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The Representation of Clifford Monoids and Semilattice Graded Weak Hopf Algebras: A Motivating Note

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Abstract: We give a brief survey of the representation of an inverse semigroup, especially of the representation of a Clifford semigroup with identity (i.e., the Clifford monoid). First we give a short background of the representation of finite groups, the inverse semigroups and the Clifford monoid which is a regular monoid as a semilattice of groups. Hopf algebras and Semilattice graded weak Hopf algebras can be considered as the generalization of group algebras and Clifford monoid algebras, respectively. We describe how the semilattice graded weak Hopf algebra can be considered as the generalization of Hopf algebra as the Clifford monoids are considered as the generalizations to groups. In this note, we also discuss the Clifford monoid algebra and its relationship with the semilattice graded weak Hopf algebra like the group algebra has a relationship with the Hopf algebra. We shed light on the importance of the representation of Clifford monoids, its algebras and the representation of Semilattice graded weak Hopf algebras. One of the main objects of this note is to get inspired from the rich structure theory of groups to make it possible develope the theory of inverse semigroups and of Clifford semigroups with identity. The second object is to go a step further in jumping from the group of grouplike elements of a Hopf algebra and getting motivation to obtain the Clifford semigroup with identity which is the set of grouplike elements of a semilattice graded weak Hopf algebra. The main purpose of this survey is to get inspiration to develope the representation theory of weak Hopf algebras and of semilattice graded weak Hopf algebras and characterizing such algebras.

Key words: Irreducible representation, Clifford monoid, Hopf algebra, weak Hopf algebra

INTRODUCTION

For the representation of finite groups we refer to Alperin and Bell (1995), Clifford and Preston (1961), Green (1976) and Zhu (1994). Here we give a brief introduction of the semigroup, the regular semigroup, the inverse semigroup, the Clifford semigroup, the Hopf algebra, the weak Hopf algebra and the semilattice graded weak Hopf algebra. We avoid to give definition of various terms to reduce the length of the note but of those which are the most relevant, the reader may find the remaining concepts in the mentioned references.

A groupoid (S, μ) is a non-empty set S on which a binary operation μ: $S\times S\to S$ is defined (Howie, 1995). We say (S, μ) to be a semigroup if the operation μ is associative i.e.,

$$((x, y)\mu, z)\mu = (x,(y, z)\mu)\mu \qquad (1)$$

For all x, y, z in S. If we take μ as a multiplication then we write

$$(x, y)\mu = xy$$

and hence the associativity (1) is

$$((xy)z) = (x(yz)) \qquad (1')$$

The semigroup S is commutative if for all x, y in S, we have

$$xy = yx \qquad (2)$$

An element a of a semigroup S is regular if there exists an element x in S such that axa = a. The semigroup S is regular if all its elements are regular. A semigroup S is called completely regular if there exists a unary operation $a\mapsto a^{-1}$ on S such that $(a^{-1})^{-1} = a$, $aa^{-1}a = a$, $aa^{-1} = a^{-1}a$, or equivalently; a completely regular semigroup is an inverse semigroup S in which, for every a in S, $aa^{-1} = a^{-1}a$. An element a' of a semigroup S is called an inverse of an element a of S if $aa'a = a$ and $a'aa' = a'$. A semigroups S is said to an inverse semigroup if each of its elements has a

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unique inverse element in S. Also by Petrich (1984) and Silva (1992), S is an inverse semigroup if S is regular whose idempotents commute.

By Howie (1995), a Clifford monoid is a regular semigroup S with identity 1 such that all of its idempotents lie in its center, or equivalently; a Clifford monoid is a regular monoid which can be expressed as the semilattice of maximal subgroups $\{G_\alpha \mid \alpha \in Y\}$ of a regular monoid S such that $S = \cup_{\alpha\in Y} G_\alpha$ and $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ for all α, $\beta \in Y$, where, Y is a semilattice. Further, for all $\alpha, \beta \in Y$ with $\alpha\beta = \beta$, there exists a homomorphism $\varphi_{\alpha,\beta}: G_\alpha \to G_\beta$ with $\varphi_{\alpha,\alpha}$ as the identity map on G_α and if $\alpha\beta = \beta$ and $\beta\gamma = \gamma$ then the composition of homomorphisms $\varphi_{\beta,\gamma}\,\varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma}$. The multiplication in S for all a, b in S is defined to be $a.b = \varphi_{\alpha,\alpha\beta}(a)\,\varphi_{\beta,\alpha\beta}(b)$. The partial ordering $\leq$ in Y is such that $\beta \leq \alpha$ if and only if, $\alpha\beta = \beta$ for all $\alpha, \beta \in Y$, where $\leq$ is the natural partial ordering of Y.

Li (1998), introduced weak Hopf algebra with weak antipode and discussed its properties (Li, 1999). He also proved that the grouplike elements of a weak Hopf algebra is a regular monoid.

A bialgebra H over a field k is called a weak Hopf algebra if there exists an element T in the convolution algebra Hom_k (H, H), such that, id*T*id = id and T*id*T=T, where * denotes the convolution product in Hom_k (H, H), then T is called weak antipode of H (Li, 1998; Li and Cao, 2005). A weak Hopf algebra H with a weak antipode T is called a semilattice graded weak Hopf algebra if $H = \oplus_{\alpha\in Y} H_\alpha$ for the semilattice Y, such that $H_\alpha H_\beta \subseteq H_{\alpha\beta}$ for $\alpha\beta \leq \alpha, \beta$; $\alpha, \beta \in Y$, where each H_α; $\alpha \in Y$ is a sub-weak Hopf algebra which is a Hopf algebra with antipode $T|_{H\alpha}$. There are Hopf algebra homomorphisms $\varphi_{\alpha,\beta}$ from H_α to H_β if $\alpha\beta = \beta$ such that for all a in H_α and b in H_β, the multiplication $a^\circ b$ in H can be given by $a^\circ b = \varphi_{\alpha,\alpha\beta}(a)\varphi_{\beta,\alpha\beta}(b)$, the set of grouplike elements G(H) of H is a Clifford monoid (Li and Cao, 2005).

THE REPRESENTATION OF FINITE GROUPS

We give a review as a brief introduction to the theory of representation of the finite groups. By the representation theory we means the classification of homomorphisms of finite groups into groups of matrices or of linear transformations (Curtis and Reiner, 1962). Frobenius and Burnside played an important role in the foundation of the representations of finite groups.

Definition 1: Let G be a finite group and M a vector space over a field k. A representation of group G with representation space M is a homomorphism $T: g \to T(g)$ of G into GL(M). Two representations T and T′ with representation spaces M and M′, respectively are equivalent if there exists a k-isomorphism S of M onto M′ such that $T'(g)S = ST(g)$ i.e., $T'(g)Sm = ST(g)m$ for all $m \in M$ and $g \in G$. Let N be a subspace of M and $T(g)n \in N$ for all $g \in G$ and $n \in N$, then N is called a G- subspace of M. The dimension [M: k] of M over k is called degree of T (Curtis and Reiner, 1962).

A matrix representation T of G with non-zero representation space M is irreducible if the only G-subspaces of M are {0} and M; otherwise T is called reducible. The representation T is called completely reducible if for every G-subspace N of M there exists another G-subspace N′ such that $M = N \oplus N'$ as a vector space direct sum.

The Wedderburn Structure Theorem plays an important role in the theory of irreducible representation. We give its statement as under:

Theorem 1: The algebra A is semisimple if and only if, it is isomorphic with a direct sum of matrix algebras over a division algebras (Alperin and Bell, 1995).

Corollary 1: The algebra A is simple if and only if, it is isomorphic with a matrix algebra over a division algebra (Alperin and Bell, 1995).

Another important result which play a very basic role in the irreducible representation is the well known Schur Lemma. For details we refer to Serre (1977). For the reader's interest we give its statement here.

Theorem 2: [Se, Schur Lemma]. Let $\rho_V: G \to GL(V)$ and $\rho_W: G \to GL(W)$ be two Irreducible representations of a finite group G and let f be a linear map from V into W such that $\rho_W(g)^\circ f = f^\circ \rho\, \rho_V(g)$ for all $g \in G$. Then:

- If ρ_V and ρ_W are not isomorphic, we have f = 0.
- If V = W and $\rho_V = \rho_W$, f is a homothety, i. e., a scalar multiple of identity.

The well known orthogonality relation between the characters of all irreducible representations of a finite group form the characters into a complete orthonormal system. The irreducible representations and the character theory has a vital role in the representation of finite groups. We also refer Serre (1997) for the Mackey's Irreducible Criterion for the induced irreducible representation of a group. Further, we include here the Frobenius Reciprocity Theorem for the restriction of representation U of a group G to a subgroup H of G and the induced representation of G from the representation of a subgroup of G. The statement of the theorem is as follows:

Theorem 3: Let U be a kG-module and let H be a subgroup of G and let V be a representation of H. Then as a k-vector spaces we have Hom_{kH} (V, Res_H^G (U))≅Hom_{kG} (Ind_H^G(V), U).

Where, Res_H^G(U)) and Ind_H^G(V) denote, respectively the restriction of the representation U to H and the induction to G of the representation V of H.

Green (1976) worked on the indecomposable modules and R. Brauer developed the blocks theory of modules over the group algebras [CR81/87]. They also has given a close relationship between the representation theory of algebras and the representation of finite groups.

In fact, the theory of group representation is rich in its structures. It is natural to ask about the generalization of this theory for the representation of semigroups and for the associative algebras, in particular for the representations of Clifford monoids and its algebras. Moreover, for the representation theory of Hopf algebras and for the semilattice graded weak Hopf algebras.

THE REPRESENTATION OF INVERSE SEMIGROUPS

In this study, we give some background of various kinds of representations of inverse semigroups.

A fundamental inverse semigroup is an inverse semigroup having no nontrivial idempotent separating congruences. Such semigroups play an important role in the structure theory of inverse semigroups. Munn (1970) representation is an action of an inverse semigroup on its underlying semilattice. Munn showed how to construct a fundamental semigroup T_E from the semilattice E such that T_E contains a semilattice of idempotents. The elements of T_E being the partial isomorphism of E. Some effective actions of inverse semigroups are introduced by O' Carroll (1977a, b).

An ordered representation, on a poset (X, ≤), of an inverse semigroup S is a pair (γ, P), where γ:S>→(X, ≤) is a homomorphism and P:X→E(S) is a surjective isotone function satisfying:

(OR1) Domγ(e)= P^{-1}([e]) for all e∈E(S).
(OR2) If $s^{-1}s$ = P(x) then $s^{-1}s$=P(γ(s)(x)).

Lawson (1996) discussed the order representations and showed that every ordered representation of an inverse semigroup S determines and is determined by a special kind of cover of S.

Clifford (1941) defined a completely regular semigroups, i.e., for all elements a in an inverse semigroups S, there exists e and a′ in S such that ea = ae and aa′= a′a = e. Vagner (1952, 1953) used the name generalized groups for the inverse semigroups and developed theory of such semigroups and Preston (1954a-c) has given the representation theory of inverse semigroups. Both discussed their ideas about the inverse semigroups independently.

By Howie (2002), an inverse semigroup is probably better to regard as generalization of a group rather than specialization of semigroup. He also expressed that we can regard inverse semigroups as generalization of groups, since there are many significant examples of inverse semigroups which are not groups. There are numerous ways to define an inverse semigroups. These semigroups are more natural being isomorphic to the set of partial symmetries.

Schwarz has given the decomposition of a finite commutative semigroup S admitting relative inverses which were latter called the Clifford semigroup by Howie (1995) into a disjoint union of character groups of a certain maximal subgroups of the semigroup S. Lin (1965) generalized the Schwarz decomposition theorem to the so-called pseudo-invertible semigroups. Lin also generalized the case of complex character semigroups to the general semigroups.

Let S be an inverse monoid with E(S) as the semilattice of idempotents. The set {(s, t)∈S×S|se = et, e∈E(S)} is the set of least group congruence σ on S. S is called E-unitary if sσ =1 lies in E(S) for each s in S. S is said to be a Clifford monoid if and only if, se=es for all s in S and e in E(S), (Howie, 1995; Silva and Clifford, 1992). By Petrich (1984), the class of all Clifford monoids form a variety of inverse monoid. Further, we can say that the class of idempotents of all Clifford monoids form the subvariety of all Clifford monoids.

THE REPRESENTATION OF CLIFFORD MONOIDS

Clifford (1941) proved the famous structure theorems; Theorem 4 and 5 for the semigroups which Clifford used the name semigroups admitting relative inverses by virtue of these results Howie first time named such semigroups, the Clifford semigroups. In fact, such semigroups are completely regular semigroups as termed by Petrich (1973). The Clifford semigroups are the completely regular semigroups whose idempotents lie in its centre.

Representation of a Clifford monoid is a homomorphism φ: S→ I_X where I_X is the symmetric inverse monoid on a finite set X, with |X| ≤ |S|, also (Howie, 1995). If φ is one-one, then φ is called the representation faithful. In particular φ: S→ I_S described in Howie (1995). Th. 9 is called Vagner-Preston representation of S. By Howie (1995). Th. 9, for every representation φ: S→ I_X the image φ(S) is a Clifford monoid as a submonoid of I_X.

A Clifford monoid, known as the semilattice of groups, is the regular semigroup S whose idempotents lie in its centre. It has been structured by Clifford in (1941) and latter Howie (1995), redescribed that a Clifford semigroup is a strong semilattice of groups. Petrich (1984) and Howie (1995), we can say that, if S is a Clifford monoid then S is disjoint union of groups G_α, for $\alpha\in Y$(a semilattice) and there is a map

$\varphi_{\alpha,\beta}$: $G_\alpha \to G_\beta$, $\mapsto g_\alpha\, g_\alpha\, e_\beta$, where $e_\beta \in G_\beta$,
and for all $g_\alpha \in G_\alpha$ and $e_\beta \in G_\beta$, we have

$$\begin{aligned}(g_\alpha\, \varphi_{\alpha,\beta})(g_\beta\, \varphi_{\alpha,\beta}) &= (g_\alpha\, e_\beta)(g_\beta\, e_\beta)\\ &= g_\alpha\, g_\beta\, e_\beta\\ &= (g_\alpha\, g_\beta)\, e_\beta\\ &= (g_\alpha\, g_\beta)\, \varphi_{\alpha,\beta}\end{aligned}$$

and $\varphi_{\alpha,\beta}$ is a homomorphism. Clearly, $\varphi_{\alpha,\alpha}$ is an identity (isomorphism) $G_\alpha \to G_\alpha$. Moreover, for all $\alpha\geq\beta\geq\gamma$,

$$\begin{aligned}(g_\alpha)\, \varphi_{\alpha,\beta}\, \varphi_{\beta,\gamma}\} &= (g_\alpha\, e_\beta)e_\gamma\\ &= (g_\alpha)(e_\beta\, e_\gamma)\\ &= (g_\alpha)(e_{\beta\gamma})\\ &= g_\alpha\, e_\gamma\\ &= (g_\alpha)\varphi_{\alpha,\gamma}\end{aligned}$$

Thus, S is a strong semilattice Y of groups G_α; $\alpha\in Y$, i. e., S[Y; G_α, $\varphi_{\alpha,\beta}$], the representation [Y; G_α, $\varphi_{\alpha,\beta}$] is called Clifford representation of the Clifford monoid S.

By Petrich (1984), let S_X be a free Clifford monoid on a nonempty set X, then S_X can be described as a quotients of free monoid with involution by the least Clifford monoid congruence. A congruence ρ on a monoid S such that S/ρ is a Clifford monoid is a Clifford congruence. The intersection of all such congruences is the least congruence of S. For any monoid S, η denotes the least semilattice congruence on S. From the varieties of Clifford monoids we can construct the varieties of Clifford monoids algebras.

Let M be an arbitrary inverse submonoid of I_X for some nonempty set X and consider the following relation:

$$\tau_M = \{(x, y)\in X\times X|(\nexists\ \rho\in M)\ x \in \mathrm{dom}\rho \text{ and } x\rho = y\}$$

then ρ_M is called the transitivity relation of M. We have the following fact.

Lemma 1: If M is an Howie (1995) inverse submonoid of a symmetric inverse monoid I_X, then ρ_M is symmetric and transitive on X.

In general ρ_M is not an equivalence relation, since there may exist elements x in X that are not in dom ρ_M in H. Thus there exists x in S such that $(x, x)\notin\rho_M$. However, if we have $(x, y)\in \rho_M$ for some y then by symmetry and transitivity we have that $(x, x)\in \rho_M$. Thus

$$\mathrm{Dom}\ \rho_M = X_{\rho M} = \{x\in X|(x,x)\in \rho_M\}$$

Then ρ_M is an equivalence relation on $X_{\rho M}$.

M is said to be an effective inverse submonoid of I_X if $X_{\rho M} = \rho_M$; then certainly ρ_M is an equivalence on X. The ρ_M -classes in $X_{\rho M}$ are called the transitivity classes of M and M is called transitive if ρ_M is universal relation on $X_{\rho M}$. Thus M is effective and transitive if and only if, for all a in X there exists e in M such that xρ = y. The representation φ: S→ I_X is called an effective (transitive) representation if Sφ is an effective (transitive) inverse submonoid of I_X.

Let $\{X_i: i\in I\}$ be a family of pairwise disjoint sets and let $X=\cup_{i\in I} X_i$. Let S be an inverse monoid and suppose that for each $i\in I$ we have representation φ_i: S→ I_{Xi}. For each s in S we may regard the one-one partial map $s\varphi_i$ as a subset of $X_i\times X_i$. Then $\cup_{i\in I}\, s\varphi_i$ is a partial one-one map of X, whose domain is $\cup_{i\in I}\, \mathrm{dom}\varphi_i$; this map is denoted by φ and is called its sum of representations φ_i. It is expressed as $\varphi = \oplus_{i\in I}\varphi_i$. If I={1, 2, ..., n}, we can write $\varphi = \varphi_1\oplus...\oplus\varphi_n$. Because the definition (Howie, 1995, Def. 5.8.1) is in term of set-theoretic union, the infinite commutative and associative laws hold for the direct sum ⊕.

If φ: S→ I_X and ψ: S> I_X representations of inverse monoid S, then φ and ψ are equivalent if there exists bijection θ: X>Y with the property that, for each s in S,

$$s\psi = \{(x\theta, x'\theta)\in Y\times Y|(x, x')\in s\varphi\},$$ or in otherwords

dom(sψ) = (dom(sφ)θ and for x in dom(sφ), (x(sφ))θ = (xθ)(sψ).

According to Clifford and Preston (1961) the matrix representation of the semigroups has been discussed in details and the theory of representation of semigroups is developed. The semisimplicity of the algebras of the finite commutative semigroups which is union of groups. The following theorem gives this characterization.

Theorem 4: Clifford and Preston (1961) [Th. 9] Let S be a finite commutative semigroup and k be field. Then k[S] is semisimple if and only if, S is union of groups, the orders of which are not divisible by characteristic of k.

The semisimplicity of finite inverse semigroups is given in the following theorem.

Theorem 5: Clifford and Preston (1961) [Th. 9]. The algebra k[S] of a finite inverse semigroup S over a field k is semisimple if and only if , the characteristic of k is zero or a prime not dividing the order of any subgroup of S.

Let k be a field and $(k)_n$ denote the algebra of all n×n matrices with entries in k. Let S be a semigroup. By a representation G of S of degree n(a positive integer) over k is a homomorphism of S into a multiplicative semigroup of $(k)_n$ i.e., each element a of S corresponds to an n×n matrix (or a linear transformation) G(a) such that $\Gamma(ab) = \Gamma(a)\,\Gamma(b)$ for all a, b in S. If Γ is an isomorphism of S upon a semigroup of $(k)_n$ then it is called faithful.

We see by Clifford and Preston (1961) each representation Γ^* of S over a field k denotes in a natural sense an extension of a representation Γ of a group G over k. If Γ is as above, among all extensions, there is a least degree uniquely determined by Γ within equivalence which is called basic extension Γ^{o*} of Γ.

A representation Γ of a semigroup S is called proper if:

- $\Gamma(z) = 0$ if S has a zero z.
- Γ is not decomposable into two representations one of which is null (Clifford and Preston, 1961).

We have the following result for the indecomposable constituents of basic extension as the extensions of the indecomposable constituents of Γ.

Theorem 6: Clifford and Preston (1961) [Th. 9]. A proper representation Γ of a group G is extendible to S if and only if each of its indecomposable constituents is extendible. If Γ is extendible, then the indecomposable constituents of a the basic extension Γ^{o*} of Γ are the basic extensions of the indecomposable constituents of Γ. In particular, Γ is indecomposable if and only if, Γ^{o*} is indecomposable; in fact any proper extension to S of an indecomposable representation Γ of G is also indecomposable.

Since, the role of irreducible representation is very important in representation theory due to the fact that all other representations can be obtained from the irreducible constituents. Following results are useful in obtaining the irreducible representations as the basic extensions to that of a semigroup S of the extendible irreducible representation of a group G. We add these results to understand the whole picture of the representation theory of semigroups and in particular of Clifford semigroups with identity.

Theorem 7: Clifford and Preston (1961) [Th. 9]. Let Γ be an extendible representation of a group G and let $\Gamma^{?}$ be any extension of Γ to S. Then the non-null irreducible constituents of $\Gamma^{?}$ are the basic extensions of the non-null irreducible constituents of Γ. The basic extension Γ^{o*} of Γ is irreducible if and only if, Γ is irreducible; thus we get all the irreducible representations of S as the basic extensions of S of the extendible irreducible representations of G.

Theorem 8: Clifford and Preston (1961) [Th. 9]. Full reducibility holds for the representations of S over the field k if and only if,

- Full reducibility holds for the extendible representations of G over k.
- The only proper extension to S of a proper representation of G is the basic extension.

Corollary 2: Clifford and Preston (1961) [Cor. 2]. Let S be a finite semigroup and assume that the characteristic of k does not divide the order of G. Then the semigroup algebra k[S] is semisimple if and only if, the only proper representation of S extending any given proper representation of G is its basic extension.

The special case of above theorem is that the Clifford monoid algebra of any finite Clifford monoid is semisimple if and only if, the group algebra of each maximal subgroup is semisimple.

The characterization of commutative semigroups is given in contrast to that of abelian groups. S. Schwarz and Hewitt and Zuckerman in their collaboration in 1955 and in 1956 developed character theory of commutative semigroups, latter Howie described their work in Howie (1995). The character of commutative semigroup with identity which is union of a semilattice of groups. The well known orthogonality relation for the characters also holds. The character of a Clifford monoid is obtained similarly. Thus it is interesting to define the character of the semilattice graded weak Hopf algebras using the definition of character of Clifford monoid and getting inspiration from Larson (1971) who developed the theory of characters of Hopf algebras from the character theory of groups.

Since, the inverse semigroups are generalized named as generalized groups by Vagner (1952, 1953). The algebras of inverse semigroups may be named as the generalized group algebras by analogy. One can travel from group algebras to the algebras of inverse semigroups with identity passing through the Clifford monoid algebras as for as their representation and characterization are concerned.

THE CHARACTERIZATION AND THE REPRESENTATION OF HOPF ALGEBRA

We refer to Montgomery (1993) for the definitions of coalgebras, antipode, Hopf algebra, Hopf ideals, Hopf modules, comodules and for some other commonly used terminologies related to Hopf algebras. We only give definitions of some unavoidable terms like integrals in Hopf algebras and semisimplicity etc.

SEMISIMPLE AND COSEMISIMPLE FINITE DIMENSIONAL HOPF ALGEBRA

A Hopf algebra H is semisimple if every left H-module (right H-module) can be decomposed into indecomposable irreducible left H-submodules (right H-submodules). By Montgomery (1993) a left integral in a Hopf algebra H is an element t in H such that ht = ε(h)t, for all h in H, where ε is the counit of the Hopf algebra H.A right integral in H is defined by dual statement. The space of left integrals of H can be defined to be the subset

$$_H\int^l = \{t\in H \mid ht = \varepsilon(h)t \ (\forall\ h\in H)\}$$

of H. The space of right integrals denoted $_H\int^r$ of H can be defined by the dual statement. A Hopf algebra H is called unimodular if $_H\int^l = {}_H\int^r$.

Now we recall the well known theorem in representation theory which is termed as Maschke's Theorem, we state as follows:

Theorem 9: Montgomery (1993) [Th. 2.2.1]. Let H be a finite dimensional Hopf algebra, then H is semisimple if and only if $\varepsilon(_H\int^l) \neq 0$ if and only if $\varepsilon(_H\int^r) \neq 0$.

By Kaplansky (1975), if H is a finite dimensional semisimple Hopf algebra over an algebraically closed field k of characteristic zero. Then H is Frobenius type, i. e., if V is an irreducible representation of H then dimV divides dimH. Further, Kaplansky has conjectured that a Hopf algebra of prime dimension over k(as above) is a group algebra. By him, (Kaplansky, 1975), the square of antipode of a finite dimensional semisimple Hopf algebra is identity. Zhu (1994) proved the Kaplansky's conjecture for a Hopf algebra of prime dimension, he showed that a prime p-dimensional Hopf algebra over an algebraically closed field k is isomorphic to the group algebra k $\mathbb{Z}_p$ of the cyclic group $\mathbb{Z}_p$.

Let A be a finite dimensional Hopf algebra over k, if B is a subHopf algebra of A, then by Nichols-Zoeller Theorem Montgomery (1993) A is a free left (or right) B-module. Larson and Radford (1995) proved that if A is semisimple then B is semisimple and if A and its dual A^* are semisimlpe then B and its dual B^* are semisimple.

A coalgebra C is simple if it has no proper subcoalgebras and C is cosemisimple if it is direct sum of simple subcoalgebras. If C is finite dimensional, then it is cosemisimple if and only if its dual C^* is semisimple. We know by Montgomery (1993) that a cocommutative and cosemisimple Hopf algebra over an algebraically closed field of characteristic zero is a group algebra. Larson (1971) has generalized the Maschke's Theorem, i.e., if Chark does not divide the dimension of finite dimensional involutory Hopf algebra, then the Hopf algebra and its dual are semisimple. He also discussed the character theory of Hopf algebras and proved that the orthogonality relation holds for finite dimensional Hopf algebras.

Green (1976) hat the indecomposable components of a coalgebra are blocks with respect to the equivalence relations on the simple comodules using injective covers.

POINTED HOPF ALGEBRA, PATH COALGEBRA AND THE QUIVER REPRESENTATION

Let C be a coalgebra with the comultiplication Δ and counit ε. An element c in C is called grouplike if $\Delta(c) = c\otimes c$ and $\varepsilon(c) = 1$, the set of grouplike elements is denoted by G(C). For h, g∈G(C), an element c is called (g, h)-primitive (or skew-primitive) if $\Delta(c)=c\otimes g+h\otimes c$, the set of (g, h)-primitive elements of coalgebra C is denoted by $P_{g,h}(C)$. If C = B is a bialgebra and g = h = 1, then the elements of $P(B) = P_{1,1}(B)$ are simply called the primitive elements of B. A Hopf algebra H is called Pointed if $G(H) = H_0$, where H_0 is the coradical (Montgomery, 1993) of H.

Andruskiewitsch and Schneider give a conjecture that a finite dimensional Pointed Hopf algebra over k of characteristic zero is generated as an algebra by its grouplike and skew-primitive elements. They give a partial proof of the conjecture in the form of Theorem 1.3, they prove that the conjecture holds for finite dimensional coradically graded Hopf algebra with coradical of odd prime dimension. As far as the classification of pointed Hopf algebras is concerned, the grouplike and skew-primitive elements play an vital role, even, in the case of the Hopf algebras which are quantum groups. First we introduce briefly the quiver and path algebra.

A quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple(which is an oriented graph) consisting of Q_0 the set of vertices, Q_0 the set of arrows and the two maps s, t: $Q_1\to Q_1$ which associate to each arrow $\alpha\in Q_0$ its source $s(\alpha)\in Q_0$ and its target $t(\alpha)\in Q_1$, respectively. An arrow $\alpha\in Q_1$ of source $a = s(\alpha)$ and target $b = t(\alpha)$ is usually denoted by α:a>b. A quiver is simply denoted by Q. A sequence of connected arrows is called a path of the quiver the vertices can be identified as trivial paths. If Q is a quiver with vertex set Q_0 and arrow set Q_1 The path coalgebra kQ of Q is defined to be the k-span of all paths in Q with coalgebra structure $\Delta(p) = \Sigma_{c(p),\ p=p2p1}p^2\otimes p^1$, where

p_2p_1 is the concatenation $a_ta_{t-1}...a_{s+1}a_s...a_1$ of the path $p_2 = a_ta_{t-1}...a_{s+1}$ and $p_1 = a_s...a_1$ where $a_i \in Q$ and $t = |p|$ denotes the length of the path p and the starting vertex of a_{i+1} is the end of a_i in path p.

Cibils (1993) used quivers to give the irreducible representation of a finite dimensional non-commutative and non-cocommutative Hopf algebras. These Hopf algebras are considered as quantum groups. He gave some results on the indecomposable modules decomposition. He also presents the finite representation type of algebra using the Auslander-Reiten quiver as the indecomposable modules and relations given by almost split sequences. Cibils and Rosso (2002) defined the Hopf quivers and classified finite dimensional pointed Hopf algebras using the path coalgebra.

An algebra A is said to be left (right) hereditary if any left (right) ideal of A is projective as an A-module. A is hereditary if it is both left and right hereditary. Chin (2002) proved that every pointed hereditary coalgebra over a field k(algebraically closed) is a path coalgebra of quiver. Due to the isomorphism of pointed hereditary coalgebra and the path coalgebra given by Chin (2002), we can give the quiver representation through path coalgebra for the representation of pointed hereditary Hopf algebras.

It looks interesting to ask about the development of the above representation and characterization for the finite dimensional pointed semilattice graded weak Hopf algebras over an algebraically closed field of characteristic zero.

REPRESENTATION APPROACH TOWARDS SEMILATTICE GRADED WEAK HOPF ALGEBRA

A commutative semigroup S is separative if for all a, b in S, $ab = a^2 = b^2$ implies a = b. Let S be a separative or cancellative commutative semigroup with identity (Clifford and Preston, 1961). Let η be a relation on S such that $a\eta b \Leftrightarrow \exists$ x, y such that $ax = b^m$, $by = an$, then the homomorphic image $S' = S/\eta = \cup_{\alpha\in Y} S_\alpha$, where Y is a semilattice and S_α are called archimedean components of S and S is called archimedean semigroup. By Clifford and Preston (1961) [Th. 4.12], S/η is a maximal semilattice homomorphic image of S. By Clifford and Preston (1961) [Th. 4.13] every commutative semigroup S is uniquely expressible as a semilattice Y of archimedean components $S_\alpha(\alpha\in Y)$, $Y \cong S/\eta$ with S_α the equivalence classes of S mod η.

If $aib \Leftrightarrow ab^m = b^{m+1}$, $ban = an^{-1}$, a, b in S, m, $n \in \mathbb{Z}^+$, if S is separative then a = b. Now, by Clifford and Preston (1961) [Th. 4.17] a commutative semigroup S can be embedded into a semigroup S which is union of groups if and only if, S is separative if and only if, its archimedean components are cancellative.

Let S be a regular separative(or cancellative) commutative semigroup with identity then S can be expressed uniquely as the semilattice Y of its archimedean semigroups S_α ($\alpha\in Y$). Of course, each of its archimedean components S_α is an equivalence class mod η. Since, S can be embedded in a semigroup Q which is union of groups G_α ($\alpha\in Y$, a semilattice), where G_α is the quotient group of S_α, for each α in Y such that $ab^{-1}\in G_\alpha$ for every a, b in S_α. Thus $S_\alpha \subseteq G_\alpha$, (Clifford and Preston, 1961).

Let k be algebraically closed field such that its characteristic does not divide order of any subgroup of S. Then the group algebra k G_α is a pointed Hopf algebra for each $\alpha\in Y$. The semigroup algebra $kQ = k\langle\cup_{\alpha\in Y} G_\alpha\rangle = \oplus_{\alpha\in Y} kG_\alpha$ is a Clifford monoid algebra so by Li (2004) and Li and Cao (2005) kQ is a pointed semilattice graded weak Hopf algebra.

It is interesting to note that in a weak Hopf algebra (Li, 1998), the set of grouplike elements is the regular semigroup with identity. Weak Hopf algebra can be considered as the generalization of Hopf algebra, the set of grouplike elements of the Hopf algebra is a group. In understanding the structural relationship between the weak Hopf algebra and the Hopf algebra one can get through the structural relationship of above algebras by the structure of the semilattice graded weak Hopf algebra, set of whose grouplike elements is the Clifford monoid. A Clifford monoid is an inverse semigroup (with 1) whose idempotents lies in its centre. Thus a Clifford monoid can also be named as generalized group in the Vagner's (1952, 1953).

After getting through the above sketch one may be able to obtain the representation of Weak Hopf algebras from the representation of Hopf algebras by developing the representation of the semilattice graded weak Hopf algebra. Thus, it looks useful to discuss first the representation of semilattice graded weak Hopf algebras.

The theory of representation and characterization of Hopf algebras has been developed in several aspects and in particular that of quantized Hopf algebras has also been well developed and is being applied to physics and to some integrable systems. Some research on weak Hopf algebras and semilattice graded weak Hopf algebras is also done in the obvious sense but there is still a capacity in this area of doing work similar to that of quantized Hopf algebra, rather much is to be done of its representation theory and the characterization theory.

Since, the Clifford monoids appear to be the set of grouplike elements of semilattice graded weak Hopf algebras. Further, we know that a Clifford monoids give us special class of groups which form a semilattice of these groups. In the case of pointed semilattice graded weak

Hopf algebra, the algebra generated by grouplike elements is isomorphic to coradical and the set of grouplike elements is always a Clifford monoid.

Analogous to the theory of representation and characterization of Hopf algebra one may investigate the same for the semilattice graded weak Hopf algebras.

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