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Pricing European Currency Options in a Fractional Brownian Motion with Jumps

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Abstract: Assuming that the interest is given, the principle of fair insurance premium-actuarial approach is used to deal with pricing formula of option on foreign currency option under the assumption that foreign currency option price process in a fractional Brownian motion with jumps and the pricing formulas of European foreign currency option are obtained. It has certain reference significance to avoiding foreign exchange risk.

Key words: Actuarial approach, fractional Brownian motion, jump-diffusion process, foreign currency option, China

INTRODUCTION

A currency option is a contract which gives the owner the right but not the obligation to buy or sell the indicated amount of foreign currency at a specified pricing on affixed date. Since, the currency option can be used as a tool for investment and hedging, it is one of the best ways for corporations or individuals to hedge against adverse movements in exchange rates and the theoretical models for pricing currency options have been carried out. The standard European currency option valuation model has been presented by Garman and Kohlhagen (1983) (Black and Scholes, 1973). However, some studies have provided evidence of the mispricing for currency options by the Garman-Kohlhagen (Hereafter G-K) Model. The most important reason why this model may not be entirely satisfactory could be that currencies are different from stocks in important respects and the geometric Brownian motion cannot capture the behavior of currency return. The empirical research on asset return indicates that discontinuities or jumps are believed to be an essential component of financial asset prices. And there is strong evidence that the stock return has little or no autocorrelation. Since, fractional Brownian motion has two important properties called self-similarity and long-range dependence, it has the ability to capture the typical tail behavior of stock prices or indexes (Aase, 1988; Peters, 1989). To capture the behaviors of spot exchange rate, the combination of Poisson jumps and fractional Brownian motion is introduced in this study.

PRICING MODEL FOR CURRENCY OPTION IN A JUMP FRACTIONAL ENVIRONMENT

Definition: Consider a probability space (Ω, F^H, P_H) on which all the random variables and processes are defined (Hu and Oksendal, 2003). A fractional Brownian motion $\{B_H(t)\}_{t\in R^+}$ with Hurst parameter $H\in(0, 1)$ is a centered Gaussian process with mean zero and covariance:

$$E_{P_H}\left[B_H(t)B_H(s)\right] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\};\ s,t \in R^+$$

Where the parameter H is the self-similarity index and $F_t^h = \sigma\{B_H(s), 0 \le s \le t\}$ with $F_T^H = F^H$. To derive the currency option pricing formula in a jump fractional market, researchers make the following assumptions (Xiao *et al*., 2010):

- There are no transaction costs or taxes and all securities are perfectly divisible
- Security trading is continuous
- The short-term domestic interest rate r_d and foreign interest rate r_f are known and constant though time
- There are no riskless arbitrage opportunities

The spot exchange rate follows a fractional Brownian motion with random jumps under the probability measure P_H. Thus:

$$\begin{cases} dS(t) = S(t)\begin{bmatrix}\left(\mu - \lambda\mu_{J(t)}\right)dt + \\ \sigma dB_H(t) + \left(e^{J(t)} - 1\right)dN_t\end{bmatrix};0 \le t \le T \\ S(0) = S \end{cases} \quad (1)$$

Let B_t^d and B_t^f denote the domestic and foreign price of risk free bond, respectively. Then, B_t^d and B_t^f satisfy Eq. 2 and 3:

$$dB_t^d = B_t^d r^d dt, B_T^d = 1B_t^d = e^{-r^d(T-t)} \quad (2)$$

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$$dB_t^f = B_t^f r^f dt, B_T^f = 1 B_t^f = e^{-r^f(T-t)} \quad (3)$$

where, S(t) denotes the spot exchange rate at time t of one unit of the foreign currency measured in the domestic currency; the drift $\mu(t)$ and volatility σ are assumed to be constants; B_H (t) is a fractional Brownian motion; N_t is a Poisson process with rate λ; $e^{J(t)}$-1 is jump size at t which is a sequence of independent identically distributed and $J(t) \sim N(-\sigma_J^2/2, \sigma_J^2)$. In addition, all three sources of randomness, the fractional Brownian motion $\{B_H(t): t \geq 0\}$, the Poisson process $\{N_t: t \geq 0\}$ and the jump size $e^{J(t)}$-1 are assumed to be independent.

From the fractional Gisanov formula, taking variable transformation for Eq. 1:

$$\hat{B}_H(t) = \frac{\mu - \lambda\mu_{J(t)} + r_f - r_d}{\sigma} t + B_H(t) \quad (4)$$

Substituting Eq. 4 into Eq. 1, then Eq. 1 can be transformed into the following standard form:

$$\begin{cases} dS(t) = S(t)\begin{bmatrix} (r_d - r_f)dt + \\ \sigma d\hat{B}_H(t) + \left(e^{J(t)} - 1\right)dN_t \end{bmatrix}; 0 \leq t \leq T \\ S(0) = S \end{cases} \quad (5)$$

It is clear that $\hat{B}_H(t)$ is a new fractional Brownian motion with random jumps under the probability measure $\hat{P}_H$.

Lemma: Using Ito formula, the solution for stochastic differential Eq. 5 is (Li *et al*., 2005):

$$S(t) = S\exp\left\{(r_d - r_f)t - \frac{1}{2}\sigma^2 t^{2H} + \sigma\hat{B}_H(t) + \sum_{i=1}^{N_t} J(t_i)\right\} \quad (6)$$

And the mean:

$$\begin{aligned} E(S(t)) &= E\left[S\exp\left\{(r_d - r_f)t - \frac{1}{2}\sigma^2 t^{2H} + \sigma\hat{B}_H(t) + \sum_{i=1}^{N_t} J(t_i)\right\}\right] \\ &= S\exp\left\{(r_d - r_f)t - \frac{1}{2}\sigma^2 t^{2H}\right\} E\left[\exp\left\{\sigma\hat{B}_H(t)\right\}\right] \\ &\quad E\left[\exp\left\{\sum_{i=1}^{N_t} J(t_i)\right\}\right] \\ &= S\exp\left\{(r_d - r_f)t - \frac{1}{2}\sigma^2 t^{2H}\right\}\exp\left\{\frac{1}{2}\sigma^2 t^{2H}\right\}\exp\left\{\frac{1}{2}n\sigma_J^2 t\right\} \\ &= S\exp\left\{\left[(r_d - r_f) + \frac{1}{2}n\sigma_J^2\right]t\right\} \end{aligned} \quad (7)$$

ACTUARIAL APPROACH TO PRICING CURRENCY OPTIONS

Bladt and Rydberg (1998) proposed the actuarial approach to option pricing which transform option pricing into a problem of equivalent of the fair insurance premium. This is no economic assumptions in the actuarial approach; so, it is valid not only to the arbitrage-free, equilibrium and complete market but also to the arbitrage, non-equilibrium and incomplete market.

Definition 1: The expectation return rate $\beta(t)$ of S(t) on $t \in [0, T]$ is defined to $\int_0^T \beta(s)dt$ as follows (Bladt and Rydberg, 1998):

$$\frac{E(S(t))}{S(0)} = \exp\left(\int_0^t \beta(s)ds\right)$$

Definition 2: Let C (k,T) denotes the European call option and P(k, T) denotes the European putt option whose spot exchange rate is S(t), the strike price is k and the expiration date is T. Then, the value of European option is defined by actuarial approach as follows:

$$C(k,T) = E\left(\left(\exp\left(-\int_0^T \beta(t)dt\right)S(T)B_0^f - kB_0^d\right)I_A\right)$$

$$P(k,T) = E\left(\left(kB_0^d - \exp\left(-\int_0^T \beta(t)dt\right)S(T)B_0^f\right)I_B\right)$$

The necessary and sufficient condition to execute of European call option and the put option on the expiration date is, respectively; Condition A:

$$\exp\left(-\int_0^T \beta(t)dt\right)S(T)B_0^f > kB_0^d$$

Condition B:

$$kB_0^d > \exp\left(-\int_0^T \beta(t)dt\right)S(T)B_0^f$$

Theorem: Suppose the spot exchange rate $(S(t): t \geq 0)$ satisfy Eq. 1, B_t^d and B_t^f satisfy Eq. 2 and 3 then, respectively the value of the European call and put option at the time 0 is:

$$\begin{aligned} C(k,T) &= E\left(\left(\exp\left(-\int_0^T \beta(t)dt\right)S(T)B_0^f - kB_0^d\right)I_A\right) \\ &= E\left(\left(\exp\left(-(r_d - r_f)T - \frac{N_T\sigma_J^2}{2}T\right)S(T)B_0^f - kB_0^d\right)I_A\right) \end{aligned}$$

$$C(k,T) = SB_0^f \sum_{n=0}^{\infty}\left[\frac{(\lambda T)^n}{n!}\exp\left(\sum_{i=0}^{n} J(t_i) - \lambda T - \frac{n\sigma_J^2}{2}T\right)\right]\phi(b_n) - kB_0^d \sum_{n=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\phi(b'_n) \quad (8)$$

$$\begin{aligned} P(k,T) &= E\left(\left(kB_0^d - \exp\left(-\int_0^T \beta(t)dt\right)S(T)B_0^f\right)I_B\right) \\ &= E\left(\left(kB_0^d - \exp\left(-(r_d - r_f)T - \frac{N_T\sigma_J^2}{2}T\right) S(T)B_0^f\right)I_B\right) \\ &= kB_0^d \sum_{n=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\phi(-b'_n) - SB_0^f \times \sum_{n=0}^{\infty}\left[\frac{(\lambda T)^n}{n!}\exp\left(\sum_{i=0}^{n} J(t_i) - \lambda T - \frac{n\sigma_J^2}{2}T\right)\right]\phi(-b_n) \end{aligned} \quad (9)$$

Where:

$$y_n = \frac{m - \sum_{i=0}^{n} J(t_i) + \frac{n\sigma_J^2}{2}}{\sigma};$$

$$m = \ln\frac{kB_0^d}{SB_0^f} + \frac{1}{2}\sigma^2 T^{2H};$$

$$b_n = \frac{\sigma T^{2H} - y_n}{T^H};\quad b'_n = \frac{-y_n}{T^H}$$

Proof: Using Ito formula, the solution for stochastic differential Eq. 5 is:

$$S(T) = S\exp\left\{(r_d - r_f)T - \frac{1}{2}\sigma^2 T^{2H} + \sigma\hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i)\right\}$$

The $\exp\left(-\int_0^T \beta(t)dt\right)S(T)B_0^f > kB_0^d$ takes equivalent forms:

$$\exp\left\{-(r_d - r_f)T - \frac{N_T\sigma_J^2}{2}T\right\} \times S\exp\left\{(r_d - r_f)T - \frac{1}{2}\sigma^2 T^{2H} + \sigma\hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i)\right\} \times B_0^f > kB_0^d$$

Then, researchers have:

$$\sigma\hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i) - \frac{N_T\sigma_J^2}{2}T > m$$

$$C_1(k,T) = E\left[\exp\left(-\int_0^T \beta(t)dt\right)S(T)B_0^f I_{\left\{\exp\left(-\int_0^T \beta(t)dt\right)S(T)B_0^f > kB_0^d\right\}}\right]$$

$$= E\left[\exp\left\{-(r_d - r_f)T - \frac{N_T\sigma_J^2}{2}T\right\} \times S\exp\left\{(r_d - r_f)T - \frac{1}{2}\sigma^2 T^{2H} + \sigma\hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i)\right\} \times B_0^f I_{\left\{\sigma\hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i) - \frac{N_T\sigma_J^2}{2}T > m\right\}}\right]$$

$$= SB_0^f \exp\left\{-\frac{1}{2}\sigma^2 T^{2H}\right\} E\left[\exp\left\{\sigma\hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i) - \frac{N_T\sigma_J^2}{2}T\right\} \times I_{\left\{\sigma\hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i) - \frac{N_T\sigma_J^2}{2}T > m\right\}}\right]$$

$$= SB_0^f \exp\left\{-\frac{1}{2}\sigma^2 T^{2H}\right\} E\left[E\left[\exp\left\{\sigma\hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i) - \frac{N_T\sigma_J^2}{2}T\right\} \times I_{\left\{\sigma\hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i) - \frac{N_T\sigma_J^2}{2}T > m\right\}} \middle| N_T\right]\right]$$

$$= SB_0^f \exp\left\{-\frac{1}{2}\sigma^2 T^{2H}\right\} \sum_{N=0}^{\infty} P(N_T = n) E\left[\exp\left\{\sigma\hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i) - \frac{N_T s_J^2}{2}T\right\} \times I_{\left\{\sigma\hat{B}_H(T) + \sum_{i=1}^{n} J(t_i) - \frac{n\sigma_J^2}{2}T > m\right\}} \middle| n\right]$$

$$= SB_0^f \exp\left\{-\frac{1}{2}\sigma^2 T^{2H}\right\} \sum_{n=0}^{\infty}\left[\frac{(\lambda T)^n}{n!}\exp\left\{\sum_{i=1}^{n} J(t_i) - \lambda T - \frac{n\sigma_J^2}{2}T\right\}\right] E\left[\exp\left\{\sigma\hat{B}_H(T)\right\} \times I_{\left\{\sigma\hat{B}_H(T) + \sum_{i=1}^{n} J(t_i) - \frac{n\sigma_J^2}{2}T > m\right\}}\right]$$

$$= SB_0^f \sum_{n=0}^{\infty}\left[\frac{(\lambda T)^n}{n!}\exp\left\{\sum_{i=1}^{n} J(t_i) - \lambda T - \frac{n\sigma_J^2}{2}T\right\}\right] \frac{1}{\sqrt{2\pi}T^H}\int_{y_n}^{+\infty} e^{-\frac{(x-\sigma T^{2H})^2}{2T^{2H}}}dx$$

$$= SB_0^f \sum_{n=0}^{\infty}\left[\frac{(\lambda T)^n}{n!}\exp\left\{\sum_{i=1}^{n} J(t_i) - \lambda T - \frac{n\sigma_J^2}{2}T\right\}\right] \times P(Z > y_n)$$

$$C_1(k,T) = SB_0^f \sum_{n=0}^{\infty} \left[\frac{(\lambda T)^n}{n!} \exp\left\{ \sum_{i=1}^{n} J(t_i) - \lambda T - \frac{n\sigma_J^2}{2} T \right\} \right]$$

$$P\left(\frac{Z - \sigma T^{2H}}{T^H} > \frac{y_n - \sigma T^{2H}}{T^H} \right)$$

$$= SB_0^f \sum_{n=0}^{\infty} \left[\frac{(\lambda T)^n}{n!} \exp\left\{ \sum_{i=1}^{n} J(t_i) - \lambda T - \frac{n\sigma_J^2}{2} T \right\} \right] \phi(b_n)$$

And:

$$C_2(k,T) = E\left[kB_0^d I_{\left\{ \exp\left(-\int_0^T \beta(t)dt \right) S(T) B_0^f > kB_0^d \right\}} \right]$$

$$= kB_0^d P\left[\sigma \hat{B}_H(T) + \sum_{i=1}^{N_T} J(t_i) - \frac{N_T \sigma_J^2}{2} T > m \right]$$

$$= kB_0^d \sum_{N=0}^{\infty} P(N_T = n) P$$

$$\left[\sigma \hat{B}_H(T) + \sum_{i=1}^{n} J(t_i) - \frac{n\sigma_J^2}{2} T > m \right]$$

$$= kB_0^d \sum_{n=0}^{\infty} \frac{e^{-\lambda T} (\lambda T)^n}{n!} \frac{1}{\sqrt{2\pi} T^H} \int_{Y_n}^{+\infty} e^{\frac{-\left[x - \frac{MT}{\sigma} \right]^2}{2T^{2H}}} dx$$

$$= kB_0^d \sum_{n=0}^{\infty} \frac{e^{-\lambda T} (\lambda T)^n}{n!} P\left(\frac{\hat{B}_H(T)}{T^H} > \frac{y_n}{T^H} \right)$$

$$= kB_0^d \sum_{n=0}^{\infty} \frac{e^{-\lambda T} (\lambda T)^n}{n!} \phi\left(\frac{-y_n}{T^H} \right)$$

$$= kB_0^d \sum_{n=0}^{\infty} \frac{e^{-\lambda T} (\lambda T)^n}{n!} \phi(b'_n)$$

Where:

$$y_n = \frac{m - \sum_{i=1}^{n} J(t_i) + \frac{n\sigma_J^2}{2}}{\sigma};$$

$$m = \ln \frac{kB_0^d}{SB_0^f} + \frac{1}{2} \sigma^2 T^{2H};$$

$$b_n = \frac{\sigma T^{2H} - y_n}{T^H}; \quad b'_n = \frac{-y_n}{T^H}$$

And the $\phi(.)$ is the cumulative normal distribution function. From Eq. 9, researchers have:

$$C(k,T) = E\left(\left(\exp\left(-\int_0^T \beta(t)dt \right) S(T) B_0^f - kB_0^d \right) I_A \right)$$

$$= C_1(k,T) - C_2(k,T)$$

$$= SB_0^f \sum_{n=0}^{\infty} \left[\frac{(\lambda T)^n}{n!} \exp\left(\sum_{i=0}^{n} J(t_i) - \lambda T - \frac{n\sigma_J^2}{2} T \right) \right] \phi(b_n)$$

$$- kB_0^d \sum_{n=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^n}{n!} \phi(b'_n)$$

In the same way, researchers obtain Eq. 9. This completes the proof.

CONCLUSION

In the actuarial approach, there is no economic assumption of the financial market which means the result is valid to all types of market. This method is simpler than the classical B-S Model, for it is not necessary to seek an equivalent martingale measure. On the basis of the general actuarial approach, this study assumes that the spot exchange rate is driven by Poisson jumps and fractional Brownian motion. It has certain reference significance to avoiding foreign exchange risk.

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