2 V(L,t) \quad (66)$$

Hence, for normal modes:

$$V_n(0) = 0 = V_n^{11}(0) \text{ and } V_n^{11}(L) - k_1 V_n^1(L) = 0 = V_n^{111}(L) + k_2 V_n(L) \quad (67)$$

which implies that:

$$V_k(0) = 0 = V_k^{11}(0) \text{ and } V_k^{11}(L) - k_1 V_k^1(L) = 0 = V_k^{111}(L) + k_2 V_k(L)$$

Thus, it can be shown that:

$$A_n = 0 = C_n \text{ and}$$

$$B_n = \frac{k_1 \cos\theta_n + \frac{\theta_n}{L}\sin\theta_n}{\frac{\theta_n}{L}\sinh\theta_n - k_1\cosh\theta_n} = \frac{\frac{\theta_n^3}{L^3}\cos\theta_n - k_2\sin\theta_n}{\frac{\theta_n^3}{L^3}\cosh\theta_n + k_2\sinh\theta_n} \quad (68)$$

and from Eq. (68) one obtains:

$$\tan\theta_n = \tanh\theta_n \quad (69)$$

As the frequency equation for the dynamical problem, such that (Gbadeyan and Oni, 1995).

$$\theta_1 = 3.927,\ \theta_2 = 7.069,\ \theta_3 = 10.210 \quad (70)$$

Using Eq. (68) and (69) in Eq. (64) and (65) one obtains the displacement response, respectively to a moving mass and a moving force of simply-elastic ends Bernoulli-Euler beam on a constant foundation.

Example (ii): For the case when the beam is elastically supported both at x = 0 and x = L, the conditions are expressed as:

$$V^{11}(0,t)-k_1V^1(0,t)=0=V^{111}(0,t)+k_2V(0,t)$$
$$V^{11}(L,t)-k_1V^1(L,t)=0=V^{111}(L,t)+k_2V(L,t) \quad (71)$$

Similarly, for normal modes:

$$V_n^{11}(L) - k_1 V_n^1(L) = 0 = V_n^{111}(L) + k_2 V_n(L) \text{and } V_n^{11}(0) - k_1 V_n^1(0) = 0 = V_n^{111}(0) + k_2 V_n(0) \quad (72)$$

which implies that:

$$V_k^{11}(0) - k_1 V_k^1(0) = 0 = V_k^{111}(0) + k_2 V_k(0) \text{and } V_k^{11}(L) - k_1 V_k^1(L) = 0 = V_k^{111}(L) + k_2 V_k(L) \quad (73)$$

Using (72), it can be shown that:

$$C_n = \frac{\left[\frac{\theta_n}{L} - k_1 r_2\right]\sin\theta_n + \left[k_1 + \frac{r_2\theta_n}{L}\right]\cos\theta_n - \frac{r_1\theta_n}{L}\sinh\theta_n + k_1 r_1 \cosh\theta_n}{k_1 r_1 \sin\theta_n - \frac{r_1\theta_n}{L}\cos\theta_n + \left[\frac{r_3\theta_n}{L} - k_1\right]\sinh\theta_n + \left[\frac{\theta_n}{L} - k_1 r_3\right]\cosh\theta_n}$$

$$= \frac{-\left[\frac{r_2\theta_n^3}{L^3} + k_2\right]\sin\theta_n + \left[\frac{\theta_n^3}{L^3} - k_2 r_2\right]\cos\theta_n - k_2 r_1 \sinh\theta_n - \frac{r_1\theta_n^3}{L^3}\cosh\theta_n}{\frac{r_1\theta_n^3}{L^3}\sin\theta_n + k_2 r_1 \cos\theta_n + \left[\frac{\theta_n^3}{L^3} + k_2 r_3\right]\sinh\theta_n + \left[\frac{r_3\theta_n^3}{L^3} + k_2\right]\cosh\theta_n}, \quad (74)$$

$$A_n = r_1 C_n + r_2 \text{ and } B_n = r_3 C_n + r_1 \quad (75)$$

Where,

$$r_1 = \frac{\frac{\theta_n^4}{L^4} + k_1 k_2}{\frac{\theta_n^4}{L^4} - k_1 k_2}; \quad r_2 = \frac{-\frac{2k_1\theta_n^3}{L^3}}{\frac{\theta_n^4}{L^4} - k_1 k_2} \text{ and}$$

$$r_3 = \frac{\frac{-2k_2\theta_n}{L}}{\frac{\theta_n^4}{L^4} - k_1 k_2}$$

Using (74), the frequency equation for the dynamical problem is obtained as

$$\tan\theta_n = \tanh\theta_n \quad (76)$$

This is similar to (69), hence one has

$$\theta_1 = 3.927,\ \theta_2 = 7.069,\ \theta_3 = 10.210 \quad (77)$$

Substituting Eq. (74), (75) and (77) into Eq. (64) and (65) one obtains the displacement response, respectively to a moving mass and a moving force of Bernoulli-Euler beam elastically supported at both ends and resting on a constant foundation.

DISCUSSION OF THE ANALYTICAL SOLUTIONS

In studying undamped system such as this, it is desirable to examine the phenomenon of resonance. Equation (65) clearly shows that the Bernoulli-Euler beam on a constant foundation and traversed by a moving force reaches a state of resonance whenever:

$$Y_{FN} = \eta_n = \frac{\theta_n c}{L} \quad (78)$$

while, Eq. (64) shows that the same beam under the action of a moving mass experiences resonance when

$$\beta_n = \gamma_0 = \frac{\theta_n c}{L} \quad (79)$$

Where,

$$\beta_n = Y_{FN}\left[1 - \frac{\varepsilon\mu}{2V_n}\left(R_2(n,n) - \frac{c^2}{Y_{FN}^2}R_1(n,n)\right)\right] \quad (80)$$

From Eq. (79) and (80), it can be shown that:

$$\frac{Y_{FN}\left[V_n - \frac{\varepsilon}{2}\left(\mu R_2(n,n) - \frac{\mu c^2 R_1(n,n)}{Y_{FN}^2}\right)\right]}{V_n} = \frac{\theta_n c}{L} \quad (81)$$

Since,

$$V_n > V_n - \frac{\varepsilon}{2}\left(\mu R_2(n,n) - \frac{\mu c^2 R_1(n,n)}{Y_{FN}^2}\right)$$

for all n.

It can be deduced from Eq. (81) that, for the same natural frequency, the critical speed (and the natural frequency) for the system of Bernoulli-Euler beam traversed by a moving mass is smaller than that of the same system traversed by a moving force. Thus, for the same natural frequency of the Bernoulli-Euler beam, the resonance is reached earlier when we consider the moving mass system than when we consider the moving force system.

RESULTS AND DISCUSSION

In this study, calculations of practical interest in dynamics of structures and Engineering design are presented for all the illustrative examples considered. An elastic Bernoulli-Euler beam of length 12.192 m has been considered. Furthermore, it is assumed that the moving load travels at the constant velocity of 8.123m/s. Again EI and M/Lμ are chosen to be 6.068 × $10^6 m^3/s^2$ and 0.25, respectively. The results are as shown on the various graphs below for the various classes of boundary conditions considered.

The effect of foundation modulli (K) on the transverse deflection of the Bernoulli-Euler beam, with one end simply supported and the other end elastically supported, in both cases of moving force and moving mass is displayed in Fig. 1 and 2. The graphs show that the response amplitude of the beam decreases as the value of the foundation modulli increases. Values of K between 0 and 100000 N/m^3 are used.

Figures 3 and 4 display the effect of axial force (N) on the transverse deflection of the beam, simply supported at one end and elastically supported at the other end, in both cases of moving force and moving mass, respectively. The graphs show that an increase in the value of axial force (N) decreases the deflection of the beam. Values of N between 0 and 1million Newtons are used.

For the purpose of comparison, the displacement curves of the moving force and moving mass for the beam, with one end simply supported and the other end elastically supported, with K = 100000 N/m^3 and N = 100000 N are illustrated in Fig. 5. It can be seen that the response amplitude of a moving mass is greater than that of a moving force problem. This result also holds for other choice of K and N.

It is observed in Table 1 that as the value of the foundation modulli K increases, the displacement of the elastic-elastic Bernoulli-Euler beam decreases for both cases of moving force and moving mass, respectively. Also, Table 2 shows that as the value of the axial force N increases, the deflection of the beam, elastically supported at both ends, decreases for both cases of moving force and moving mass, respectively.

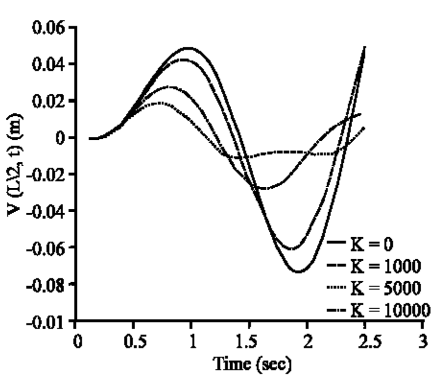

Fig. 1: Deflection of moving force for simply-elastically supported Bernoulli-Euler beam for various values of K

Comparing the displacement response of the moving force and moving mass for a Bernoulli-Euler beam elastically supported at both ends, for fixed values of K and N, it is evident from Table 1 and 2 that the response amplitude of the moving mass problem is greater than that of the moving force problem.

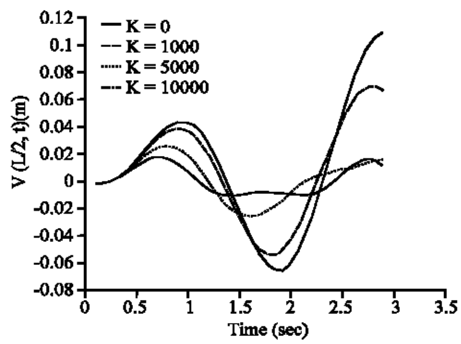

Fig. 2: Displacement of moving masss for simply-elastically supported Bermoulli-Euler beam for various values of K

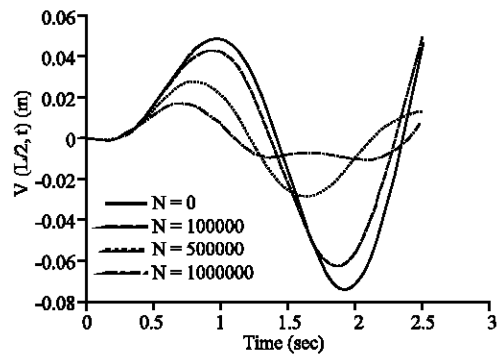

Fig. 3: Deflection of moving force for simply-elastically supported Bernoulli-Euler beam for various values of N

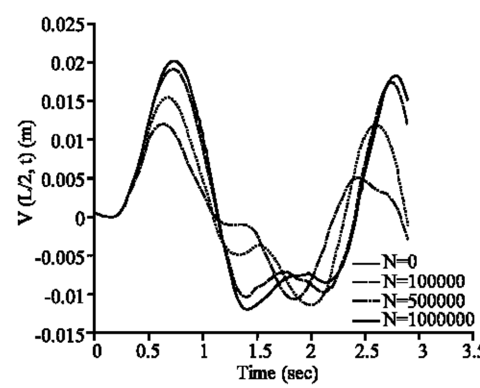

Fig. 4: Displacement of moving mass for Simply-Elastically supported Bernoulli-Euler Beam or various values of N

Table 1: The elastic-elastic Bernoulli-Euler beam decreases for both cases of moving force and moving mass

		Moving force			Moving mass		
S/n	T(sec.)	K = 0	K= 500000	K=1000000	K = 0	K= 500000	K=1000000
1	0	0	0	0	0	0	0
2	0.1	-2.64E-04	-2.65E-04	-2.49E-04	-2.64E-04	-2.65E-04	-2.49E-04
3	0.2	-1.95E-03	-1.65E-03	-1.31E-03	-1.94E-03	-1.66E-03	-1.32E-03
4	0.3	-6.48E-03	-4.23E-03	-2.62E-03	-6.47E-03	-4.25E-03	-2.65E-03
5	0.4	-1.52E-02	-7.02E-03	-3.42E-03	-0.01519	-7.07E-03	-3.46E-03
6	0.5	-2.91E-02	-9.08E-03	-4.18E-03	-2.91E-02	-9.17E-03	-4.22E-03
7	0.6	-4.85E-02	-1.05E-02	-5.68E-03	-4.85E-02	-0.01065	-5.72E-03
8	0.7	-7.32E-02	-1.24E-02	-7.49E-03	-7.34E-02	-1.26E-02	-7.54E-03
9	0.8	-0.10257	-1.57E-02	-8.85E-03	-0.10294	-1.58E-02	-8.94E-03
10	0.9	-0.13543	-0.01998	-1.03E-02	-0.13608	-2.01E-02	-1.04E-02
11	1	-0.17075	-0.02452	-1.27E-02	-0.17177	-2.47E-02	-1.28E-02
12	1.1	-0.20789	-2.91E-02	-1.59E-02	-0.20938	-2.94E-02	-1.60E-02
13	1.2	-0.24719	-3.46E-02	-0.01934	-0.24921	-3.50E-02	-1.95E-02
14	1.3	-0.29031	-4.29E-02	-2.36E-02	-0.29291	-0.04319	-2.38E-02
15	1.4	-0.34076	-5.48E-02	-0.02995	-0.34391	-5.52E-02	-3.02E-02
16	1.5	-0.40414	-7.07E-02	-3.86E-02	-0.40781	-7.12E-02	-3.89E-02
17	1.6	-0.48846	-9.06E-02	-4.94E-02	-0.49259	-9.14E-02	-0.04987
18	1.7	-0.60429	-0.11606	-6.34E-02	-0.60884	-0.11708	-6.40E-02
19	1.8	-0.76501	-0.14995	-0.08252	-0.76998	-0.1512	-0.08316
20	1.9	-0.98702	-0.19568	-0.10796	-0.99254	-0.19722	-0.10882

Table 2: The value of the axial force N increases, the deflection of the beam, elastically supported at both ends, decreases for both cases of moving force and moving mass

		Moving force			Moving mass		
S/n	T(sec.)	N = 0	N=2000000	N=4000000	N = 0	N=2000000	N=4000000
1	0	0	0	0	0	0	0
2	0.1	-2.74E-04	-2.73E-04	-2.70E-04	-2.74E-04	-2.73E-04	-2.71E-04
3	0.2	-1.96E-03	-1.90E-03	-1.81E-03	-1.96E-03	-1.90E-03	-1.81E-03
4	0.3	-6.21E-03	-5.71E-03	-5.12E-03	-6.22E-03	-5.72E-03	-5.13E-03
5	0.4	-1.36E-02	-1.17E-02	-9.68E-03	-1.36E-02	-1.17E-02	-9.73E-03
6	0.5	-2.38E-02	-1.89E-02	-1.43E-02	-2.39E-02	-1.90E-02	-1.44E-02
7	0.6	-3.59E-02	-2.60E-02	-1.79E-02	-0.0361	-2.62E-02	-1.81E-02
8	0.7	-4.84E-02	-3.20E-02	-2.06E-02	-4.88E-02	-3.23E-02	-2.08E-02
9	0.8	-6.01E-02	-3.67E-02	-2.34E-02	-0.0606	-3.71E-02	-2.36E-02
10	0.9	-7.03E-02	-4.12E-02	-2.77E-02	-7.11E-02	-4.17E-02	-2.79E-02
11	1	-7.98E-02	-4.73E-02	-3.43E-02	-8.08E-02	-4.77E-02	-3.46E-02
12	1.1	-9.04E-02	-5.68E-02	-4.33E-02	-9.14E-02	-5.72E-02	-4.36E-02
13	1.2	-0.10462	-7.09E-02	-5.41E-02	-0.1057	-7.14E-02	-5.45E-02
14	1.3	-0.12591	-9.03E-02	-6.65E-02	-0.127	-9.09E-02	-6.71E-02
15	1.4	-0.15756	-0.1153	-8.18E-02	-0.1587	-0.11613	-8.25E-02
16	1.5	-0.20273	-0.14655	-0.1023	-0.20403	-0.14774	-0.10321
17	1.6	-0.26445	-0.18603	-0.13103	-0.26612	-0.18766	-0.13207
18	1.7	-0.34603	-0.23738	-0.17066	-0.34835	-0.23947	-0.17193
19	1.8	-0.45187	-0.30584	-0.22348	-0.45516	-0.30843	-0.22515
20	1.9	-0.58832	-0.39786	-0.29199	-0.59291	-0.40104	-0.29431

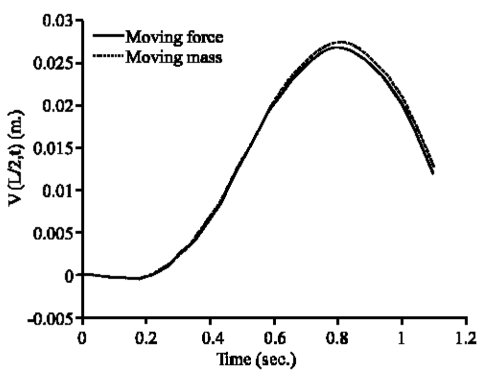

Fig. 5: Comparison of moving force and moving mass for simply-elastically supported Bernoulli-Euler beam

CONCLUSION

The problem of the dynamic response to moving concentrated masses of Bernoulli-Euler beam on constant elastic foundation has been studied in this work. An approximate analytical solution is obtained to the governing 4th order partial differential equation with variable and singular coefficients. Two illustrative examples are presented for this class of problems involving non-classical boundary conditions. For both illustrative examples considered, analytical and numerical analysis mainly in plotted curves show that as the foundation modulli and the axial force increase, the response amplitudes of the beam decrease, for fixed axial force and foundation modulli, the response amplitude for the moving mass problem is greater than that of the moving force problem and for the same natural frequency, the critical speed for the moving mass problem is smaller than that for the moving force problem and so resonance is reached earlier in the moving mass problem.

Finally, this work has suggested valuable method of approximate analytical solution for this class of problems for all variants of both non-classical and combination of classical and non-classical boundary conditions.

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