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Accurate Collocation Multistep Method for Integration of First Order Ordinary Differential Equations

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Abstract: In this study we present a collocation multistep method for Integration of first order ordinary differential equations. It is consistent (order seven) zero stable and convergent. When compared with existing multistep method, it is found to be more accurate.

Key words: Collocation, multistep, integration, consistent, zero stable and convergent

INTRODUCTION

The solution of initial value problems of ordinary differential equations of the form

$$y' = f(x,y), \; y(x_0) = y_0 \qquad (1)$$

Where, y, f $\in R^n$, x^o [a,b], has been discussed by various researchers among them are Lie and Norsett, (1989), Onumanyi *et al.* (1994, 1999), Onumanyi and Yusuph (2002), Sirisena (2004), Lambert, (1973) and Gear, (1971). However experience has shown in Lie and Norsett (1989) and Onumanyi *et al.* (1994) that the traditional multistep methods including the hybrid ones can be made continuous through the idea of multistep collocation. These earlier works have focused on the construction of continuous multistep methods by employing the multistep collocation. The continous multistep methods produce piecewise polynomial solutions over k –steps [x_n, x_{n+k}] for the first order systems of Ordinary Differential Equation (ODEs). Sirisena *et al.* (2004) developed a continuous new Butcher type two-step block hybrid multistep method for problem (1). The results obtained showed a class of discrete schemes of order 5 and error constants ranging from $C_6 = 1.45\times10^{-5}$ to $C_6 = 1.790 \times 10^{-4}$. In this study, we propose a continuous Butcher type three- step block hybrid method employing multistep collocation approach, which yields a class of 2 discrete schemes of order 7 with error constants.

$C_8 = -\dfrac{27}{777420}$ and $C_8 = \dfrac{155525}{1273724928}$ for solving problem (1.1).

DEVELOPMENT OF THE METHODS

In this study we discussed the development of continuous scheme and its discrete schemes using Sirisena (1997) where a K-step multistep collocation method with m collocation points was obtained as follows:

$$\bar{y}(x) = \sum_{j=0}^{t-1} \alpha_j(x) y(x_{n+j}) + h_j \sum_{j=0}^{m-1} \beta_j(x) f(\bar{x}_j, \bar{y}(\bar{x}_j)) \qquad (2)$$

Where:

$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^i \qquad (3)$$

$$h\beta_j(x) = \sum_{i=0}^{t+m-1} h\beta_{j,i+1}\, x^i \qquad (4)$$

are the continuous coefficients of the method and x_{n+j}, j = 0, 1...t-1 in (2) are t ($0<t\le k$) arbitrary chosen nterpolation points from (x_n, … x_{n+k}) and , $\bar{x}_j$ j = o, 1…, m-2 are the m collocation points belonging to $\{x_n\ldots, x_{n+k}\}$.

To determine α_j (x) and β_j (x), we use a matrix equation of the form

$$DC = I \qquad (5)$$

Where,
I is an identity matrix
While D ad C are the matrices defined as in Sirisena (1997).

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$$D = \begin{bmatrix} 1 & x_n & x_n^2 \cdots & x_n^{t+m-2} \\ 1 & x_{n+1} & x_{n+1}^2 \cdots & x_{n+1}^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+t-1} & x_{n+T-1}^2 \cdots & x_{n+t-1}^{t+m-2} \\ 0 & 1 & 2\overline{x}_o \cdots & (t+m\text{-}2)\overline{x}^{t+m-3} \\ \vdots & \vdots & \vdots & \vdots \\ . & . & . & . \\ .0 & 1 & 2\overline{x}_{m-1} \cdots & (t+m)\overline{x}_{m-1}^{t+m-3} \end{bmatrix} \quad (6)$$

and

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} \cdots & \alpha_{t-1,1} & h\beta_{o,1} \cdots & h\beta_{n-1,1} \\ \alpha_{o,2} & \alpha_{1,2} \cdots & \alpha_{t-1,2} & h\beta_{0,2} \cdots & h\beta_{m-1,2} \\ . & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{o,t+m} & \alpha_{1,t+m} \cdots & \alpha_{t-1,t+m} & h\beta_{o,t+m} \cdots & h\beta_{m+1,t+m} \end{bmatrix} \quad (7)$$

The columns of the matrix $C = D^{-1}$ consists of the continuous coefficients, i.e.,

$$\alpha_j(x);\ j = o, 1 \ldots k\text{-}1 \text{ and } \beta_j\ (x);\ j = 0, 1 \ldots.\ k\text{-}1.$$

In this study

$$k = t = 3,\ m = 6,\ \overline{x}_o = x_n,\ \overline{x}_n,\ \overline{x}_1, = x_{p+1,}\ \overline{x}_2 = x_{n+2.}$$

Then Eq. 2 becomes

$$\overline{y}(x) = \alpha_o(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y + h\left[\beta_o(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_{5/2}(x)\ f_{n+5/2}\right] \quad (8)$$

Thus, the matrix D in (6) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \\ 0 & 1 & 2x_{n+5/2} & 3x_{n+5/2}^2 & 4x_{n+5/2}^3 & 5x_{n+5/2}^4 & 6x_{n+5/2}^5 & 7x_{n+5/2}^6 \end{bmatrix} \quad (9)$$

We obtained $C = D^{-1}$ in (9) to determine $\alpha_i(x)$; $i = 0(1)2$ and $h\beta_i\ (x)$; $i = 0, 1, 2, 3, 5/2$ in (8) as follows:

$$\alpha_o(x) = \frac{1}{2468h^7}\left[-20115h^5(x-x_n)^2 + 39189h^4(x-x_n)^3 - 32910h^3(x-x_n)^4 + 14169h^2 (x\text{-}x_n)^5 - 3065h(x-x_n)^6 + 264(x-x_n)^7 + 2468h^2\right] \quad (10)$$

$$\alpha_1(x) = \frac{1}{617h^7}\left[-180h^5(x\text{-}x_n)^2 + 6148h^4(x-x_n)^3 - 10005h^3(x\text{-}x_n)^4 + 6156h^2(x\text{-}x_n)^5 - 1670h(x\text{-}x_n)^6 + 168(x-x_n)^7\right] \quad (11)$$

$$\alpha_2(x) = \frac{1}{2468h^7}\Big[20835h^5(x\text{-}x_n)^2 - 6378h^4(x-x_n)^3 + 72930h^3(x\text{-}x_n)^4 - 38793h^2(x\text{-}x_n)^5 + 9745h(x\text{-}x_n)^6 - 936(x-x_n)^7\Big] \quad (12)$$

$$h\beta_o(x) = \frac{1}{37020h^6}\Big[37020h^6(x\text{-}x_n) - 136364h^5(x-x_n)^2 + 200531h^4(x\text{-}x_n)^3 - 150680h^3(x\text{-}x_n)^4 + 61199h^2(x\text{-}x_n)^5 - 12782h(x-x_n)^6 + 1076(x\text{-}x_n)^7\Big] \quad (13)$$

$$h\beta_1(x) = \frac{1}{7404h^6}\Big[-68220h^5(x\text{-}x_n)^2 + 182932h^4(x-x_n)^3 + 1861147h^3(x\text{-}x_n)^4 - 90946h^2(x\text{-}x_n)^5 - 21483h(x\text{-}x_n)^6 - 1972(x-x_n)^7\Big] \quad (14)$$

$$h\beta_2(x) = \frac{1}{2468h^6}\Big[-11250h^5(x\text{-}x_n)^2 + 35645h^4(x-x_n)^3 - 42556h^3(x\text{-}x_n)^4 + 23805h^2(x\text{-}x_n)^5 - 6272h(x\text{-}x_n)^6 + 628(x-x_n)^7\Big] \quad (15)$$

$$h\beta_3(x) = \frac{1}{7404h^6}\Big[-980h^5(x\text{-}x_n)^2 + 3308h^4(x-x_n)^3 - 4289h^3(x\text{-}x_n)^4 + 2666^2(x\text{-}x_n)^5 - 797h\,(x\text{-}x_n)^6 + 92(x-x_n)^7\Big] \quad (16)$$

$$h\beta_{5/2}(x) = \frac{1}{9255h^6}\Big[9216h^5(x\text{-}x_n)^2 - 30464h^4(x-x_n)^3 + 38400h^3(x\text{-}x_n)^4 - 22976h^2(x\text{-}x_n)^5 + 6528h(x\text{-}x_n)^6 - 704(x-x_n)^7\Big] \quad (17)$$

Putting Eq. 10-17 into Eq. 8, we obtained a continuous scheme.

$$\bar{y}(x) = \frac{y_n}{2468h^7}\Big[-20115h^5(x\text{-}x_n)^2 + 39189h^4(x-x_n)^3 - 32910h^3(x\text{-}x_n)^4 + 14169h^2(x\text{-}x_n)^5 - 3065h\,(x\text{-}x_n)^6 + 264(x-x_n)^7 + 2468h^7\Big]$$

$$+ \frac{y_{n+1}}{617h^7}\Big[-100h^5(x\text{-}x_n)^2 + 6148h^4(x-x_n)^3 - 10005h^3(x\text{-}x_n)^4 + 6156h^2(x\text{-}x_n)^5 - 1670h\,(x\text{-}x_n)^6 + 168(x-x_n)^7\Big]$$

$$+ \frac{y_{n+2}}{2468h^7}\Big[20835h^5(x\text{-}x_n)^2 - 63781h^4(x-x_n)^3 + 72930h^3(x\text{-}x_n)^4 - 38793h^2(x\text{-}x_n)^5 + 9745h\;\;(x\text{-}x_n)^6 + 936(x-x_n)^7 + \frac{f_n}{37020h^6}\Big(37020h^6(x-x_n)\Big)\Big]$$

$$-136364h^5(x\text{-}x_n)^2 + 200531h^4(x\text{-}x_n)^3 - 150680h^3(x\text{-}x_n)^{-4} + 61199h^2(x-x_n)^5$$

$$-12782h^5(x\text{-}x_n)^6 + 1076(x\text{-}x_n)^7 + \frac{f_{n+1}}{7404h^6}\Big[-62280h^5(x\text{-}x_n)^2 + 182932h^4(x-x_n)^3$$

$$-186147h^3(x\text{-}x_n)^4 + 90946(x\text{-}x_n)^5 - 21483h(x\text{-}x_n)^6 + 1972(x-x_n)^7\Big]$$

$$+\frac{f_{n+2}}{2468h^6}\left[-11250h^5(x-x_n)^2+35645h^4(x-x_n)^3-42556h^3(x-x_n)^4+23805h^2(x-x_n)^5\right.$$
$$\left.-6272h(x-x_n)^6+628(x-x_n)^7\right]+\frac{f_{n+3}}{7404h^6}\left[-980h^5(x-x_n)^2\right.$$
$$\left.+3308h^4(x-x_n)^3+4289h^3(x-x_n)^4+2666h^2(x-x_n)^5-797h(x-x_n)^6+92(x-x_n)^7\right]$$
$$+\frac{f_{n+5/2}}{9255h^6}\left[921h^5(x-x_n)^2-30464h^4(x-x_n)^3+38400h^3(x-x_n)^4-22976h^2(x-x_n)^5\right.$$
$$\left.+6528h(x-x_n)^6-704(x-x_n)^7\right] \quad (18)$$

On evaluating (18) at $x = x_{n+3}$ and $x = x_{n+5/2}$, we obtained the following 2 discrete equations.

$$y_{n+3}-\frac{783}{617}y_{n+2}+\frac{135}{617}y_{n+1}+\frac{31}{617}y_n=\frac{h}{18510}\left[-234f_n-2970f_{n+1}-810f_{n+2}+2790f_{n+3}+13824f_{n+5/2}\right] \quad (19)$$

and

$$y_{n+5/2}-\frac{4077}{157952}y_n-\frac{29000}{157952}y_{n+1}+\frac{124875}{157952}y_{n+2}=\frac{h}{157952}\left[-990f_{n+1}+16125f_{n+1}+67500f_{n+2}-1125f_{n+3}+32640f_{n+5/2}\right] \quad (20)$$

The schemes (18) and (19) has order p = 7, error constants.

$$C_8=\frac{15525}{1273724928} \text{ and } C_8=\frac{-27}{777420}, \text{ respectively}$$

Since the order p > 1, then the Eq. (19) and (20) are consistent as in Lambert (1973). Equation 19 and 20 are two equations with four unknowns. For the two equations to constitute two member block hybrid method, we need to eliminate two unknowns either by using existing standard one-step method or using the analytical solution or develop two more equations. What we adopted is discussed in the next section.

STARTING VALUES

We adopted the explicit sixth order Runge-Kutta scheme in (Lambert, 1973) to evaluate y_{n+j}; j=1 and 2; n = o i.e.

$$y_{n+j}=y_{n+j-1}+\frac{h}{840}\left[41k_1+216k_3+27k_4+272k_5+27k_6+216k_7+41k_8\right] \quad (21)$$

Where,

$$k_1\ f(x_n, y_{n+j-1})$$
$$k_2=f(x_n+h/9,\ y_{n+j-1}+h/9k_1)$$
$$k_3=f\left\{x_n+h/6,\ y_{n+j-1}+h/24_1(k_1+3k_2)\right\}$$
$$k_4=f\left\{x_n+h/3,\ y_{n+j-1}+h/6(k_1+3k_2+4k_3)\right\}$$
$$k_5=f\left\{x_n+h/2,\ y_{n+j-1}+h/8(-5k_1+27k_2-24k_3-6k_4)\right\}$$
$$k_6=f\left\{x_n+2h/2,\ y_{3+j-1}+h/9(22k_1-981k_2+867k_3-102k_4+k_5)\right\}$$
$$k_7=f\left\{x_n+5h/6,\ y_{3+j-1}+h/48(-183k_1+678k_2-472k_3-66k_4+80k_5+3k_6)\right\}$$
$$k_8=f\left\{x_n+h, y_{n+j-1}+h/82(716k_1-2079k_2+1002k_3+834k_4-454k_5-9k_6+72k_7)\right\} \quad (22)$$

Table 1: Comparison of Errors

X	Sirisena *et al.* (2004)	Proposed scheme
0.1	2.0×10^{-9}	2.1×10^{-10}
0.2	2.0×10^{-9}	2.2×10^{-10}
0.3	1.0×10^{-9}	6.0×10^{-10}
0.4	2.0×10^{-9}	1.0×10^{-10}
0.5	1.0×10^{-9}	4.1×10^{-9}
0.6	3.0×10^{-9}	7.0×10^{-10}
0.7	2.0×10^{-9}	1.5×10^{-9}
0.8	3.0×10-9	$7.0\text{x}10^{-10}$
0.9	3.0×10^{-9}	1.4×10^{-9}
1.0	3.0×10^{-9}	8.0×10^{-10}

Table 2: Comparison of Errors

X	Sirisena *et al.* (2004)	Our new method
0.1	2.0×10^{-9}	0.0
0.2	2.1×10^{-9}	0.0
0.3	1.7×10^{-9}	6.0×10^{-10}
0.4	0.0	2.0×10^{-11}
0.5	6.7×10^{-9}	7.0×10^{-10}
0.6	0.0	1.0×10^{-10}
0.7	1.0×10^{-9}	8.0×10^{-10}
0.8	0.0	$2.0\text{x}10^{-10}$
0.9	0.0	9×10^{-10}
1.0	0.0	$4.0\text{x}10^{-10}$

Table 3: Comparison of Errors.

X	Sirisena *et al.* (2004)	Our method
0.1	3.6×10^{-4}	1.7×10^{-5}
0.2	1.5×10^{-4}	1.6×10^{-5}
0.3	5.9×10^{-5}	9.3×10^{-6}
0.4	1.6×10^{-5}	4.6×10^{-6}
0.5	4.3×10^{-5}	1.8×10^{-6}
0.6	2.1×10^{-5}	4.2×10^{-7}
0.7	5.7×10^{-7}	1.8×10^{-7}
0.8	1.6×10^{-6}	2.3×10^{-6}
0.9	5.1×10^{-6}	$3.8\text{x}10^{-7}$
1.0	2.8×10^{-6}	3.2×10^{-7}

CONVERGENCE AND STABILITY ANALYSIS

In this study, we discuss the stability and convergence properties of the schemes (19) and (20).

Zero stability of (19):

$$P|\xi| = \xi^3 - \frac{783}{617}\xi + \frac{135}{617} + \frac{31}{617} = 0$$

$$\xi_1 = 1,\ \xi_2 = 0.4285 \text{ and } \xi_3 = 0.1587$$

Since $\xi_1 \# \xi_2 \# \xi_3$ and

$|\xi_1| \le |\xi_2| \le |\xi_3| \le 1$, then the method is zero stable according to Lambert (1973), Zero stability of (20)

$$\ell(\xi) = \xi^{5/2} - \frac{124875}{15795}\xi^2 - \frac{29000}{157952}\xi^3 - \frac{4077}{157952} = 0$$

$$\xi_1 = 1,\ \xi_2 = 0.0258 \text{ and } \xi_3 = 0.2094$$

Since, $\xi_1 \# \xi_2 \# \xi_3$

The method is zero stable according to Lambert (1973). According to Lambert (1973) the necessary and sufficient condition for a linear multistep method to be convergent are that it be consistent and zero stable. Therefore, proposal schemes are convergent.

NUMERICAL EXPERIMENT

In this study, we use the proposal schemes (19) and (20) with Eq. 21 and 22 as starting values to solve the examples stated below. The errors arising from the computed and theoretical values are compared with Sirisena *et al.* (2004) as shown in Table 1-3.

Example 1

$$y' = -y,\ y(o) = 1,\ 0 \le x \le 1,\ h = 0.1$$
$$y(x) = e\text{-}x$$

Example 2

$$y' = x\text{-}y,\ y(o) = 0,\ 0 \le x \le 1,\ h = 0.1$$
$$y(x) = x + e\text{-}x\ \text{-}1$$

Example 3

$$y' = 8(y\text{-}x) + 1,\ y(o) = 2,\ 0 \le x \le 1,\ h = 0.1$$
$$y(x) = x + 2e^{-8x}$$

From the above presented tables, our new method is more accurate when compared with Sirisena *et al.* (2004). The proposed method uses two difference equations per step while Sirisena *et al.* (2004) used three difference equations per step.

CONCLUSION

Our new proposed scheme are consistent and convergent. And it compares favourably with the existing scheme.

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