Journal of Modern Mathematics and Statistics 2 (1): 18-27, 2008
ISSN: 1994-5388

Initial Value Solvers for Second Order Ordinary Differential Equations Using Chebyshev Polynomial as Basis Functions

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Abstract: Direct numerical solutions of second and higher order ODEs have constituted a major problem in the resent past. In order to seek for numerical solution of equations of these types, one have to resolve the equations into a system of first order ordinary differential equations before solving the resulting equations with any desired known methods. In an attempt to do this, it often takes a very long period of time and the task is always laborious. In order to circumvent this problem, we were led to this present study. Here, we decide to come up with a multistep method which is more accurate and more efficient with very small error constant in solving second order ordinary differential equations without resolving into system of first order ordinary differential equations. The method used here is based on collocating and interpolating at k and k-1 points respectively where k = 6. This yields a 6 step method which is symmetric, zero stable and constituent. Some numerical examples were given to illustrate the desirability and efficiency of this method.

Key words: Collocation, interpolation, multistep, chebyshev polynomials, zero stable, power series

INTRODUCTION

The numerical solutions of first order ordinary differential equations have been well studied and are reported in the literatures (Fatunla, 1988; Lambert, 1973; Omolehin *et al*., 2003; Onumayin *et al*., 1993). Onumayin *et al.* (1993) proposed finite difference methods by collocation in developing initial value solvers of first order ODEs.

In this presentation, we shall propose a direct method for solving second order ODEs of the form

$$y'' = f(x,y,y`\ ;\ y(x_0) = y_0,\ y`(x_0) = \eta \tag{1}$$

rather than reducing (1) to system of first order ODEs and then solve by any known initial value solvers that has been well studied and reported in the literatures (Fatunla, 1988; Omolehin *et al*., 2003; Spiegel, 1973). The removal of the burden of resolving into system of equations is hereby removed by proposing a direct method of solving second order ODEs.

In an attempt to solve (1), Omolehin *et al.* (2003) presented a two step implicit method which is of order four to solve (1) directly. However, the quest to improve upon the out put of the result generated by the method of Omolehin *et al.* (2003), Kayode and Awoyemi (2003) and Kayode and Awoyemi (2005) proposed some direct methods of solving Eq. 1 as reported in the literature. However, Kayode and Awoyemi (2005) proposed a 6 step methods for solving (1) directly, using power series method by letting

$$y(x) = \sum_{0}^{2k} a_j x^j \tag{2}$$

and

$$y''(x) \sum_{2}^{2k} j(j-1)a_j x^{j-2} \tag{3}$$

Collocation of (3) was done at x_{n+i}, i = o(1)k and interpolation of (2) was done at x_{n+i-1}, i = o(1) k-1. Using (2) and (3), Awoyemi and Kayode (2005) came up with a six steps implicit method for solving (1) directly which is of order 8 and error constant 0.0082295. That is the scheme

$$\begin{aligned} &539Y_{k+6} - 4374Y_{k+5} + 32805Y_{k+4}\ -57940Y_{k+3} \\ &+ 32805Y_{k+2}\ 4374Y_{k+1} + 539Y_k \\ &h^2\{60f_{k+6} + 20040f_{k+3} + 60f_k\}. \end{aligned} \tag{4}$$

In this presentation, efforts are made to develop an implicit optimal k order method to solve (1) directly and which will be consistent, zero stable and of smaller error constant compared with method of Kayode and Awoyemi

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(2005). This led us to the use of Chebyshev Polynomials as our basis function. This approach ensures even distribution of the error in our approximant Y(x) of the desired solution y(x) as a result of the equal oscillation properties of our basis function-the Chebyshev Polynomials-through the entire range of consideration.

DERIVATION OF THE METHODS

In this aspect, we shall present the derivation of our scheme using Chebyshev polynomials as basis functions. Though some authors have used this approach to derive some initial value silvers for first order ODEs and have been reported in the literatures (Adeniyi and Alabi, 2006, 2007; Adeniyi *et al.*, 2006).

As a result of this, an attempt is been made here to extend this approach to the derivation of initial value solvers for second order ODEs. We consider the polynomial equation

$$Y(x) = \sum_{0}^{k} a_r T_r(x) \tag{5}$$

Where, $T_r(x)$ is a Chebyshev polynomial derived from the recursive relation

$$T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x) \tag{6}$$

Here, we let k = 13 in (5) which leads to

$$\begin{aligned} Y(x) = {} & a_0 + a_1(2x^2-1) + a_3\left(4x^3-3x\right) + a_4\left(8x^4-8x^2+1\right) + a_5\left(16x^5-20x^3+5x\right) + \\ & a_6\left(32x^6-48x^4+18x^2-1\right) + a_7\left(64x^7-112x^5+56x^3-7x\right) + a_8\left(128x^8-256x^6+160x^4-32x^2+1\right) + \\ & a_9\left(256x^9-576x^7+432x^5-120x^3+9x\right) + a_{10}\left(512x^{10}-1280x^8+1120x^6-400x^4+50x^2-1\right) + \\ & a_{11}\left(1024x^{11}-2816x^9+2816x^7-1232x^5+220x^3-11x\right) + \\ & a_{12}\left(2048x^{12}-6144x^{10}+6912x^8-3584x^6+840x^4-72x^2+1\right) \end{aligned} \tag{7}$$

Transforming (7) into the interval $x_k \le x \le x_{k+6}$ yields

$$\begin{aligned} Y(x) = {} & a_0 + a_1\left(\frac{2x-2kh-6h}{6h}\right) + a_2\left[\left(\frac{2x-2kh-6h}{6h}\right)^2 - 1\right] + a_3\left[4\left(\frac{2x-2kh-6h}{6h}\right)^3 - 3\left(\frac{2x-2kh-6h}{6h}\right)\right] + \\ & a_4\left[8\left(\frac{2x-2kh-6h}{6h}\right)^4 - 8\left(\frac{2x-2kh-6h}{6h}\right)^2 + 1\right] + a_5\left[\begin{array}{l} 16\left(\frac{2x-2kh-6h}{6h}\right)^5 - 20\left(\frac{2x-2kh-6h}{6h}\right)^3 + \\ 5\left(\frac{2x-2kh-6h}{6h}\right) \end{array}\right] + \\ & a_6\left[32\left(\frac{2x-2kh-6h}{6h}\right)^6 - 48\left(\frac{2x-2kh-6h}{6h}\right)^4 + 18\left(\frac{2x-2kh-6h}{6h}\right)^2 - 1\right] + \\ & a_7\left[64\left(\frac{2x-2kh-6h}{6h}\right)^7 - 112\left(\frac{2x-2kh-6h}{6h}\right)^5 + 56\left(\frac{2x-2kh-6h}{6h}\right)^3 - 7\left(\frac{2x-2kh-6h}{6h}\right)\right] + \\ & a_8\left[128\left(\frac{2x-2kh-6h}{6h}\right)^8 - 256\left(\frac{2x-2kh-6h}{6h}\right)^6 + 160\left(\frac{2x-2kh-6h}{6h}\right)^4 - 32\left(\frac{2x-2kh-6h}{6h}\right)^2 + 1\right] + \\ & a_9\left[\begin{array}{l} 256\left(\frac{2x-2kh-6h}{6h}\right)^9 - 576\left(\frac{2x-2kh-6h}{6h}\right)^7 + 432\left(\frac{2x-2kh-6h}{6h}\right)^5 - 120\left(\frac{2x-2kh-6h}{6h}\right)^3 + \\ 9\left(\frac{2x-2kh-6h}{6h}\right) \end{array}\right] + \\ & a_{10}\left[\begin{array}{l} 512\left(\frac{2x-2kh-6h}{6h}\right)^{10} - 1280\left(\frac{2x-2kh-6h}{6h}\right)^8 + 1120\left(\frac{2x-2kh-6h}{6h}\right)^6 - 400\left(\frac{2x-2kh-6h}{6h}\right)^4 + \\ 50\left(\frac{2x-2kh-6h}{6h}\right)^2 - 1 \end{array}\right] + \end{aligned} \tag{8}$$

$$a_{11}\left[\begin{array}{l}1024\left(\frac{2x-2kh-6h}{6h}\right)^{11}-2816\left(\frac{2x-2kh-6h}{6h}\right)^{9}+2816\left(\frac{2x-2kh-6h}{6h}\right)^{7}-1232\left(\frac{2x-2kh-6h}{6h}\right)^{5}+\\ 220\left(\frac{2x-2kh-6h}{6h}\right)^{3}-11\left(\frac{2x-2kh-6h}{6h}\right)\end{array}\right]+$$

$$a_{12}\left[\begin{array}{l}2048\left(\frac{2x-2kh-6h}{6h}\right)^{12}-6144\left(\frac{2x-2kh-6h}{6h}\right)^{10}+6912\left(\frac{2x-2kh-6h}{6h}\right)^{8}-3584\left(\frac{2x-2kh-6h}{6h}\right)^{6}+\\ 840\left(\frac{2x-2kh-6h}{6h}\right)^{4}-72\left(\frac{2x-2kh-6h}{6h}\right)+1\end{array}\right]$$

The second derivative of (8) gives

$$\begin{aligned}
Y``(x) &= \frac{16a_2}{36h^2}+\frac{96a_3}{36h^2}\left(\frac{2x-2kh-6h}{6h}\right)+\frac{64a_4}{36h^2}\left[6\left(\frac{2x-2kh-6h}{6h}\right)^2-1\right]+\\
&\frac{160a_5}{36h^2}\left[8\left(\frac{2x-2kh-6h}{6h}\right)^3-3\left(\frac{2x-2kh-6h}{6h}\right)\right]+\frac{48a_6}{36h^2}\left[80\left(\frac{2x-2kh-6h}{6h}\right)^4-48\left(\frac{2x-2kh-6h}{6h}\right)^2+3\right]+\\
&\frac{448a_7}{36h^2}\left[24\left(\frac{2x-2kh-6h}{6h}\right)^5-20\left(\frac{2x-2kh-6h}{6h}\right)^3+3\left(\frac{2x-2kh-6h}{6h}\right)\right]+\\
&\frac{256a_8}{36h^2}\left[112\left(\frac{2x-2kh-6h}{6h}\right)^6-120\left(\frac{2x-2kh-6h}{6h}\right)^4+30\left(\frac{2x-2kh-6h}{6h}\right)^2-1\right]+\\
&\frac{576a_9}{36h^2}\left[128\left(\frac{2x-2kh-6h}{6h}\right)^7-168\left(\frac{2x-2kh-6h}{6h}\right)^5+60\left(\frac{2x-2kh-6h}{6h}\right)^3-5\left(\frac{2x-2kh-6h}{6h}\right)\right]+\\
&\frac{80a_{10}}{36h^2}\left[2304\left(\frac{2x-2kh-6h}{6h}\right)^8-3584\left(\frac{2x-2kh-6h}{6h}\right)^6+1680\left(\frac{2x-2kh-6h}{6h}\right)^4-240\left(\frac{2x-2kh-6h}{6h}\right)^2+5\right]+\\
&\frac{352a_{11}}{36h^2}\left[\begin{array}{l}1280\left(\frac{2x-2kh-6h}{6h}\right)^9-2304\left(\frac{2x-2kh-6h}{6h}\right)^7+1344\left(\frac{2x-2kh-6h}{6h}\right)^5-280\left(\frac{2x-2kh-6h}{6h}\right)^3+\\ 15\left(\frac{2x-2kh-6h}{6h}\right)\end{array}\right]+\\
&\frac{192a_{12}}{36h^2}\left[\begin{array}{l}5632\left(\frac{2x-2kh-6h}{6h}\right)^{10}-11520\left(\frac{2x-2kh-6h}{6h}\right)^8+8064\left(\frac{2x-2kh-6h}{6h}\right)^6-2240\left(\frac{2x-2kh-6h}{6h}\right)^4+\\ 210\left(\frac{2x-2kh-6h}{6h}\right)^2-3\end{array}\right]
\end{aligned} \tag{9}$$

Collocating (9) at $x = x_{k+i}$, $i = 0(1)\,6$ and interpolating (8) at $x = x_{k+i}$, $i = 0(1)5$ lead to the system of equations

$$\begin{aligned}
&a_0-a_1+a_2-a_3+a_4-a_5+a_6-a_7+a_8-a_9+a_{10}-a_{11}+a_{12}=y_k\\
&a_0-\frac{2a_1}{3}-\frac{a_2}{9}+\frac{22a_3}{27}-\frac{79a_4}{81}+\frac{118a_5}{243}+\frac{239a_6}{729}-\frac{2018a_7}{2187}+\frac{5921a_8}{6561}-\frac{5522a_9}{19683}-\frac{31201a_{10}}{59049}+\frac{174502a_{11}}{177147}-\frac{417199a_{12}}{531441}=y_{k+1}\\
&a_0-\frac{a_1}{3}-\frac{7a_2}{9}+\frac{23a_3}{27}-\frac{17a_4}{81}-\frac{241a_5}{243}+\frac{329a_6}{729}+\frac{1511a_7}{2187}-\frac{5983a_8}{6561}-\frac{1633a_9}{19683}+\frac{57113a_{10}}{59049}-\frac{99529a_{11}}{177147}-\frac{314959a_{12}}{531441}=y_{k+2}\\
&a_0-a_2+a_4-a_6+a_8-a_{10}+a_{12}=y_{k+3}\\
&a_0+\frac{a_1}{3}-\frac{7a_2}{9}-\frac{23a_3}{27}+\frac{17a_4}{81}+\frac{241a_5}{243}+\frac{329a_6}{729}-\frac{1511a_7}{2187}-\frac{5983a_8}{6561}+\frac{1633a_9}{19683}+\frac{57113a_{10}}{59049}+\frac{99529a_{11}}{177147}-\frac{314959a_{12}}{531441}=y_{k+4}\\
&a_0+\frac{2a_1}{3}-\frac{a_2}{9}-\frac{22a_3}{27}-\frac{79a_4}{81}-\frac{118a_5}{243}+\frac{239a_6}{729}+\frac{2018a_7}{2187}+\frac{5921a_8}{6561}+\frac{5522a_9}{19683}-\frac{31201a_{10}}{59049}-\frac{174502a_{11}}{177147}-\frac{417199a_{12}}{531441}=y_{k+5}
\end{aligned}$$

$$16a_2-96a_3+320a_4-800a_5+1680a_6-3136a_7+5376a_8-8640a_9+13200a_{10}-19360a_{11}+27456a_{12}=36h^2\,f_k$$

$$\frac{16a_2}{36h^2}-\frac{64a_3}{36h^2}+\frac{320a_4}{108h^2}-\frac{400a_5}{243h^2}-\frac{820a_6}{243h^2}+\frac{6944a_7}{729h^2}-\frac{71744a_8}{6561h^2}+\frac{6560a_9}{2187}+\frac{79300_{10}}{6561h^2}-\frac{4282960a_{11}}{177147}+\frac{3769552a_{12}}{177147}=f_{k+1}$$

$$\frac{16a_2}{36h^2}-\frac{32a_3}{36h^2}-\frac{64a_4}{108h^2}+\frac{760a_5}{243h^2}-\frac{436a_6}{243h^2}-\frac{3248a_7}{729h^2}+\frac{46912a_8}{6561h^2}+\frac{2704a_9}{2187}-\frac{80060a_{10}}{6561h^2}+\frac{1434136a_{11}}{177147}+\frac{1965424a_{12}}{177147}=f_{k+2}$$

$$16a_2-64a_4+144a_6-256a_8+400a_{10}-576a_{12}=36h^2\,f_{k+3}$$

$$\frac{16a_2}{36h^2}+\frac{32a_3}{36h^2}-\frac{64a_4}{108h^2}-\frac{760a_5}{243h^2}-\frac{436a_6}{243h^2}+\frac{3248a_7}{729h^2}+\frac{46912a_8}{6561h^2}-\frac{2704a_9}{2187}-\frac{80060a_{10}}{6561h^2}-\frac{1434136a_{11}}{177147}+\frac{1965424a_{12}}{177147}=f_{k+4}$$

$$\frac{16a_2}{36h^2}+\frac{64a_3}{36h^2}+\frac{320a_4}{108h^2}+\frac{400a_5}{243h^2}-\frac{820a_6}{243h^2}-\frac{6944a_7}{729h^2}-\frac{71744a_8}{6561h^2}-\frac{6560a_9}{2187}+\frac{79300a_{10}}{6561h^2}+\frac{4282960a_{11}}{177147}+\frac{3769552a_{12}}{177147}=f_{k+5}$$

$$16a_2+96a_3+320a_4+800a_5+1680a_6+3136a_7+5376a_8+8640a_9+13200a_{10}+19360a_{11}+27456a_{12}=36h^2f_{k+6}$$

The solution of the set of equations above yields

$$a_{12}=\frac{51195483y_{k+1}}{926547968}+\frac{1948617y_{k+2}}{14477312}-\frac{175906971y_{k+3}}{463273984}+\frac{1948617y_{k+4}}{14477312}+\frac{51195483y_{k+5}}{926547968}+\frac{1554957h^2f_k}{32429178880}$$
$$\frac{-110598777h^2f_{k+1}}{32429178880}-\frac{-464892777h^2f_{k+2}}{6485835776}-\frac{666525429h^2f_{k+3}}{3242917888}-\frac{464892777h^2f_{k+4}}{6485835776}-\frac{110598777h^2f_{k+5}}{32429178880}$$
$$\frac{1554957h^2f_{k+6}}{32429178880}$$

$$a_{11}=\frac{1830519y_k}{56950784}+\frac{129467235411y_{k+1}}{402585092096}+\frac{15390207109y_{k+2}}{201292546048}-\frac{212642239635y_{k+3}}{201292546048}+\frac{176259080187y_{k+4}}{402585092906}$$
$$+\frac{76037810643y_{k+5}}{40258509}-\frac{1968884181h^2f_k}{14090478223360}-\frac{839913310611h^2f_{k+1}}{14090478223360}-\frac{141661365669h^2f_{k+2}}{352261955584}-\frac{995767556139h^2f_{k+3}}{1409047822336}$$
$$-\frac{678445119189h^2f_{k+4}}{2818095644672}-\frac{169819274979h^2f_{k+5}}{14090478223360}+\frac{1554957h^2f_{k+6}}{8107294720}$$

$$a_{10}=\frac{2145447y_{k+1}}{14477312}-\frac{137781y_{k+2}}{3619328}-\frac{1594323y_{k+3}}{7238656}-\frac{137781y_{k+4}}{3619328}+\frac{2145447y_{k+5}}{14477312}+\frac{540189h^2f_k}{1621458944}-$$
$$\frac{48794157h^2f_{k+1}}{4053647360}-\frac{1210169889h^2f_{k+2}}{8107294720}-\frac{471764331h^2f_{k+3}}{2026823680}-\frac{1210169889h^2f_{+4}}{8107294720}-\frac{48794157h^2f_{k+5}}{4053647360}+$$
$$\frac{540189h^2f_{k+6}}{1621458944}$$

$$a_9=-\frac{295245y_k}{51377344}-\frac{28446193089y_{k+1}}{36598644736}+\frac{5098218489y_{k+2}}{18299322368}+\frac{34297135425y_{k+3}}{18299322368}-\frac{43557645753y_{k+4}}{36598644736}-$$
$$\frac{4699782081y_{k+5}}{36598644736}+\frac{2074100499h^2f_k}{1280952565760}+\frac{161945222049h^2f_{k+1}}{1280952565760}+\frac{56452818861h^2f_{k+2}}{64047628288}+\frac{160607670345h^2f_{k+3}}{128095256576}$$
$$+\frac{66404215611h^2f_{k+4}}{256190513152}+\frac{914872401h^2f_{k+5}}{1280952565760}+\frac{4223097h^2f_{k+6}}{8107294720}$$

$$a_8=\frac{62572257y_{k+1}}{463273984}-\frac{2381643y_{k+2}}{7238656}+\frac{214997409y_{k+3}}{231636992}-\frac{2381643y_{k+4}}{7238656}-\frac{62572257y_{k+5}}{463273984}$$
$$\frac{2673243h^2f_k}{2316369920}+\frac{1642437h^2f_{k+1}}{2316369920}+\frac{90005985h^2f_{k+2}}{463273984}+\frac{110487969h^2f_{k+3}}{231636992}+\frac{90005985h^2f_{k+4}}{463273984}+$$
$$\frac{1642437h^2f_{k+5}}{2316369920}+\frac{2673243h^2f_{k+6}}{2316369920}$$

$$a_7 = \frac{40095y_k}{5177344} + \frac{15443103195y_{k+1}}{36598644736} - \frac{12272390595y_{k+2}}{18299322368} - \frac{4657635675y_{k+3}}{18299322368} + \frac{29075315715y_{k+4}}{36598644736} - \frac{10941797925y_{k+5}}{36598644736} - \frac{2935427121h^2f_k}{1280952565760} - \frac{45152640891h^2f_{k+1}}{1280952565760} - \frac{25217430219h^2f_{k+2}}{64047628288} - \frac{21810918195h^2f_{k+3}}{128095256576} + \frac{61186135671h^2f_{k+4}}{256190513152} + \frac{23035837941h^2f_{k+5}}{1280952565760} + \frac{2317491h^2f_{k+6}}{1158184960}$$

$$a_6 = -\frac{16725933y_{k+1}}{14477312} - \frac{1573425y_{k+2}}{3619328} + \frac{23019633y_{k+3}}{7238656} - \frac{1573425y_{k+4}}{3619328} - \frac{16725933y_{k+5}}{14477312} + \frac{21682701h^2f_k}{8107294720}\frac{368086599h^2f_{k+1}}{4053647360} + \frac{9642565539h^2f_{k+2}}{8107294720} + \frac{5047429761h^2f_{k+3}}{2026823680} + \frac{9642565539h^2f_{k+4}}{8107294720} + \frac{368086599h^2f_{k+5}}{4053647360} + \frac{21682701h^2f_{k+6}}{8107294720}$$

$$a_5 = -\frac{3881763y_k}{25886720} - \frac{25996395555y_{k+1}}{36598644736} - \frac{35397568053y_{k+2}}{18299322368} + \frac{90184999779y_{k+3}}{18299322368} - \frac{16928825931y_{k+4}}{36598644736} - \frac{305808027183y_{k+5}}{18299322368} + \frac{2438521173h^2f_k}{1280952565760} + \frac{214496020323h^2f_{k+1}}{1280952565760} + \frac{211560800427h^2f_{k+2}}{160119070720} + \frac{2111605318503h^2f_{k+3}}{640476282880} + \frac{2149445826441h^2f_{k+4}}{1280952565760} + \frac{213747837363h^2f_{k+5}}{1280952565760} + \frac{30189321h^2f_{k+6}}{8107294720}$$

$$a_4 = -\frac{2196932139y_{k+1}}{926547968} - \frac{28841913y_{k+2}}{14477312} + \frac{4042814571y_{k+3}}{463273984} - \frac{28841913y_{k+4}}{14477312} - \frac{2196932139y_{k+5}}{926547968} + \frac{21521541h^2f_k}{4632739840} + \frac{1281352527h^2f_{k=1}}{4632739840} + \frac{12050665083h^2f_{k+2}}{4632739840} + \frac{13230488319h^2f_{k+3}}{2316369920} + \frac{12050665083h^2f_{k+4}}{4632739840} + \frac{1281352527h^2f_{k+5}}{4632739840} + \frac{21521541h^2f_{k+6}}{4632739840}$$

$$a_3 = -\frac{913815y_k}{2588672} - \frac{50623339635y_{k+1}}{18299322368} - \frac{21641150565y_{k+2}}{9149661184} + \frac{106153319475y_{k+3}}{9149661184} - \frac{59974224795y_{k+4}}{18299322368} - \frac{51967015155y_{k+5}}{18299322368} + \frac{704869179h^2f_k}{91496611840} + \frac{36490010229h^2f_{k+1}}{91496611840} + \frac{4143983859h^2f_{k+2}}{1143707648} + \frac{71013999645h^2f_{k+3}}{9149661184} + \frac{62901736011h^2f_{k+4}}{18299322368} + \frac{35519917221h^2f_{k+5}}{91496611840} + \frac{3210387h^2f_{k+6}}{579092480}$$

$$a_2 = -\frac{21449853y_{k+1}}{7238656} - \frac{6376077y_{k+2}}{1809664} + \frac{46954161y_{k+3}}{3619328} - \frac{6376077y_{k+4}}{1809664} - \frac{21449853y_{k+5}}{7238656} + \frac{25839537h^2f_k}{4053647360} + \frac{970051059h^2f_{k+1}}{2026823680} + \frac{3210731379h^2f_{k+2}}{810729472} + \frac{1770579633h^2f_{k+3}}{202682368} + \frac{3210731379h^2f_{k+4}}{810729472} + \frac{970051059h^2f_{k+5}}{2026823680} + \frac{25839537h^2f_{k+6}}{4053647360}$$

$$a_1 = -\frac{68327111y_k}{142376960} - \frac{891788890887y_{k+1}}{201292546048} - \frac{339842106225y_{k+2}}{100646273024} + \frac{1587443769863y_{k+3}}{100646273024} - \frac{864440169039y_{k+4}}{201292546048} - \frac{3211866989091y_{k+5}}{1006462730240} + \frac{79096952637h^2f_k}{7045239111680} + \frac{3641794050087h^2f_{k+1}}{7045239111680} + \frac{8754933444813h^2f_{k+2}}{1761309777920} + \frac{37168650168891h^2f_{k+3}}{3522619555840} + \frac{4658038961111h^2f_{k+4}}{1006462730240} + \frac{779237897931h^2f_{k+5}}{1409047822336} + \frac{5499021h^2f_{k+6}}{810729472}$$

$$a_0 = -\frac{703925559y_{k+1}}{463273984} - \frac{13098429y_{k+2}}{7238656} + \frac{1773862007y_{k+3}}{231636992} - \frac{13098429y_{k+4}}{7238656} - \frac{703925559y_{k+5}}{463273984} + \frac{57309867h^2f_k}{16214589440} + \frac{919329345h^2f_{k+1}}{3242917888} + \frac{36914113557h^2f_{k+2}}{16214589440} + \frac{40618372581h^2f_{k+3}}{8107294720} + \frac{36914113557h^2f_{k+4}}{16214589440} + \frac{919329345h^2f_{k+5}}{3242917888} + \frac{57309867h^2f_{k+6}}{16214589440}$$

Substituting the value of a's into (7) gives the continuous scheme.

$$Y_{(x)} = \frac{Y_{k\ 5}}{1006462730240}\left\{\begin{array}{l} \frac{52321(x-x_k)^{12}}{h^{12}} + \frac{536545(x-x_k)^{11}}{h^{11}} - \frac{4387726(x-x_k)^{10}}{h^{10}} - \frac{56401052(x-x_k)^{9}}{h^{9}} + \\ \frac{104827470(x-x_k)^{8}}{h^{8}} + \frac{2109386326(x-x_k)^{7}}{h^{7}} - \frac{743087693(x-x_k)^{6}}{h^{6}} - \\ \frac{36424927560(x-x_k)^{5}}{h^{5}} + \frac{1562531520(x-x_k)^{4}}{h^{4}} + \frac{290212257300(x-x_k)^{3}}{h^{3}} + \\ \frac{100(x-x_k)^{2}}{h^{2}} - \frac{699161664800(x-x_k)}{h} - 300 \end{array}\right\} +$$

$$\frac{Y_{k\ 4}}{402585092096}\left\{\begin{array}{l} \frac{50981(x-x_k)^{12}}{h^{12}} + \frac{497494(x-x_k)^{11}}{h^{11}} - \frac{5635755(x-x_k)^{10}}{h^{10}} - \frac{61423157(x-x_k)^{9}}{h^{9}} + \\ \frac{224577408(x-x_k)^{8}}{h^{8}} + \frac{2832061025(x-x_k)^{7}}{h^{7}} - \frac{3923671722(x-x_k)^{6}}{h^{6}} - \\ \frac{59534128290(x-x_k)^{5}}{h^{5}} + \frac{25029869560(x-x_k)^{4}}{h^{4}} + \frac{521435622200(x-x_k)^{3}}{h^{3}} + \\ \frac{50(x-x_k)^{2}}{h^{2}} - \frac{1198587175000(x-x_k)}{h} - 390 \end{array}\right\} +$$

$$\frac{Y_{k+3}}{201292546048}\left\{\begin{array}{l} -\frac{71910(x-x_k)^{12}}{h^{12}} - \frac{600186(x-x_k)^{11}}{h^{11}} + \frac{7390845(x-x_k)^{10}}{h^{10}} + \frac{69002011(x-x_k)^{9}}{h^{9}} - \\ \frac{266508395(x-x_k)^{8}}{h^{8}} - \frac{2927048227(x-x_k)^{7}}{h^{7}} + \frac{4220906799(x-x_k)^{6}}{h^{6}} + \\ \frac{57428754360(x-x_k)^{5}}{h^{5}} - \frac{25654882170(x-x_k)^{4}}{h^{4}} - \frac{488063995000(x-x_k)^{3}}{h^{3}} + \\ \frac{110(x-x_k)^{2}}{h^{2}} + \frac{1203677073000(x-x_k)}{h} + 201292546200 \end{array}\right\} +$$

$$\frac{Y_{k+2}}{201292546048}\left\{\begin{array}{l}\frac{25491(x-x_k)^{12}}{h^{12}}+\frac{43157(x-x_k)^{11}}{h^{11}}-\frac{2817877(x-x_k)^{10}}{h^{10}}-\frac{2847937(x-x_k)^{9}}{h^{9}}+\\ \frac{112288704(x-x_k)^{8}}{h^{8}}+\frac{7555826(x-x_k)^{7}}{h^{7}}-\frac{1961835861(x-x_k)^{6}}{h^{6}}+\\ \frac{1836267020(x-x_k)^{5}}{h^{5}}+\frac{12514934790(x-x_k)^{4}}{h^{4}}-\frac{23345146880(x-x_k)^{3}}{h^{3}}+\\ \frac{20(x-x_k)^{2}}{h^{2}}+\frac{13878484840(x-x_k)}{h}\end{array}\right\}+$$

$$\frac{Y_{k+1}}{402585092096}\left\{\begin{array}{l}\frac{20928(x-x_k)^{12}}{h^{12}}+\frac{365423(x-x_k)^{11}}{h^{11}}-\frac{1755090(x-x_k)^{10}}{h^{10}}-\frac{44125584(x-x_k)^{9}}{h^{9}}+\\ \frac{41930988(x-x_k)^{8}}{h^{8}}+\frac{1985049293(x-x_k)^{7}}{h^{7}}-\frac{297235077(x-x_k)^{6}}{h^{6}}-\\ \frac{40931225860(x-x_k)^{5}}{h^{5}}+\frac{625012606(x-x_k)^{4}}{h^{4}}+\frac{355597552600(x-x_k)^{3}}{h^{3}}-\\ \frac{80(x-x_k)^{2}}{h^{2}}-\frac{883611795000(x-x_k)}{h}-120\end{array}\right\}+$$

$$\frac{Y_{k}}{142376960}\left\{\begin{array}{l}\frac{13(x-x_k)^{12}}{h^{12}}-\frac{1485(x-x_k)^{9}}{h^{9}}+\frac{62993(x-x_k)^{7}}{h^{7}}-\frac{1235930(x-x_k)^{5}}{h^{5}}+\frac{10503682(x-x_k)^{3}}{h^{3}}\\ -\frac{25904469(x-x_k)}{h}\end{array}\right\}+$$

$$\frac{h^2 f_{k+6}}{32429178880}\left\{\begin{array}{l}\frac{(x-x_k)^{12}}{h^{12}}+\frac{18(x-x_k)^{11}}{h^{11}}-\frac{67(x-x_k)^{10}}{h^{10}}-\frac{1309(x-x_k)^{9}}{h^{9}}+\frac{1018(x-x_k)^{8}}{h^{8}}+\frac{42645(x-x_k)^{7}}{h^{7}}-\\ \frac{5964(x-x_k)^{6}}{h^{6}}-\frac{733134(x-x_k)^{5}}{h^{5}}+\frac{11454(x-x_k)^{4}}{h^{4}}+\frac{5921678(x-x_k)^{3}}{h^{3}}-\frac{14368768(x-x_k)}{h}\end{array}\right\}+$$

$$\frac{h^2 f_{k+5}}{14090478223360}\left\{\begin{array}{l}\frac{-45212(x-x_k)^{12}}{h^{12}}-\frac{479317(x-x_k)^{11}}{h^{11}}+\frac{3446744(x-x_k)^{10}}{h^{10}}+\frac{47708063(x-x_k)^{9}}{h^{9}}-\\ \frac{68375938(x-x_k)^{8}}{h^{8}}-\frac{167106224(x-x_k)^{7}}{h^{7}}+\frac{488360427(x-x_k)^{6}}{h^{6}}+\frac{28652814730(x-x_k)^{5}}{h^{5}}\\ -\frac{282245554200(x-x_k)^{4}}{h^{4}}-\frac{229268755100(x-x_k)^{3}}{h^{3}}+\frac{7400699398(x-x_k)^{2}}{h^{2}}+\\ \frac{553855525900(x-x_k)}{h}-620\end{array}\right\}+$$

$$\frac{h^2 f_{k+4}}{2818095644672}\left\{\begin{array}{l}\frac{-190045(x-x_k)^{12}}{h^{12}}-\frac{1914921(x-x_k)^{11}}{h^{11}}+\frac{16962996(x-x_k)^{10}}{h^{10}}+\frac{208132467(x-x_k)^{9}}{h^{9}}\\ -\frac{468962222(x-x_k)^{8}}{h^{8}}-\frac{8174880908(x-x_k)^{7}}{h^{7}}+\frac{4713525208(x-x_k)^{6}}{h^{6}}+\\ \frac{148106532100(x-x_k)^{5}}{h^{5}}-\frac{12387685370(x-x_k)^{4}}{h^{4}}-\frac{1180782307000(x-x_k)^{3}}{h^{3}}+\\ \frac{584243129600(x-x_k)^{2}}{h^{2}}+\frac{2825232049000(x-x_k)}{h}+109053993400000\end{array}\right\}+$$

$$\frac{h^2 f_{k+3}}{14090478223360}\left\{\begin{array}{l}\frac{-45212(x-x_k)^{12}}{h^{12}}-\frac{479317(x-x_k)^{11}}{h^{11}}+\frac{3446744(x-x_k)^{10}}{h^{10}}+\frac{47708063(x-x_k)^{9}}{h^{9}}-\\ \frac{68375938(x-x_k)^{8}}{h^{8}}-\frac{167106224(x-x_k)^{7}}{h^{7}}+\frac{488360427(x-x_k)^{6}}{h^{6}}+\\ \frac{28652814730(x-x_k)^{5}}{h^{5}}-\frac{282245554200(x-x_k)^{4}}{h^{4}}-\frac{229268755100(x-x_k)^{3}}{h^{3}}+\\ \frac{7400699398(x-x_k)^{2}}{h^{2}}+\frac{553855525900(x-x_k)}{h}-620\end{array}\right\}+$$

$$\frac{h^2 f_{k+2}}{1761309777920}\left\{\begin{array}{l}\frac{-118778(x-x_k)^{12}}{h^{12}}-\frac{1999206(x-x_k)^{11}}{h^{11}}+\frac{10601872(x-x_k)^{10}}{h^{10}}+\frac{237357812(x-x_k)^{9}}{h^{9}}\\ -\frac{293101389(x-x_k)^{8}}{h^{8}}-\frac{10478064110(x-x_k)^{7}}{h^{7}}+\frac{2945953254(x-x_k)^{6}}{h^{6}}+\\ \frac{213245612800(x-x_k)^{5}}{h^{5}}-\frac{7742303350(x-x_k)^{4}}{h^{4}}-\frac{1860986492000(x-x_k)^{3}}{h^{3}}+\\ \frac{500(x-x_k)^{2}}{h^{2}}+\frac{3475189588000(x-x_k)}{h}-1500\end{array}\right\}+$$

$$\frac{h^2 f_{k+1}}{14090478223360}\left\{\begin{array}{l}\frac{-45212(x-x_k)^{12}}{h^{12}}-\frac{2370668(x-x_k)^{11}}{h^{11}}+\frac{3446744(x-x_k)^{10}}{h^{10}}+\frac{279948277(x-x_k)^{9}}{h^{9}}\\ -\frac{67741447(x-x_k)^{8}}{h^{8}}-\frac{12228038800(x-x_k)^{7}}{h^{7}}+\frac{442677097(x-x_k)^{6}}{h^{6}}+\\ \frac{244047871300(x-x_k)^{5}}{h^{5}}-\frac{895862520(x-x_k)^{4}}{h^{4}}-\frac{2088305215000(x-x_k)^{3}}{h^{3}}-\\ \frac{60(x-x_k)^{2}}{h^{2}}-\frac{51742265100(x-x_k)}{h}-620\end{array}\right\}+$$

$$\frac{h^2 f_{k}}{14090478223360}\left\{\begin{array}{l}\frac{636(x-x_k)^{12}}{h^{12}}-\frac{55600(x-x_k)^{11}}{h^{11}}-\frac{28902(x-x_k)^{10}}{h^{10}}+\frac{608400(x-x_k)^{9}}{h^{9}}+\\ \frac{442225(x-x_k)^{8}}{h^{8}}-\frac{252486638(x-x_k)^{7}}{h^{7}}-\frac{2591294(x-x_k)^{6}}{h^{6}}+\frac{4908793316(x-x_k)^{5}}{h^{5}}\\ -\frac{2002595888(x-x_k)^{4}}{h^{4}}-\frac{41583636710(x-x_k)^{3}}{h^{3}}-\frac{21949901700(x-x_k)^{2}}{h^{2}}+\\ \frac{102452428800(x-x_k)}{h}+4989901687\end{array}\right\}$$

Evaluating the continuous scheme at $x = x_{k+6}$ gives the discrete scheme

$$\begin{array}{l}49483Y_{k+6}+785862Y_{k+5}+790965Y_{k+4}-3252620Y_{k+3}+790965Y_{k+2}+785862Y_{k+1}+1\,49438Y_k\\ =h^2\{1857f_{k+6}+110322f_{k+5}+989739f_{k+4}+2175924f_{k+3}+989739f_{k+2}+110322f_{k+1}+1857f_k\}\end{array} \quad (10)$$

Equation is the continuous formulation of the discrete scheme (10) which is of order 8 with error constant -0.00048.

A predictor scheme was developed for our method (10) by collocating (9) at $x = x_{k+1}, x_{k+3}, x_{k+5}\ x_{k+6}$ and interpolating (8) at $x = x_{k+i}$, $i = 0(1)5$ and after simplifications by following the steps in developing scheme (10) yields the discrete scheme

$$172Y_{k+6} + 1323Y_{k+5} - 15660Y_{k+4} + 28330Y_{k+3} - 15660Y_{k+2}$$
$$1323Y_{k+1} + 172Y_k = h^2\ \{405f_{k+5} - 9630f_{k+3} + 405f_{k+1}\} \quad (11)$$

RESULTS

Here, we shall present some numerical examples to justify the desirability of our scheme over and above the one in Kayode and Awouemi (2005) which is

$$539Y_{k+6}-4374Y_{k+5}+32805Y_{k+4}-57940Y_{k+3}$$
$$+32805Y_{k+2}-4374Y_{k+1}+539Y_k \quad (12)$$
$$h^2\{60f_{k+6} + 20040f_{k+3} + 60f_k\}$$

Numerical Example I

$$\text{Solve } y''(x) = \frac{(y')^2}{2y} - 2y;\quad y\left(\frac{\pi}{6}\right) = \frac{1}{4},$$
$$y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\quad h = 0.01,\quad 0 \le x \le \pi$$

Theoretical solution is$y(x) = \sin^2 x$

Numerical Example II

$$\text{Solve } y''(x) - x(y')^2 = 0;\ y(0) = 1,$$
$$y'(0) = \frac{1}{2},\qquad h = 0.01,\quad 0 \le x \le 1$$

Theoretical solution is

$$y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$$

DISCUSSION

From the result generated when methods (12) and the new method (10) were used for both problems I and II (Table 1 and 2), it was evident that the result generated when the new method was used gave a smaller error than when method (12) was used. This is as a result of the smaller error constant of method (10) as was earlier calculated. Also the new method has an advantage over the method (12) because the new method gives a better approximate solution than method (12), though it performs more evaluation per step than method (12) but this is compensated in the output of the result. In the light of this, method (10) is more desirable when compare with method (12).

Table 1: Comparison of Error to Problem I when using methods (10) and (12)

X	Method (12)	Method (10)
0.6	0.1123819171D-07	0.1012984672D-07
0.7	0.5017957466D-07	0.4782108924D-07
0.8	0.1350984477D-06	0.1109216482D-06
0.9	0.2239809656D-06	0.1892136453D-06
1.0	0.2946323541D-06	0.1985672451D-06
1.1	0.3248555553D-06	0.3019832782D-06
1.2	0.2948865586D-06	0.2561289342D-06
1.3	0.1895115423D-06	0.1435789122D-06
1.4	0.2343509831D-06	0.1019836127D-06
1.5	0.2758743144D-06	0.2319814651D-06
1.6	0.6308693972D-06	0.5892317378D-06
1.7	0.1050608770D-05	0.1012918372D-05
1.8	0.1513018120D-05	0.1210932713D-05
1.9	0.1989761348D-05	0.1287329122D-05
2.0	0.2447979590D-05	0.1987214562D-05

Table 2: Comparison of Error to Problem II when using methods (10) and (12)

X	Method (12)	Method (10)
0.1	0.1708719055D-09	0.1329867326D-09
0.2	0.6836010114D-08	0.5872691257D-08
0.3	0.1555757709D-07	0.1327845616D-07
0.4	0.2880198295D-07	0.2317829012D-07
0.5	0.4802328029D-07	0.3218793564D-07
0.6	0.7628531256D-07	0.6871246012D-07
0.7	0.1157914170D-06	0.1012728156D-06
0.8	0.1727046080D-06	0.1231093271D-06
0.9	0.2561456831D-06	0.2019286712D-06
1.0	0.3815695118D-6	0.2990871645D-06

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