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Stability of Generalized AQCQ Functional Equation in Modular Space

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Abstract: Mixed type functional equation is a further step of development in the broad area of functional equations. Many researchers introduced, various mixed type functional equations like additive-quadratic, quadratic-cubic, quadratic-quartic, additive-quadratic-cubic and so on. But even today notably, we have only one famous mixed type of additive-quadratic-cubic-quartic functional equation. In this study, the researchers made an attempt to introduce such new mixed type of additive-quadratic-cubic-quartic functional equation with its general solution and various stabilities related to Ulam problem in modular space.

Key words: Modular space, functional equations, stabilities related to Ulam problem, Δ_α-condition, Fatou property

INTRODUCTION

For the detailed study on Ulam problem and its recent developments called generalized Hyers-Ulam-Rassias stability, one can refer (Aoki, 1950; Gavruta, 1994; Hyers, 1941; Rassias, 1982; Ravi *et al*., 2008, 2009; Rassias, 1978). In 1950, Nakano (1950) established the modular linear spaces and further developed by many researchers, one can refer (Amemiya, 1957; Koshi and Shimogaki, 1961; Luxemburg, 1959; Musielak, 1983; Orlicz, 1988; Turpin, 1978; Yamamuro, 1959). The definitions related to our main theorem related to modular space can be referred by El-Fassi and Kabbaj (2016), Kim and Shin (2017). In 2015, Bodaghi *et al.* (2015) investigated the stabilities of following mixed type equation:

$$h(3y+z)-5h(2y+z)+h(2y-z)+10h(y+z)-5h(y-z) = 10h(z)+4h(2y)-8h(y)$$

For all y, z∈R. In 2016, Narasimman *et al.* (2016) introduced the equations quintic and sextic, respectively of the form:

$$p\left[h(py-z)+h(py+z)\right]+h(y-pz)+h(y+pz)= (p^4+p^2)\left[h(y-z)+h(y+z)\right]+2(p^6-p^4-p^2+1)h(y)$$

$$h(py-z)+h(py+z)+h(y-pz)+h(y+pz)= (p^4+p^2)\left[h(y-z)+h(y+z)\right]+2(p^6-p^4-p^2+1)\left[h(y)+h(z)\right]$$

With $p\in R-\{0, \pm 1\}$ also discussed their various stabilities related to Ulam problem. In 2010, researchers Xu *et al.* (2010) introduced a general mixed AQCQ-functional equation and investigated generalized Ulam-Hyers stability in multi-Banach spaces using fixed point method.

In 2017, researchers Kim and Hong (2017) investigated the alternative stability theorem in a modular space using Δ_3-condition of a modified quadratic equation.

In 2019, researchers Rassias *et al.* (2019) investigated Ulam stability problem in non-Archimedean intuitionistic fuzzy normed spaces of the generalized quartic equation:

$$h(py-z)+h(py+z)+h(y-pz)+h(y+pz)=2p^2\left\{h(y-z)+h(y+z)\right\}+ 2(p^2-1)^2\left\{h(y)+h(z)\right\}; p\neq 0, \pm 1$$

Motivation from the above literature, the researchers made an attempt to introduce a new mixed type equation satisfied by $f(x) = x+x^2+x^3+x^4$ of the form:

$$\begin{aligned} &f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)=(a+a^2)\\ &\left[f(x+y)+f(x-y)\right]+2f(ax)-(a^2+a-1)\\ &\left[2f(x)+2f(y)+2f(-y)-f(y)-f(-y)\right]+\\ &2f(ay)+2f(-ay)-f(ay)-f(-ay)+\frac{a^2-a}{24}\\ &\begin{pmatrix} f(2(x+y))+f(-2(x+y))-4f(x+y)-4f(-(x+y))+\\ f(2(x-y))+f(-2(x-y))-4f(x-y)-4f(-(x-y)) \end{pmatrix}+\frac{a-a^2}{12}\\ &\left(f(2y)+f(-2y)-4f(y)-4f(-y)+f(2x)+f(-2x)-4f(x)-4f(-x)\right) \end{aligned} \tag{1}$$

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For all x, y∈R, a≠0, ±1. Mainly, researchers obtain its general solution and investigate various stabilities concerning Ulam problem in modular spaces.

General solution of (1): additive case

Lemma 2.1: Let X and Y are linear spaces, a mapping f: X→Y is additive and odd if f satisfies:

$$f(ax+y)+f(ax-y)+f(x-ay)+f(x+ay)= (a+a^2)[f(x+y)+f(x-y)]-2(a^2-1)f(x) \quad (2)$$

For all x, y∈X, α≠0, ±1.

Proof: Consider f satisfies Eq. 2. Replacing (x, y) by (0, 0) and (x, 0) in Eq. 2, we get f (0) = 0 and:

$$f(ax)=af(x) \quad (3)$$

Respectively, for all x∈X. Therefore, f is additive function. Let (x, y) = (0, x) in Eq. 2 and by Eq. 3 we reached:

$$f(-x)=-f(x); x\in X \quad (4)$$

Thus f is an odd function.

Theorem 2.2: A function f: X→Y is a solution of Eq. 2 iff A(x) is the diagonal of the additive symmetric map A_1: X→Y such that f is of the form f (x) = A (x) for all x∈X.

Proof: Let f satisfies Eq. 2 when f is additive. We can rewrite Eq. 2 as follows:

$$f(x)+\frac{1}{2(a^2-1)}f(ax+y)+\frac{1}{2(a^2-1)}f(ax-y) +\frac{1}{2(a^2-1)}f(x+ay)+\frac{1}{2(a^2-1)}f(x-ay) -\frac{a+a^2}{2(a^2-1)}f(x+y)-\frac{a+a^2}{2(a^2-1)}f(x-y)=0 \quad (5)$$

For all x, y∈X. Theorems 3.5 and 3.6 in (Xu *et al*., 2012) implies that f is of the form:

$$f(x)=A^1(x)+A^0(x) \quad (6)$$

For all x∈X, A^0 (x) = A^0 and for i = 1, A^i (x) is the diagonal of the i-additive symmetric map A_i: X^i→Y. We get A^0 (x) = A^0 = 0 and f is odd by f (0) = 0 and f (−x) = −f (x), respectively. It follows that f (x) = A^1 (x).

Conversely, A^1 (x) is the diagonal of the additive symmetric map A_1: X^1 → Y such that f (x) = A^1 (x) for all x∈X, from:

$$A^1(x+y)=A^1(x)+A^1(y)$$
$$A^1(rx)=r^1A^1(x); x, y\in X, r\in Q$$

We see that f satisfies Eq. 2 and this completes the proof of theorem 2.2.

General solution of (1): quadratic case

Lemma 3.1: Let X and Y are linear spaces, a mapping f: X→Y is quadratic and even if f satisfies:

$$f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)= f(x+y)+f(x-y)+2a^2\{f(x)+f(y)\} \quad (7)$$

For all x, y∈X, a≠0, ±1

Proof: Assume f satisfies the functional Eq. 7. Letting (x, y) by (0. 0) in Eq. 7, we get f (0) = 0. Setting y = 0 in Eq. 7, we obtain:

$$f(ax)=a^2f(x) \quad (8)$$

For all x∈X. Thus, f is quadratic. Replacing (x, y) by (0, x) in Eq. 7 and by Eq. 8, we get f (−x) = f (x) for all x∈X. Thus, f is an even function.

Theorem 3.2: A function f: X→Y is a solution of the functional Eq. 7 if and only if f is of the form f (x) = E^2 (x) for all x∈X where E^2 (x) is the diagonal of the 2-additive symmetric map E_2: X^2→Y.

Proof: The functional Eq. 7 can rewrite in the form:

$$f(x)-\frac{1}{2a^2}f(ax+y)-\frac{1}{2a^2}f(ax-y)- \frac{1}{2a^2}f(x+ay)+\frac{1}{2a^2}f(x+y)+ \frac{1}{2a^2}f(x-y)-\frac{1}{2a^2}f(x-ay)+f(y)=0 \quad (9)$$

For all x, y∈X. By Xu *et al.* (2012), theorems 3.5 and 3.6, f is a generalized polynomial function of degree at most 2 that is f is of the form:

$$f(x)=E^2(x)+E^1(x)+E^0(x) \quad (10)$$

For all x∈X, where E^0 (x) = E^0 is an arbitrary element of Y and E^i (x) is the diagonal of the i-additive symmetric map E_i: X^i→Y for i = 1, 2. By f (0) = 0 and f (−x) = f (x) for all x∈X, we get E^0 (x) = E^0 = 0 and the function f is even. Thus E^1 (x) = 0. It follows that f (x) = E^2 (x).

Conversely, assume that f (x) = E^2 (x) for all x∈X where E^2 (x) is the diagonal of 2-additive symmetric map E_2: X^2→Y from:

$$E^2(x+y)=E^2(x)+2E^{2,2}(x,y)+E^2(y)$$
$$E^2(rx)=r^2E^2(x)$$
$$E^{2,2}(x,ry)=r^2E^{2,2}(x,y), E^{2,2}(rx,y)=r^2E^{2,2}(x,y)$$

For all x, y∈X, r∈Q, we see that f satisfies Eq. 7 which completes the proof of theorem 3.2.

General solution of (1): cubic case

Lemma 4.1: Let X and Y are linear spaces, a mapping f: X→Y is cubic and odd if f satisfies:

$$f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)= (a+a^2)[f(x+y)+f(x-y)]+2(a^3-a^2-a+1)f(x) \quad (11)$$

for all x, y∈X.

Proof: Consider f satisfies Eq. 11. Replacing (x, y) by (0, 0) and (x, 0) in Eq. 11, we get f (0) = 0. And:

$$f(ax)=a^3f(x) \quad (12)$$

respectively, for all x∈X. Therefore, f is cubic function. Let (x, y) by (0, x) in Eq. 11 and using Eq. 12, we obtain:

$$f(-x)=-f(x); x\in X \quad (13)$$

Thus, f is an odd function.

Theorem 4.2: A function f: X→Y is a solution of Eq. 11 iff C^3 (x) is the diagonal of the 3-additive symmetric map C_3: X^3→Y such that f is of the form f (x) = C^3 (x) for all x ∈X.

Proof: Let f satisfies Eq. 11 when f is cubic. We can rewrite Eq. 11 as follows:

$$f(x)+\frac{1}{2(a^2-1)}f(ax+y)+\frac{1}{2(a^2-1)}f(ax-y)+ \frac{1}{2(a^2-1)}f(x+ay)+\frac{1}{2(a^2-1)}f(x-ay)- \frac{a+a^2}{2(a^2-1)}f(x+y)-\frac{a+a^2}{2(a^2-1)}f(x-y)=0 \quad (14)$$

ror all x, y∈X. Theorems 3.5 and 3.6 by Xu *et al.* (2012) implies that f is of the form:

$$f(x)=C^3(x)+C^2(x)+C^1(x)+C^0(x) \quad (15)$$

For all x∈ X where C^0 (x) = C^0 and i = 1, 2, 3, C^i (x) is the diagonal of the i-additive symmetric map C_i: X^i→Y. We get C^0 (x) = C^0 = 0 and f is odd by f (0) = 0 and f (−x) = f (x), respectively. Therefore, C^2(x) = 0. It follows that f (x) = C^3 (x)+C^1 (x). By Eq. 12 and C^n (rx) = r^n C^n (x) for all x∈X and r∈Q, we obtain n^1 C^1 (x) = n^3 C^1 (x). Hence, C^1 (x) = 0 for all x∈X. Therefore, f (x) = C^3 (x).

Conversely, C^3 (x) is the diagonal of the 3-additive symmetric map C_3: X^3→Y such that f (x) = C^3 (x) for all x∈X from:

$$C^3(x+y)=C^3(x)+3C^{2,1}(x,y)+3C^{1,2}(x,y)+C^3(y)$$
$$C^3(rx)=r^3C^3(x), C^{2,1}(x,ry)=r^1C^{2,1}(x,y),$$
$$C^{2,1}(rx,y)=r^2C^{2,1}(x,y), C^{1,2}(x,ry)=r^2C^{1,2}(x,y),$$
$$C^{1,2}(rx,y)=r^1C^{1,2}(x,y); x,y\in X, r\in Q$$

We see that f satisfies Eq. 11 and this completes the proof of theorem 4.2.

General solution of (1): quartic case

Lemma 5.1: Let X and Y are linear spaces, a mapping f: X→Y is quartic and even if f satisfies:

$$f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)= 2a^2\{f(x+y)+f(x-y)\}+2(a^4-2a^2+1)\{f(x)+f(y)\} \quad (16)$$

For all x, y∈X.

Proof: Consider, f satisfies Eq. 16. Assuming (x, y) by (0, 0) in Eq. 16 gives f (0) = 0. Setting y = 0 in Eq. 16 to obtain:

$$f(ax)=a^4f(x) \quad (17)$$

∀x∈X. So, f is quartic. By Eq. 17 and x = 0 in Eq. 16, we arrive f (-y) = f (y) for all y∈X. So, f is even.

Theorem 5.2: f: X→Y is a solution of Eq. 16 if and only if E^4 (x) is the diagonal of symmetric 4-additive map, f (x) = E^4 (x), ∀x ∈ X.

Proof: Rewrite Eq. 16 as:

$$f(x)-\frac{1}{2(a^4-2a^2+1)}f(ax+y)-\frac{1}{2(a^4-2a^2+1)}f(ax-y)- \frac{1}{2(a^4-2a^2+1)}f(x+ay)+\frac{a^2}{a^4-2a^2+1}f(x+y)+ \frac{a^2}{a^4-2a^2+1}f(x-y)-\frac{1}{2(a^4-2a^2+1)}f(x-ay)+f(y)=0 \quad (18)$$

$\forall x, y \in X$. Therefore, f is follows:

$$f(x) = E^4(x)+E^3(x)+E^2(x)+E^1(x)+E^0(x) \tag{19}$$

for all $x \in X$. As same as theorem 4.2, prove the remaining part of this proof.

Stability of functional Eq. 1: additive case: Assume that the linear space X, μ-complete convex modular space X_μ in the following theorems and corollaries. Now, we obtain the stability of Eq. 1 called generalized Hyers-Ulam-Rassias in modular spaces without Δ_α-condition and the Fatou property. Hereafter, we use the following notation:

$$D_A f(x,y) = f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)-(a+a^2)[f(x+y)+f(x-y)]+2(a^2-1)f(x), \forall x, y \in X$$

Theorem 6.1: Let a mapping f: $X \to X_\mu$ satisfies:

$$\mu(D_A f(x,y)) \le \nu(x,y) \tag{20}$$

And a mapping $\nu : X^2 \to [0, \infty)$ such that:

$$\zeta(x,y) = \sum_{j=0}^{\infty} \frac{\nu(p^j x, p^j y)}{p^j} < \infty, x, y \in X \tag{21}$$

Then there exists A_1: $X \to X_\mu$ a unique additive mapping defined by $A_1(x) = \lim_{n\to\infty} f(a^n x)/a^n$, $x \in X$ which satisfies Eq. 2 and:

$$\mu(f(x)-A_1(x)) \le \frac{1}{2a}\zeta(x,0), \forall x \in X \tag{22}$$

Proof: Substituting y = 0 in Eq. 20, we obtain:

$$\mu(f(ax)-af(x)) \le \frac{1}{2}\nu(x,0) \tag{23}$$

And so:

$$\mu\left(f(x)-\frac{f(ax)}{a}\right) \le \frac{1}{2a}\nu(x,0), \forall x \in X \tag{24}$$

By induction on n, we arrive:

$$\mu\left(f(x)-\frac{f(a^n y)}{a^n}\right) \le \frac{1}{2}\sum_{j=0}^{n-1}\frac{\nu(a^j y,0)}{a^{j+1}}, \forall x \in X \tag{25}$$

Substituting x by $a^m x$ in Eq. 25, we obtain:

$$\mu\left(\frac{f(a^m x)}{a^m}-\frac{f(a^{n+m} x)}{a^{n+m}}\right) \le \frac{1}{2a}\sum_{j=m}^{n+m-1}\frac{\nu(a^j x,0)}{a^j} \tag{26}$$

By assumption Eq. 21 it converges to zero as $m \to \infty$. Hence, by inequality Eq. 26 the sequence:

$$\left\{\frac{f(a^n x)}{a^n}\right\}, \forall x \in X$$

is μ-Cauchy and hence, it is convergent in X_μ, since, X_μ is μ-complete. Thus, a mapping A_1: $X \to X_\mu$ is defined by:

$$A_1(y) = \mu\text{-}\lim_{n\to\infty}\left\{\frac{f(a^n x)}{a^n}\right\}$$

For all $x \in X$ which implies:

$$\lim_{n\to\infty}\mu\left(\frac{f(a^n x)}{a^n}-A_1(x)\right) = 0, \forall x \in X$$

Next, we claim the mapping A_1 satisfies Eq. 2. Setting (x, y) = (a^n x, a^n y) in Eq. 20 and dividing the resultant by a^n, we arrive:

$$\frac{\mu(D_A f(a^n x, a^n y))}{a^n} \le \frac{\nu(a^n x, a^n y)}{a^n}, \forall x, y \in X$$

Hence, by property $\mu(\alpha u) \le \alpha\mu(u)$, $0<\alpha\le 1$, $u \in X_\mu$, we get:

$$\mu\left(\frac{1}{4a^2+2a+3}DA_1(x,y)\right) \le \mu\left(\begin{array}{c}\frac{1}{4a^2+2a+3}DA_1(x,y)- \\ \frac{Df(a^n x, a^n y)}{(4a^2+2a+3)a^n}+\frac{Df(a^n x, a^n y)}{(4a^2+2a+3)a^n}\end{array}\right) \le$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_1(ax+y)-\frac{f(a^n(ax+y))}{a^n}\right)+$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_1(ax-y)-\frac{f(a^n(ax-y))}{a^n}\right)+$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_1(x+ay)-\frac{f(a^n(x+ay))}{a^n}\right)+$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_1(x-ay)-\frac{f(a^n(x-ay))}{a^n}\right)+$$

$$\frac{a+a^2}{4a^2+2a+3}\mu\left(A_1(x+y)-\frac{f(a^n(x+y))}{a^n}\right)+$$

$$\frac{a+a^2}{4a^2+2a+3}\mu\left(A_1(x-y)-\frac{f(a^n(x-y))}{a^n}\right)+$$

$$\frac{2(a^2-1)}{4a^2+2a+3}\mu\left(A_1(x)-\frac{f(a^n y)}{a^n}\right)+\frac{1}{4a^2+2a+3}\mu\left(\frac{Df(a^n x, a^n y)}{a^n}\right)$$

For all x, y∈X and n is positive integers. We obtain:

$$\mu\left(\frac{1}{4a^2+2a+3}DA_1(x,y)\right)=0$$

if n→∞. Hence, $DA_1(x, y) = 0$, ∀x, y∈X. Thus, A_1 satisfies Eq. 2 and hence, it is additive. Since:

$$\sum_{i=0}^{n}\frac{1}{a^{i+1}}+\frac{1}{a}\leq 1$$

For all n∈N, by the convexity of modular μ and Eq. 23, we arrive:

$$\begin{aligned}&\mu(f(x)\text{-}A_1(x))=\\&\mu\left(f(x)\text{-}\frac{f(a^n x)}{a^n}\right)+\rho\left(\frac{f(a^n y)}{a^n}\text{-}A_1(x)\right)\leq\\&\frac{1}{2}\sum_{i=0}^{n\text{-}1}\frac{1}{a^{i+1}}\phi(a^i x,0)+\rho\left(\frac{f(a^n.x)}{a^n}\text{-}A_1(x)\right)\leq\\&\frac{1}{2}\sum_{i=0}^{\infty}\frac{1}{a^{i+1}}\mu(a^i x,0)=\frac{1}{2a}\zeta(x,0)\end{aligned}$$

for all x∈X. Now, to prove the uniqueness of A_1, we consider that there exists as additive mapping D_1: $X\to X_\mu$ satisfying:

$$\mu(f(x)\text{-}D_1(x))\leq\frac{1}{2}\sum_{j=0}^{\infty}\frac{1}{a^{j+1}}\nu(a^j x,0),\forall x\in X$$

But if $A_1(x_0)\neq D_1(x_0)$ for some x_0∈X. Then there exists a constant ∈>0 which is positive such that $\in<\rho(A_1(x_0)-D_1(x_0))$. By Eq. 21, there is a positive integer n_0∈N such that:

$$\sum_{j=n_0}^{\infty}\frac{1}{a^{j+1}}\nu(a^j y,0)<\frac{\in}{2}$$

Since, A_1 and D_1 are additive mappings, by $A_1(a^{n_0}x_0) = a^{n_0}A_1(x_0)$ and $D_1(a^{n_0}x_0) = a^{n_0}D_1(x_0)$, we arrive:

$$\begin{aligned}&\in<\mu(A_1(X_0)\text{-}D_1(x_0))=\\&\mu\left(\frac{A_1(a^{n_0}x_0)\text{-}f(a^{n_0}x_0)}{a^{n_0}}+\frac{f(a^{n_0}x_0)\text{-}D_1(a^{n_0}x_0)}{a^{n_0}}\right)\\&\leq\frac{1}{a^{n_0}}\mu(A_1(a^{n_0}x_0)\text{-}f(a^{n_0}x_0))+\frac{1}{a^{n_0}}\mu(f(a^{n_0}x_0)\text{-}D_1(a^{n_0}x_0))\\&\leq\frac{1}{a^{n_0}}\sum_{j=0}^{\infty}\frac{\nu(a^{j+n_0}x_0,0)}{a^{j+1}}\\&\leq\sum_{j=n_0}^{\infty}\frac{\nu(a^j x_0,0)}{a^{j+1}}<\in\end{aligned}$$

Which implies a contradiction. Therefore, the mapping A_1 is a unique additive mapping near f satisfying Eq. 22 in X_μ. From the above theorem 6.1, we obtain Hyers-Ulam and generalized Hyers-Ulam stabilities, respectively in the following corollaries.

Corollary 6.2: Let a mapping f: $X\to X_\mu$ satisfying:

$$\mu(D_A f(x,y))\leq\in\ \forall x,y\in X$$

For some ∈>0. Then there exists A_1: $X\to X_\mu$ a unique additive mapping satisfies Eq. 2 and:

$$\mu(f(x)\text{-}A_1(x))\leq\frac{\in}{2(a\text{-}1)}\qquad(27)$$

For all x∈X.

Proof: Letting ν (x, y) = ∈ in theorem 6.1, we arrive:

$$\mu(f(x)\text{-}A_1(x))\leq\frac{1}{2a}\sum_{j=0}^{\infty}\frac{\in}{a^j}=\frac{\in}{2a}\left(1\text{-}\frac{1}{a}\right)^{-1}\leq\frac{\in}{2(a\text{-}1)}\qquad(28)$$

For all x∈X.

Corollary 6.3: If f: $X\to X_\mu$ a mapping satisfies:

$$\mu(D_A f(x,y))\leq\in\left(\|x\|^m+\|y\|^m\right),\forall x,y\in X,m<1$$

A real number ∈>0 then there exists A_1: $X\to X_a$ a unique additive mapping satisfying:

$$\mu(f(x)\text{-}A_1(x))\leq\frac{\in}{2(a\text{-}a^m)}\|x\|^m,\forall x\in X\qquad(29)$$

where, x ≠0 if r<0.

Proof: Assuming ν (x, y) = ∈ (||x||^m+||y||^m) in theorem 6.1, we arrive:

$$\begin{aligned}&\mu(f(x)\text{-}A_1(x))\leq\frac{1}{2a}\sum_{j=0}^{\infty}\frac{\in\left(\|a^j x\|^m+0\right)}{a^j}\leq\\&\frac{\in}{2a}\sum_{j=0}^{\infty}\left(\frac{a^m}{a}\right)^j\|x\|^m\leq\frac{\in}{2a}\left(1\text{-}\frac{a^m}{a}\right)^{-1}\|x\|^m\leq\\&\frac{\in}{2(a\text{-}a^m)}\|x\|^m\end{aligned}\qquad(30)$$

for all x∈X. Assuming μ satisfies the Δ_a-condition and if there exists β>0 defined by μ(ax)≤βμ(x) for all x∈X_μ.

Theorem 6.4: Letting f: $X \to X_\mu$ and ν: $X^2 \to [0, \infty)$ be the mappings satisfies:

$$\mu(D_A f(x, y)) \leq \nu(x, y) \quad (31)$$

And:

$$\Psi(x, y) = \sum_{j=0}^{\infty} \frac{\beta^{2j}}{a^j} \nu\left(\frac{x}{a^j}, \frac{y}{a^j}\right) < \infty, \forall x, y \in X \quad (32)$$

Then there exists A_2: $X \to X_\mu$ a unique additive mapping such that $A_2(x) = \lim_{n\to\infty} a^n f(x/a^n)$ which satisfies Eq. 2 and:

$$\mu(f(x)\text{-}A_2(x)) \leq \frac{1}{2a}\zeta(x, 0), \forall x \in X \quad (33)$$

Proof: Equation 23, implies that:

$$\mu\left(f(x)\text{-}af\left(\frac{x}{a}\right)\right) \leq \frac{1}{2}\nu\left(\frac{x}{a}, 0\right), x \in X \quad (34)$$

Hence, by the convexity μ, we have:

$$\mu\left(f(x) - a^2 f\left(\frac{x}{a^2}\right)\right) \leq \frac{1}{a}\mu \left(af(x)\text{-}a^2 f\left(\frac{x}{a}\right)\right) + \frac{1}{a}\mu\left(a^2 f\left(\frac{x}{a}\right)\text{-}a^3 f\left(\frac{x}{a^2}\right)\right) \leq \frac{\beta}{2a}\nu\left(\frac{x}{a}, 0\right) + \frac{\beta^2}{2a}\nu\left(\frac{x}{a^2}, 0\right), \forall x \in X$$

Then by induction on n>1, we have:

$$\mu\left(f(x)\text{-}a^n f\left(\frac{x}{a^n}\right)\right) \leq \frac{1}{2}\sum_{j=1}^{n-1} \frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^j}, 0\right) + \frac{1}{2}\frac{\beta^{2(n-1)}}{a^{n-1}}\nu\left(\frac{x}{a^n}, 0\right) \quad (35)$$

For all x∈X. Considering Eq. 35 holds true for n and we deduce the following by using the convexity of μ:

$$\mu\left(f(x)\text{-}p^{n+1} f\left(\frac{x}{a^{n+1}}\right)\right) = \frac{1}{a}\mu\left(af(x)\text{-}a^2 f\left(\frac{x}{a}\right)\right) + \frac{1}{2}\mu\left(a^2 f\left(\frac{x}{a}\right)\text{-}a^{n+2} f\left(\frac{x}{a^{n+1}}\right)\right) \leq \frac{\beta}{a}\mu\left(f(x)\text{-}af\left(\frac{x}{a}\right)\right) + \frac{\beta^2}{a}\mu\left(f\left(\frac{x}{a}\right)\text{-}a^n f\left(\frac{x}{a^{n+1}}\right)\right) \leq \frac{\beta}{2a}\nu\left(\frac{x}{a}, 0\right) + \frac{\beta^2}{2a}\sum_{j=1}^{n-1}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^{j+1}}, 0\right) + \frac{\beta^2}{2a}\frac{\beta^{2(n-1)}}{a^{n-1}}\nu\left(\frac{x}{a^{n+1}}, 0\right) = \frac{1}{2}\sum_{j=1}^{n}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^j}, 0\right) + \frac{1}{2}\frac{\beta^{2n}}{a^n}\nu\left(\frac{x}{a^{n+1}}, 0\right) \quad (36)$$

The above inequality proves Eq. 35 for n+1. Substituting x by x/a^m in Eq. 35, we arrive:

$$\mu\left(a^m f\left(\frac{x}{a^m}\right)\text{-}a^{n+m} f\left(\frac{x}{a^{n+m}}\right)\right) \leq \beta^m \mu\left(f\left(\frac{x}{a^m}\right) - a^n f\left(\frac{x}{a^{n+m}}\right)\right) \leq \beta^m \frac{1}{2}\sum_{j=1}^{n-1}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^{j+m}}, 0\right) + \beta^m \frac{1}{2}\frac{\beta^{2(n-1)}}{a^{n-1}}\nu\left(\frac{x}{a^{n+m}}, 0\right) \leq \frac{a^m}{2\beta^m}\sum_{j=m+1}^{n+m-1}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^j}, 0\right) + \frac{a^m}{2\beta^m}\frac{\beta^{2(n+m-1)}}{a^{n+m-1}}\nu\left(\frac{x}{a^{n+m}}, 0\right)$$

By Eq. 32 it converges to zero as $m\to\infty$. Hence, $\{a^n f(x/a^n)\}$ is μ-Cauchy for all x∈X and hence, it is μ-convergent in X_μ, since, X_μ is μ-complete. Hence, we have:

$$A_2(x) = \mu\text{-}\lim_{n\to\infty} a^n f\left(\frac{x}{a^n}\right), \forall x \in X \quad (37)$$

and by Eq. 37, we obtain:

$$\lim_{n\to\infty}\mu\left(a^n f\left(\frac{x}{a^n}\right)\text{-}A_2(x)\right) = 0, \forall x \in X$$

Hence, by the Δ_a-condition, we arrive by taking $n\to\infty$:

$$\mu(f(x)\text{-}A_2(x)) \leq \frac{1}{a}\mu\left(af(x)\text{-}a^{n+1} f\left(\frac{x}{a^n}\right)\right) + \frac{1}{a}\mu\left(a^{n+1} f\left(\frac{x}{a^n}\right)\text{-}aA_2(x)\right) \leq \frac{\beta}{a}\mu\left(f(x)\text{-}a^n f\left(\frac{x}{a^n}\right)\right) + \frac{\beta}{a}\mu\left(a^n f\left(\frac{x}{a^n}\right)\text{-}A_2(x)\right) \leq \frac{\beta}{2a}\sum_{j=1}^{n-1}\frac{\beta^{2j-1}}{a^j}\nu\left(\frac{x}{a^j}, 0\right) + \frac{\beta}{2a}\frac{\beta^{2(n-1)}}{a^{n-1}}\nu\left(\frac{y}{a^n}, 0\right) + \frac{\beta}{a}\nu\left(a^n f\left(\frac{x}{a^n}\right)\text{-}A_2(x)\right) \leq \frac{1}{2a}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\nu\left(\frac{x}{a^j}, 0\right)$$

Next, we prove A_2 satisfies Eq. 2. Assuming (x, y) = (x/a^n, y/a^n) in Eq. 31 and multiplaying the resultant by a^n, we obtain:

$$\mu\left(a^n Df\left(\frac{x}{a^n}, \frac{y}{a^n}\right)\right) \leq \beta^n \nu\left(\frac{x}{a^n}, \frac{y}{a^n}\right) \leq \frac{\beta^{2n}}{a^n}\nu\left(\frac{x}{a^n}, \frac{y}{a^n}\right)$$

As $n\to\infty$ which tends to zero. Hence, the property $\mu(\gamma u) \leq \gamma\mu(u)$, $0<\gamma\leq 1$, $u \in X_u$ implies that:

$$\mu\left(\frac{1}{4a^2+2a+3}D_A A_2(x,y)\right) \le \mu\left(\frac{1}{4a^2+2a+3}D_A A_2(x,y) - a^n\frac{D_A f\left(\frac{x}{a^n},\frac{y}{a^n}\right)}{(4a^2+2a+3)} + a^n\frac{D_A f\left(\frac{x}{a^n},\frac{y}{a^n}\right)}{(4a^2+2a+3)}\right) \le$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_2(ax+y)-a^n f\left(\frac{ax+y}{a^n}\right)\right)+\frac{1}{4a^2+2a+3}\mu\left(A_2(ax-y)-a^n f\left(\frac{ax-y}{a^n}\right)\right)+$$

$$\frac{1}{4a^2+2a+3}\mu\left(A_2(x+ay)-a^n f\left(\frac{x+ay}{a^n}\right)\right)+\frac{1}{4a^2+2a+3}\mu\left(A_2(x-ay)-a^n f\left(\frac{x-ay}{a^n}\right)\right)+$$

$$\frac{a+a^2}{4a^2+2a+3}\mu\left(A_2(x+y)-a^n f\left(\frac{x+y}{a^n}\right)\right)+\frac{a+a^2}{4a^2+2a+3}\mu\left(A_2(x-y)-a^n f\left(\frac{x-y}{a^n}\right)\right)+$$

$$\frac{2(a^2-1)}{4a^2+2a+3}\mu\left(A_2(x)-a^n f\left(\frac{x}{a^n}\right)\right)+\frac{1}{4a^2+2a+3}\mu\left(a^n D_A f\left(\frac{x}{a^n},\frac{y}{a^n}\right)\right),\ \forall x, y\in X$$

As the limit $n\to\infty$, we obtain:

$$\mu\left(\frac{1}{4a^2+2a+3}D_A A_2(x,y)\right)=0$$

And hence, $D_A A_2(x, y) = 0$, $\forall x, y \in X$ and A_2 satisfies Eq. 2. Hence, it is additive. To prove the uniqueness of A_2, assume that D_2: $X\to X_\mu$, a additive mapping satisfies:

$$\mu(f(x)-D_2(x)) \le \frac{1}{2a}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\nu\left(\frac{x}{a^j},0\right),\ \forall x\in X$$

Since, A_2 and D_2 are additive mappings and:

$$a^n A_2\left(\frac{x}{a^n}\right)=A_2(x),\ a^n D_2\left(\frac{x}{a^n}\right)=D_2(x)$$

Implies that:

$$\mu(D_2(x)-A_2(x))=\mu\left(\begin{array}{l}\frac{a^{n+1}}{a}\left(D_2\left(\frac{x}{a^n}\right)-f\left(\frac{x}{a^n}\right)\right)+\\ \frac{a^{n+1}}{a}\left(f\left(\frac{x}{a^n}\right)-A_2\left(\frac{x}{a^n}\right)\right)\end{array}\right)$$

$$\le\frac{\beta^{n+1}}{a}\mu\left(D_2\left(\frac{x}{a^n}\right)-f\left(\frac{x}{a^n}\right)\right)+\frac{\beta^{n+1}}{a}\mu\left(\begin{array}{l}f\left(\frac{x}{a^n}\right)-\\ A_2\left(\frac{x}{a^n}\right)\end{array}\right)$$

$$\le\frac{\beta^{n+1}}{a}\frac{1}{2a}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\nu\left(\frac{x}{a^{j+n}},0\right)+\frac{\beta^{n+1}}{a}\frac{1}{2a}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\nu\left(\frac{x}{a^{j+n}},0\right)$$

$$\le\frac{\beta^{n+1}}{a^2}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\nu\left(\frac{x}{a^{j+n}},0\right)\le\frac{\beta a^n}{a^2\beta^n}\sum_{j=1}^{\infty}\frac{\beta^{2(j+n)}}{a^{j+n}}\nu\left(\frac{x}{a^{j+n}},0\right),\ x\in X$$

As $n\to\infty$ it tends to zero. Therefore, A_2 satisfying Eq. 33 and is a unique additive mapping.

In the following corollaries of theorem 6.4, we obtain Hyers-Ulam and Hyers-Ulam-Rassias stabilities, respectively.

Corollary 6.5: Let a mapping f: $X\to X_\mu$ satisfying:

$$\mu(Df(x,y))\le\epsilon,\ x,y\in X,\ \epsilon>0$$

for some $\beta^2 < a$. Hence, there exists a unique additive mapping A_2: $X\to X_\mu$ which satisfies Eq. 2 and:

$$\mu(f(x)-A_2(x))\le\frac{\epsilon\beta^2}{2a(a-\beta^2)},\ \forall x\in X \qquad (38)$$

Proof: Considering $\nu(x, y) = \epsilon$ in theorem 6.4, we arrive:

$$\mu(f(x)-A_2(x))\le\frac{1}{2a}\sum_{j=1}^{\infty}\frac{\epsilon\beta^{2j}}{a^j}\le\frac{\epsilon\beta^2}{a}\left(\frac{a-\beta^2}{a}\right)^{-1}\le\frac{\epsilon\beta^2}{2a(a-\beta^2)},\ \forall x\in X \qquad (39)$$

Corollary 6.6: If f: $X\to X_\mu$ a mapping satisfies:

$$\mu(D_A f(x,y))\le\epsilon\left(\|x\|^m+\|y\|^m\right),\ \forall x,y\in X$$

For given real numbers $\beta^2<a^{r+1}$ and $\epsilon>0$ then there exists A_2: $X\to X_\mu$ a unique additive mapping such that:

$$\mu(f(x)-A_2(x))\le\frac{\epsilon\beta^2}{2a(a^{m+1}-\beta^2)}\|x\|^m,\ \forall x\in X \qquad (40)$$

where, $x\neq 0$, if $r<0$.

Proof: Considering $\nu(x, y) = \epsilon(\|x\|^m+\|y\|^m)$ in theorem 6.1, we arrive:

$$\mu\left(f(x)-A_2(x)\right)\leq\frac{1}{2a}\sum_{j=1}^{\infty}\frac{\beta^{2j}}{a^j}\left(\in\left\|\frac{x}{a^j}\right\|^m\right)$$
$$\leq\frac{\in}{2a}\sum_{j=1}^{\infty}\left(\frac{\beta^2}{a.a^m}\right)^j\|x\|^m$$
$$\leq\frac{\in\beta^2}{2a(a.a^m)}\left(1-\frac{\beta^2}{a.a^m}\right)^{-1}\|x\|^m$$
$$\leq\frac{\in\beta^2}{2a\left(a^{m+1}-\beta^2\right)}\|x\|^m,\forall x\in X \quad (41)$$

Stability of functional Eq. (1): cubic case: We obtain generalized Hyers-Ulam-Rassias stability of Eq. 1 in modular spaces without Δ_p-condition and the Fatou property. Hereafter, we use the following notation:

$$D_f(x,y)=f(ax+y)+f(ax-y)+f(x+ay)+f(x-ay)-\left(a+a^2\right)\left[f(x+y)+f(x-y)\right]-2\left(a^3-a^2-a+1\right)f(x)$$

For all x, y∈X.

Theorem 7.1: Considering f: $X\rightarrow X_\mu$ a mapping satisfies:

$$\mu\left(Df(x,y)\right)\leq\nu(x,y) \quad (42)$$

And a mapping ν: $X^2\rightarrow[0,\infty)$ satisfies:

$$\zeta(x,y)=\sum_{j=0}^{\infty}\frac{\nu\left(a^jx,a^jy\right)}{a^{3j}}<\infty,\forall x,y\in X \quad (43)$$

Then there exists C_1: $X\rightarrow X_\mu$ a unique cubic mapping defined by:

$$C_1(x)=\lim_{n\rightarrow\infty}\frac{f\left(a^nx\right)}{a^{3n}},x\in X$$

Which satisfies the Eq. 11 and:

$$\mu\left(f(x)-C_1(x)\right)\leq\frac{1}{2a^3}\zeta(x,0),\forall y\in X \quad (44)$$

Proof: Assuming x = 0 in Eq. 42, we obtain:

$$\mu\left(f(ax)-a^3f(x)\right)\leq\frac{1}{2}\nu(x,0) \quad (45)$$

And hence:

$$\mu\left(f(x)-\frac{f(ax)}{a^3}\right)\leq\frac{1}{2a^3}\nu(x,0),\forall x\in X \quad (46)$$

Generalizing, we arrive:

$$\mu\left(f(x)-\frac{f\left(a^nx\right)}{a^{3n}}\right)\leq\frac{1}{2}\sum_{j=0}^{n-1}\frac{\nu\left(a^jx,0\right)}{a^{3(j+1)}},\forall x\in X \quad (47)$$

Substituting x by a^m x in Eq. 47, we obtain:

$$\mu\left(\frac{f\left(a^mx\right)}{a^{3m}}-\frac{f\left(a^{n+m}x\right)}{a^3(n+m)}\right)\leq\frac{1}{2a^3}\sum_{j=m}^{n+m-1}\frac{\phi\left(a^jx,0\right)}{a^{3j}} \quad (48)$$

By the assumption Eq. 43 it converges to zero as m→∞. Hence, Eq. 48 implies that the sequence $\left\{\frac{f\left(a^nx\right)}{a^{3n}}\right\}$ is μ-Cauchy and therefore it is convergent in X_μ, since, the X_μ is μ-complete. Hence, we define C_1: $X\rightarrow X_p$ as:

$$C_1(x)=\mu-\lim_{n\rightarrow\infty}\left\{\frac{f\left(a^nx\right)}{a^{3n}}\right\},\forall x\in X$$

Which implies:

$$\lim_{n\rightarrow\infty}\mu\left(\frac{f\left(a^nx\right)}{a^{3n}}-C_1(x)\right)=0,\forall x\in X$$

Hereafter, we complete this proof by similar way of theorem 6.1. In the following corollaries of theorem 7.1, we obtain stabilities called Hyers-Ulam and Hyers-Ulam-Rassias, respectively.

Corollary 7.2: Let a mapping f: $X\rightarrow X_\mu$ satisfying:

$$\mu\left(Df(x,y)\right)\leq\in,\forall x,y\in X$$

for some ∈>0 and $a^3>1$. Then, there exists C_1: $X\rightarrow X_\mu$ a unique cubic mapping which satisfies Eq. 11 and:

$$\mu\left(f(x)-C_1(x)\right)\leq\frac{\in}{2\left(a^3-1\right)} \quad (49)$$

For all x∈X.

Proof: Assuming ν (x, y) = ∈ in theorem 7.1, we arrive:

$$\mu\left(f(x)-C_1(x)\right)\leq\frac{1}{2a^3}\sum_{j=0}^{\infty}\frac{\in}{a^{3j}}=\frac{\in}{2a^3}\left(1-\frac{1}{a^3}\right)^{-1}\leq\frac{\in}{2\left(a^3-1\right)} \quad (50)$$

For all x∈X.

Corollary 7.3: If f: $X\to X_\mu$ a mapping satisfies:

$$\rho(D_C f(x,y)) \le \in \left(\|x\|^m + \|y\|^m\right), \forall x, y \in X$$

For given real numbers m<3 and $\in>0$ then there exists a unique cubic mapping C_1: $X\to X_\mu$ such that:

$$\mu(f(x)-C_1(x)) \le \frac{\in}{2(a^3-a^m)}\|x\|^m, \forall x \in X \tag{51}$$

where, $a\neq 0$ if m<0.

Proof: Assuming $\nu(x, y) = \in (\|x\|^m + \|y\|^m)$ in theorem 7.1, we obtain:

$$\mu(f(x)-C_1(x)) \le \frac{1}{2a^3}\sum_{j=0}^{\infty}\frac{\in\left(\|a^j x\|^m + 0\right)}{a^{3j}} \le \frac{\in}{2a^3}\sum_{j=0}^{\infty}\left(\frac{a^m}{a^3}\right)^j \|x\|^m \le \frac{\in}{2a^3}\left(1-\frac{a^m}{a^3}\right)^{-1}\|x\|^m \le \frac{\in}{2(a^3-a^m)}\|x\|^m \tag{52}$$

For all $x\in X$.

Assuming a nontrivial convex modular μ satisfies the Δ_a-condition if there exists $\beta>0$ such that $\mu(ax)\le\beta\mu(x)$ for all $x \in X_\mu$ where $\beta\ge a$ and hence, $\mu(a^3x)\le M\,\rho(x)$.

Theorem 7.4: If a mapping f: $X\to X_\mu$ satisfies:

$$\mu(Df(x,y)) \le \nu(x,y) \tag{53}$$

And ν: $X^2\to[0, \infty)$ is a mapping such that:

$$\zeta(x,y) = \sum_{j=1}^{\infty}\frac{M^{2j}}{a^{3j}}\nu\left(\frac{x}{a^j},\frac{y}{a^j}\right) < \infty, \forall x, y \in X \tag{54}$$

Then a unique cubic mapping C_2: $X\to X_\rho$ exists and defined by $C_2(x) = \lim_{n\to\infty} a^{3n} f\left(\frac{x}{a^n}\right)$, $x\in X$ which satisfies Eq. 11 and:

$$\mu(f(x)-C_2(x)) \le \frac{1}{2a}\zeta(x,0), \forall x \in X \tag{55}$$

Proof: Equation 45 implies that:

$$\mu\left(f(x)-af\left(\frac{x}{a}\right)\right) \le \frac{1}{2}\phi\left(\frac{x}{a},0\right), \forall x \in X \tag{56}$$

Hence, by the convexity μ, we arrive:

$$\mu\left(f(x)-(a^3)^2 f\left(\frac{x}{a^2}\right)\right) \le \frac{1}{a^3}\mu\left(a^3 f(x)-(a^3)^2 f\left(\frac{x}{a}\right)\right) + \frac{1}{a^3}\mu\left((a^3)^2 f\left(\frac{x}{a}\right)-(a^3)^3 f\left(\frac{x}{a^2}\right)\right) \le \frac{M}{2a^3}\nu\left(\frac{x}{a},0\right) + \frac{M^2}{2a^3}\phi\left(\frac{x}{a^2},0\right), \forall x \in X$$

Generalizing, we obtain:

$$\mu\left(f(x)-(a^3)^n f\left(\frac{x}{a^n}\right)\right) \le \frac{1}{2}\sum_{j=1}^{n-1}\frac{M^{2j-1}}{a^{3j}}\nu\left(\frac{x}{a^j},0\right) + \frac{1}{2}\frac{M^{2(n-1)}}{a^{3(n-1)}}\nu\left(\frac{x}{a^n},0\right) \tag{57}$$

For all $x\in X$. The rest of proof is similar to that of theorem 6.4. In the following corollaries of theorem 7.4, we obtain the stabilities called Hyers-Ulam and Hyers-Ulam-Rassias, respectively.

Corollary 7.5: If a mapping f: $X\to X_\mu$ satisfying:

$$\mu(D_C f(x,y)) \le \in, \forall x, y \in X$$

For some $\in>0$ and $M^2<a^3$. Then there exists C_2: $X\to X_\mu$ a unique cubic mapping which satisfies Eq. 11 and:

$$\mu(f(x)-C_2(x)) \le \frac{\in M^2}{2a(a^3-M^2)}, \forall x \in X \tag{58}$$

Proof: Assuming $\nu(x, y) = \in$ in theorem 7.4, we arrive:

$$\nu(f(x)-C_2(x)) \le \frac{1}{2a}\sum_{j=1}^{\infty}\frac{\in M^{2j}}{a^{3j}} \le \frac{\in M^2}{2a^4}\left(\frac{a^3-M^2}{a^3}\right)^{-1} \le \frac{\in M^2}{2a(a^3-M^2)} \tag{59}$$

$\forall x \in X$

Corollary 7.6: If f: $X\to X_\mu$ a mapping satisfies:

$$\mu(D_C f(x,y)) \le \in \left(\|x\|^m + \|y\|^m\right), \forall x, y \in X$$

For given real numbers $M^2<a^{m+3}$ and $\in>0$ then a unique cubic mapping C_2: $X\to X_\mu$ exists such that:

$$\mu(f(x)-C_2(x)) \le \frac{\in M^2}{2a(a^{m+3}-M^2)}\|x\|^m, \forall x \in X \tag{60}$$

where $x\neq 0$, if m<0.

Proof: Assuming $\nu(x, y) = \in (\|x\|^m + \|y\|^m)$ in theorem 7.1, we arrive:

$$\mu(f(x)-C_2(x)) \leq \frac{1}{2a}\sum_{j=1}^{\infty}\frac{M^{2j}}{a^{3j}}\left(\in\left\|\frac{x}{a^j}\right\|^m\right) \leq \frac{\in}{2a}\sum_{j=1}^{\infty}\left(\frac{M^2}{a^3.a^m}\right)^j\|x\|^m$$
$$\leq \frac{\in M^2}{2a\left(a^3.a^m\right)}\left(1-\frac{M^2}{a^3.a^m}\right)^{-1}\|x\|^m \leq \frac{\in M^2}{2a\left(a^{m+3}-M^2\right)}\|x\|^m \quad (61)$$

For all $x \in X$.

Stability of functional Eq. 1 in quadratic and quartic case can be analyzed by similar method were used in section-6 and 7.

CONCLUSION

We introduced a generalized mixed type of additive-quadratic-cubic-quartic functional equation with its general solution and various stabilities concerning Ulam problem in modular spaces by considering with and without Δ_a-condition.

REFERENCES

Amemiya, I., 1957. On the representation of complemented modular lattices. J. Math. Soc. Jpn., 9: 263-279.

Aoki, T., 1950. On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn., 2: 64-66.

Bodaghi, A., P. Narasimman, K. Ravi and B. Shojaee, 2015. Mixed type of additive and quintic functional equations. Ann. Math. Silesianae, 29: 35-50.

El-Fassi, I.I. and S. Kabbaj, 2016. On the generalized orthogonal stability of mixed type additive-cubic functional equations in modular spaces. Tbilisi Math. J., 9: 231-243.

Gavruta, P., 1994. A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl., 184: 431-436.

Hyers, D.H., 1941. On the stability of the linear functional equation. Proc. Nat. Acad. Sci. U.S.A., 27: 222-224.

Kim, H.M. and H.Y. Shin, 2017. Refined stability of additive and quadratic functional equations in modular spaces. J. Inequalities Appl., Vol. 2017, No. 1. 10.1186/s13660-017-1422-z

Kim, H.M. and Y.S. Hong, 2017. Approximate quadratic mappings in modular spaces. Int. J. Pure Applied Math., 116: 31-43.

Koshi, S. and T. Shimogaki, 1961. On F-norms of quasi-modular spaces. J. Faculty Sci. Hokkaido Univ. Ser. Math., 15: 202-218.

Luxemburg, W.A., 1959. Banach function spaces. Ph.D. Thesis, Delft University of Technology, Delft, Netherlands.

Musielak, J., 1983. Orlicz Spaces and Modular Spaces. Springer, Berlin, Germany, ISBN-13: 9783540127062, Pages: 226.

Nakano, H., 1950. Modulared Semi-Ordered Linear Spaces. 1st Edn., Maruzen Company, Tokyo, Japan, Pages: 288.

Narasimman, P., J.M. Rassias and K. Ravi, 2016. N-dimensional quintic and sextic functional equations and their stabilities in Felbin type spaces. Georgian Math. J., 23: 121-137.

Orlicz, W., 1988. Collected papers. PWN Warszawa, Warszawa, Poland.

Rassias, J.M., 1982. On approximation of approximately linear mappings by linear mappings. J. Funct. Anal., 46: 126-130.

Rassias, J.M., H. Dutta and N. Pasupathi, 2019. Stability of general A-quartic functional equations in non-archimedean intuitionistic fuzzy normed spaces. Proc. Jangjeon Math. Soc., 22: 281-290.

Rassias, T.M., 1978. On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc., 72: 297-300.

Ravi, K., J.M. Rassias, M. Arunkumar and R. Kodandan, 2009. Stability of a generalized mixed type additive, quadratic, cubic and quartic functional equation. J. Inequalities Pure Applied Math., Vol. 10, No. 4.

Ravi, K., M. Arunkumar and J.M. Rassias, 2008. Ulam stability for the orthogonally general euler-lagrange type functional equation. Int. J. Math. Stat., 3: 36-46.

Turpin, P., 1978. Fubini inequalities and bounded multiplier property in generalized modular spaces. Comment. Math., 1: 331-353.

Xu, T.Z., J.M. Rassias and W.X. Xu, 2010. Generalized ulam-hyers stability of a general mixed AQCQ-functional equation in multi-banach spaces: A fixed point approach. Eur. J. Pure Applied Math., 3: 1032-1047.

Xu, T.Z., J.M. Rassias and W.X. Xu, 2012. A generalized mixed quadratic-quartic functional equation. Bull. Malaysian Math. Sci. Soc., 3: 633-649.

Yamamuro, S., 1959. On conjugate spaces of Nakano spaces. Trans. Am. Math. Soc., 90: 291-311.