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Gagliardo-Nirenberg Inequality as a Consequence of Pointwise Estimates for the Functions in Terms of Riesz Potential of Gradient

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Abstract: Our aim in this study is to give the Gagliardo-Nirenberg Inequality as a consequence of pointwise estimates for the function in terms of the Riesz potential of the gradient. Further, in this study we are interested to extract Sobolev type inequality in terms of Riesz potentials for ρ = 1 and to extend our work for weak type estimates when ρ = 1.

INTRODUCTION

The Hardy-Littlewood maximal function was considered as a classical tool in various areas such as potential analysis and harmonic analysis from years and later in Sobolev space theory and partial differential equations[1-4]. Sobolev space plays a significant role in dealing with existence and regularity of solutions of Partial Differential Equations[5, 6]. With the opinion by Harjuletho and Hurri-Syrjanen[7], Koskela and Saksman[8], Leoni[9], the Hardy-Littlewood maximal function bridging between functional analysis, sobolev spaces and partial differential equations. Boundedness of Maximal function has been discussed earlier with different arguments such as Boundedness and regularity of maximal functions on hardy-sobolev spaces discussed by Hajlasz[10], Luiro[4] generalized the original boundedness result and established the continuity of the centered maximal operator in $W^{1,p}(R^d)$, $1<p<\infty$P. Hajlasz and J.Onninen proved the boundedness of the spherical maximal function in the Sobolev space $W^{1,p}(\Omega)$ $p>d/d-1$. For other arguments and related results, one can[11-13]. With the strong arguments over the boundedness of Hardy-Lilltewood Maximal function, our aim, here is to discuss boundedness of Reisz potential in term of maximal functions and to give the proof for Gagliardo-Nirenberg Inequality in term of Reisz potential. We will extend our result to discuss weak type estimate for Gagliaro-Nirenberg sobolev inequality. We start by recalling the definition of maximal function.

Let $B(y, a) = \{x \in R^d: |x-y|<a\}$ is an open ball having center at $y \in R^d$ and radius $\alpha>0$ then $Mf: R^d \to [0, \infty]$ where $f \in L^1_{loc}(R^d)$ is:

$$Mf(y) = \sup_{a>0} \frac{1}{|B(y,a)|} \int_{B(y,a)} |f(x)| dx \tag{1}$$

As from Lebesgue differentiation theorem:

$$|f(y)| = \lim_{a>0} \frac{1}{|B(y,a)|} \int_{B(y,a)} |f(x)| dx$$
$$\le \sup_{a>0} \frac{1}{|B(y,a)|} \int_{B(y,a)} |f(x)| dx = Mf(y) \text{ for all } y \in R^d$$

As maximal function approach involve in our theme of work, we will begin with some basic and obvious results:

Lemma 1: If $f \in L^{\infty}(R^d)$, then $Mf \in L^{\infty}(R^d)$ and $\|Mf\|_{L^{\infty}(Rd)} \leq \|f\|_{L^{\infty}(Rd)}$:

Proof: For any α>0 and for all $y \in R^d$ we can write:

$$\frac{1}{|B(y,a)|}\int_{B(y,a)}|f(x)|dx \leq \frac{1}{|B(y,a)|}\|f\|_{L^{\infty}(R^d)}|B(y,a)| = \|f\|_{L^{\infty}(R^d)}$$

$$\sup_{a>0}\frac{1}{|B(y,a)|\int_{B(y,a)}|f(x)dx \leq \|f\|_{L^{\infty}(R^d)}}$$

Using (1, 1), we have:

$$Mf(y) \leq \|f\|_{L^{\infty}(R^d)}$$

Thus:

$$\|Mf\|_{L^{\infty}(R^d)} \leq \|f\| L^{\infty}(R^d)$$

It shows the maximal function will essentially bounded with the boundedness of original function and thus finite everywhere[8]. Hardly-Littlewood wiener theorem states that: If $f \in L^1(R^d)$, there exist c = c (d) such that:

$$\left|\left\{y \in R^d : Mf(y) > \lambda\right\}\right| \leq \frac{c}{\lambda}\|f\|_{L1(R^d)} \text{ for every } \lambda > 0 \quad (2)$$

This result indicates that Mf maps from $L^1(R^d)$ to weak $L^1(R^d)$ and $f \in L^1(R^d)$ $L^1(R^d)$ does not claim about $Mf \in L^1(R^d)$ and thus Hardy-Littlewood maximal operator is not bounded in $L^1(R^d)$. In this case we can get only weak type estimates. Hajlasz and Onninen raised same type of question by Hajlasz and Onninen[10]. Later on Tanaka[14] gave its answer positively for d = 1.

If $f \in L^p(R^d)$, $1< p \leq \infty$, so is $Mf \in L^p(R^d)$ then there exist c = c (d, p) such that:

$$\|Mf\|_{L^p(R^d)} \leq c\|f\|_{L^p(R^d)} \quad (3)$$

This result shows that Hardly-Littlewood operator is bounded in $L^p(R^d)$ for p>1.

Lemma 2: If $\nu \in C^1_0(R^d)$ then for every $y \in R^d$ and ω_{d-1}, the (d-1)-dimensional measures of ∂B (1, 0) we have:

$$\nu(y) = \frac{1}{\omega_{d-1}}\int_{R^d}\frac{D\nu(x).(y-x)}{|y-x|^d}dx$$

Proof: We will use fundamental theorem of calculus and start the case by taking one dimensional case. If $\nu \in C^1_0(R)$, then there exists interval $[l, m] \subset R$ and ν (y) = 0 for all $y \in R/[l, m]$. We can write with the help of fundamental theorem of calculus:

$$\nu(y) = \nu(l) + \int_l^y \nu'(x)dx = \int_{-\infty}^y \nu'(x)dx \quad (4)$$

In another way can write for initial condition ν (l) = 0 as:

$$0 = \nu(m) = \nu(y) + \int_y^m \nu'(x)dx = \nu(y) + \int_y^{\infty}\nu'(x)dx$$

Or

$$\nu(y) = -\int_y^{-\infty}\nu'(x)dx \quad (5)$$

From (Eq. 3 and 4), we have:

$$2\nu(y) = \int_{-\infty}^y \nu'(x)dx - \int_y^{\infty}\nu'(x)dx = \int_{-\infty}^{\infty}\frac{\nu'(x)(y-x)}{|y-x|}dx + \int_y^{\infty}\frac{\nu'(x)(y-x)}{|y-x|}dx = \int_{-\infty}^{\infty}\frac{\nu'(x)(y-x)}{|y-x|}dx$$

Thus:

$$\nu(y) = \frac{1}{2}\int_R \frac{\nu'(x)(y-x)}{|y-x|}dx \text{ for all } y \in R$$

Now, we will extend the result for R^n, If $x \in R^n$ and $\xi \in \partial B(1, 0)$, fundamental theorem of calculate help us to write:

$$\nu(y) = -\int_0^{\infty}\frac{\partial}{\partial t}(\nu(y+t\xi))dt = \int_0^{\infty}D\nu(y+t\xi).\xi dt$$

Fubini theorem implies:

$$\omega_{d-1}\nu(y) = \nu(y)\int_{\partial B(1,0)}1ds(\xi) = \int_{\partial B(1,0)}\int_0^{\infty}D\nu(y+t\xi).\xi dt\, dS(\xi)$$

Applying Fubini:

$$= -\int_0^{\infty}\int_{\partial B(1,0)}D\nu(y+t\xi).\xi dS(\xi)dt$$

Considering x = tξ, dS (ξ) = t^{1-d} dS (x)

We have:

$$= -\int_0^\infty \int_{\partial B(1,0)} D\nu(y+x).\frac{x}{t}\frac{1}{t^{d-1}} dS(x)dt$$
$$= -\int_0^\infty \int_{\partial B(1,0)} D\nu(y+x).\frac{x}{|x|^d} dS(x)dt$$

It terms of polar coordinates, we can express as:

$$= \int_{R^d} \frac{D\nu(y+x).x}{|x|^d} dx$$
$$= \int_{R^d} \frac{D\nu(z).z-y}{|z-y|^d} dz$$

Replacing z = y+x, dx = dz:

$$= \int_{R^d} \frac{D\nu(x).y-x}{|y-x|^d} dx$$

Thus,

$$\nu(y) = \frac{1}{\omega_{d-1}} \int_{R^d} \frac{D\nu(x).(y-x)}{|y-x|d} dx \quad (6)$$

This is the representation formula for a company supported continuously differential function un in term of its gradient. By Cauchy-Schwarz inequality and Lmma 2, we can write:

$$\begin{aligned} |\nu(y)| &= \left|\frac{1}{\omega_{d-1}} \int_{R^d} \frac{D\nu(x).y-x}{|y-x|^d}\right| \\ &\le \frac{1}{\omega_{d-1}} \int_{R^d} \frac{|D\nu(x)\|y-x\|}{|y-x|^d} dx \\ &\le \frac{1}{\omega_{d-1}} \int_{R^d} \frac{|D\nu(x)|}{|y-x|^{d-1}} dx \\ &\le \frac{1}{\omega_{d-1}} \int_{R^d} \frac{|D\nu|}{|y-x|^{d-a}} dx \end{aligned} \quad (7)$$

Where, I_1 f denotes the reisz potential for α = 1. For 0<α<d, reisz potential of order α can be deduced as:

$$I_\alpha f = \int_{R^d} \frac{f(x)}{|y-x|d-\alpha} \quad (8)$$

One who interested in fundamental properties of Riesz potentials, e.g.[3]. In case of compactly supported smooth functions, the above result is useful for pointwise bound of functions in term of Reisz potential of the gradient. Some researchers have obtained some results for Reisz protentional for example Armin Schikorra and Daniel Spectory[15] established new L^1-type estimate for Riesz potential. By Harjulehto and Hurri-Syrjanen[16], Petteri Harjulehto, Ritva Hurri-Syrjanen, obtained Pointwise estimates to the modified Riesz potential. Using the similar results, they obtained Poincare Inequality for irregular domain. The inequality of Gagliardo-Nirenberg-Sobolev type was established for non-isotropic Generalized Riesz Potential depending on λ-distance by Cinar[13]. Our point of interest here is to discuss boundedness of Reisz Potential by Maximal Operator and then to obtain Gagliardo-Nirenberg inequality. Before moving to the main results, we shall elaborate few technical lemmas for Reisz potential for α = 1.

Lemma 3: If $\psi \subset R^d$ is a measureable set and |ψ|<∞, then:

$$\int_\psi \frac{1}{|y-x|^{d-1}} dx \le c(d)|\psi|\frac{1}{d}$$

Proof: Consider a ball B = B (y, a) with |B| = |ψ. This implies that |ψ/B| = |B/ψ|. We are able to write:

$$\int_{\psi/B} \frac{1}{|y-x|^{d-1}} dx \le |\psi/B|\frac{1}{a^{d-1}}$$

And

$$|\psi/B|\frac{1}{a^{d-1}} \le \int_{\psi/B} \frac{1}{|y-x|^{d-1}} dx$$

As |B| =|ψ| and |ψ/B| = |B/ψ|, then by comparing above inequalities:

$$\begin{aligned} \int_\psi \frac{1}{|y-x|^{d-1}} dx &= \int_{\psi/B} \frac{1}{|y-x|^{d-1}} dx + \int_{\psi\cap B} \frac{1}{|y-x|^{d-1}} dx \\ &\le \int_{B/\psi} \frac{1}{|y-x|^{d-1}} dx + \int_{\psi\cap B} \frac{1}{|y-x|^{d-1}} dx \\ &= \int_B \frac{1}{|y-x|^{d-1}} = c(d)a = c(d)|B|\frac{1}{d} = c(d)|\psi|\frac{1}{d} \end{aligned}$$

Lemma 4: Let 1 ≤ρ<∞ and assuming |Ω|<∞ then:

$$\|I_1(f|X_\Omega\|_{L^p(\Omega)} \le c(d,p)|\Omega|\frac{1}{d}\|f\|_{L^p(\Omega)}$$

Prof: For ρ>1, by applying holder inequality, Lemma 3 can be expressed as:

$$\int_\Omega \frac{|f(x)|}{|y-x|^{d-1}}dx = \int_\Omega \frac{|f(x)|}{|y-x|^{\frac{1}{p}(d-1)}}\frac{1}{|y-x|^{\frac{1}{p}(d-1)}}dx$$

$$\leq \left(\int_\Omega \frac{|f(x)|^p}{|y-x|^{(d-1)}}\right)^{\frac{1}{p}}\left(\int_\Omega \frac{1}{|y-x|\frac{1}{p}(d-1)}dx\right)^{\frac{1}{p}}$$

$$\leq c\,|\Omega|^{\frac{1}{dp'}}\left(\int_\Omega \frac{|f(x)|^p}{|y-x|^{(d-1)}}dx\right)^{\frac{1}{p}}$$

$$\leq c\,|\Omega|^{\frac{p-1}{dp'}}\left(\int_\Omega \frac{|f(x)|^p}{|y-x|^{(d-1)}}dx\right)^{\frac{1}{p}}$$

For p = 1, the above inequality is satisfied. Hence, using Lemma (3) and Fubini's theorem:

$$\int_\Omega \left|I_1(|f|X_\Omega)(y)\right|^p dyc\,|\Omega|^{\frac{p-1}{d}}\int_\Omega\int_\Omega \frac{|f(x)|^p}{|y-x|^{(d-1)}}dxdy$$

$$\leq c|\Omega|^{\frac{p-1}{d}}|\Omega|^{\frac{1}{d}}\int_\Omega |f(x)|^p dx$$

This result gives the idea if $|\Omega|<\infty$, then I_1: $L^p(\Omega)\rightarrow L^p(\Omega)$ is bounded for $1\leq p<\infty$. We are now able to discuss the boundedness of Reisz potential by the Hardy-Littlewood maximal function for general α.

Lemma 5: If $0<\alpha<d$, then for every $y\in R^d$ and $\alpha>0$, there exist c = c (d, a) such that:

$$\int_{B(y,a)} \frac{|f(x)|^p}{|y-x|^{(d-a)}}dx \leq ca^\alpha Mf(y)$$

Proof: Let we denote $A_j = B(y, a2^i)$, I = 0, 1, 2, 3. We can express:

$$\int_{B(y,a)} \frac{|f(x)|^p}{|y-x|^{(d-a)}}dx\sum_{i=0}^{\infty}\int_{A_i/Ai+1}\frac{|f(x)|}{|y-x|(d-a)}dx$$

$$\leq \sum_{i=0}^{\infty}\left(\frac{a}{2^i+1}\right)^{a-d}\int_{A_i}|f(x)|dx =$$

$$\Omega_d\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{a-d}\left(\frac{a}{2^i}\right)^a\frac{1}{\Omega_d}\left(\frac{a}{2^i}\right)^{-d}\int_{Ai}|f(x)|dx$$

$$\Omega_d\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^{a-d}\left(\frac{a}{2^i}\right)^a\frac{1}{|A_i|}\int_{Ai}|f(x)|dx$$

$$\leq cMf(y)a^a\sum_{i=0}^{\infty}\left(\frac{1}{2^a}\right)^i$$

Thus:

$$\int_{B(y,a)} \frac{|f(x)|}{|y-x|^{(d-a)}}dx \leq ca^a Mf(y) \tag{9}$$

Or:

$$I_a f(y) \leq ca^a Mf(y)$$

This implies clearly our objective about Lemma 5. With the strong basis of above work and some important results we are going to extend our work toward the main result for sobolev inequality for Reisz potential.

Theorem 1: For p>1 and α>0 there exists c = c (d, p, c) such that for all $f\in L^p(R^d)$.

We have:

$$\|I_a f\|_{L^{p^*}(R^d)} \leq c\|f\|L^p(R^d)$$

where ap<d and p* = pd/d-ap.

Proof: Observe that the claim is exactly right for f = 0. Consider f ≠0 on a set of positive measure, it is obvious that Mf>0 everywhere then:

$$\int_{R^d/B(y,a)} \frac{|f(x)|}{|y-x|(d-a)}dx \leq \left(\int_{R^d/B(y,a)}|f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_{R^d/B(y,a)}|y-x|(a-d)^{p'}dx\right)\frac{1}{p'} \tag{10}$$

We can compute one part of product on right side as:

$$\int_{R^d/B(y,a)}|y-x|^{(a-d)p'}dx = \int_a^\infty\int_{\partial B(y,\varphi)}|y-x|^{(a-d)p'}dS(x)d\varphi$$

$$= \int_a^\infty \varphi^{(a-d)p'}\int_{\partial B(y,\varphi)}1dS(x)d\varphi$$

Since, $\int_{\partial B(y,\varphi)} 1\, dS(x)\, d\varphi = \omega_{d-1}\,\omega^{d-1}$ above inequality can be reduced as:

$$\int_{R^d/B(y,a)}|y-x|^{(a-d)p'}dx = \omega_{d-1}\int_a^\infty \varphi^{(a-d)p'+n-1}d\varphi$$

Taking integral:

$$= \frac{\omega_{d-1}}{(a-d)p'-d}a^d-(d-a)p' \tag{11}$$

Exponent in Eq .11 can be expressed in the form:

$$d-(d-a)p' = d-(d-a)\frac{p}{p-1} = \frac{ap-d}{p-1}$$

Implies:

$$\int_{R^d/B(y,a)} |y\text{-}x|^{(a\text{-}d)p'} dx = \frac{\omega_{d\text{-}1}}{(a\text{-}d)p'\text{-}d} a \frac{ap\text{-}d}{p\text{-}1}$$

Applying (Eq. 11 and 10) can be rewritten as for any constant c:

$$\int_{R^d/B(y,a)} \frac{|f(x)|}{|y\text{-}x|^{(d\text{-}a)}} dx \le c\, a \frac{ap\text{-}d}{p\text{-}1} \|f\| L^p(R^d)$$

Recalling lemma (5), we have setting:

$$|I_a f(y)| \le \int_{R^d} \frac{|f(x)|}{|y\text{-}x|^{(d\text{-}a)}} dx$$

$$= \int_{B(y,a)} \frac{|f(x)|}{|y\text{-}x|^{(d\text{-}a)}} dx + \int_{R^d/B(y,a)} \frac{|f(x)|}{|y\text{-}x|^{(d\text{-}a)}} dx$$

$$\le c\left(a_a\, Mf(y) + a^{\frac{ap\text{-}d}{p\text{-}1}} \|f\|_{L^p(R^d)} \right)$$

$$a = \frac{Mf(y)\frac{p}{d}}{\|f\|_{L^p(R^d)}}$$

We get:

$$|I_a f(y)| \le cMf(y) 1\text{-}\frac{ap}{d} \|f\|^{\frac{ap}{d}}_{L^p(R^d)} \qquad (14)$$

Taking exponent p* = dp/d-ap on both sides, we have:

$$|I_a f(y)|^{p^*} \le cMf(y)^p \|f\| \frac{ap}{d} p^*_{Lp(R^d)}$$

By lemma (1):

$$\int_{R^d} |I_a f(y)| p^* dx \le c\|f\| \frac{ap}{d} p^*_{L^p(R^d)} \int_{R^d} Mf(y)^p dy$$

$$\le c\|f\| \frac{ap}{d} p^*_{L^p(R^d)} \|Mf\|^p_{L^p(R^d)}$$

$$\le c\|f\| \frac{ap}{d} p^*_{L^p(R^d)} \|f\|^p_{L^p(R^d)}$$

Hence,

$$\|I_a f\|_{L^{p^*}(R^d)} \le c\|f\| \frac{ap}{d} + \frac{p}{p^*}_{L^p(R^d)}$$

Here, p* = pd/d-ap is the Sobolev cojugate if α = 1. Also, for p = 1 we can produce weak type estimates. From (Eq. 14), for p = 1 there exist c = c (d, a) such that:

$$|I_a f(y)| \le cMf(y) 1\text{-}\frac{a}{d} \|f\|^{\frac{a}{d}}_{L^1(R^d)}$$

Taking maximal function approach for p = 1, we have:

$$\left|\left\{ y \in R^d : |I_a f(y)| > t \right\}\right| \le \left|\left\{ y \in R^d : cMf(y) \frac{d\text{-}a}{d} \|f\| \frac{a}{d}_{L1(R^d)} > t \right\}\right|$$

$$\le \left|\left\{ y \in R^d : Mf(y) > c \frac{d}{td\text{-}a} \|f\|_{L^1(R^d)} \text{-}\frac{a}{d}.\frac{d}{d\text{-}a} \right\}\right|$$

$$\le ct^{\text{-}\frac{d}{d\text{-}a}} \|f\|_{L_1(R^d)}^{\frac{a}{d\text{-}a}} \|f\| \frac{a}{d\text{-}a}_{L_1(R^d)}$$

This can also for every t>0. Hence,

$$\left|\left\{ y \in R^d : |I_a f(y) > t| \right\}\right| \le ct^{\text{-}\frac{d}{d\text{-}a}} \|f\| \frac{a}{d\text{-}a}_{L^1(R^d)}$$

It will help us to deduce the proof Sobolev Gagliardo-Nirenberg inequality.

MAIN RESULT

Theorem 2: For every, $\nu \in C^1_0(R^d)$ if $1 \le p < d$, there exist c = c (d, p) such that:

$$\|\nu\|_{L_{p^*}}(R^d) \le c\|D\nu\|_{L^p(R^d)},\ p^* = \frac{pd}{d\text{-}p}$$

Proof: We will split the proof into two cases

Case I: First we will take the case for p>1

For every, $y \in R^d$, we can write from (Eq. 7):

$$\|\nu\|_{L_{p^*}}(R^d) \le \frac{1}{\omega_{d\text{-}1}} I_1(|D\nu|)(y) \text{ where } 1 < p < d$$

Changing the inequality for $L^{p^*}(R^d)$:

$$\|\nu\|_{L_{p^*}} L_{p^*}(R^d) \le c\|I_1(D\nu)\|_{L_{p^*}(R^d)}$$

By using above theorem (Eq. 6) for a = 1, we can write:

$$\|\nu\|_{L^{p*}(R^d)} \leq c\|D\nu\|_{L^p(R^d)}$$

This is the required inequality for p<1

Case II: For p = 1, let we consider pairwise disjoint sets like:

$$B_i = \{y \in R^d : 2^i < |\nu(y)| \leq 2^{i+1}\}, i \in Z$$

Suppose g : R→R, g (t) = max {0, min {t, 1}} be an auxiliary function and ν_i: R^d→[0, 1]:

$$\nu_i(x) = g\left(2^{1-i}|\nu(y)|-1\right) = \begin{cases} 0 & |\nu(y)| \leq 2^{i-1} \\ 2^{1-i}|\nu(y)|-1 & 2^{i-1} < |\nu(y)| \leq 2^i \\ 1 & |\nu(y)| > 2^i \end{cases}$$

Using lemma, if $\nu_i \in W^{1,1}$ (R^d) and I ∈ Z then $D\nu_i$ = 0 almost a.e in R^d/B_{i-1}. We can write:

$$|B_i| \leq \left|\{y \in R^d : |\nu(y)| \leq 2^i\}\right|$$

If

$$|\nu(y)| \leq 2^i \text{ it implies that } 2^{1-i}|\nu(y)|-1>1$$

Above result can be written as:

$$\begin{aligned}
&|B_i| = \left|\{y \in R^d : \nu_i(y) = 1\}\right| \\
&\leq \left|\{y \in R^d : I_1(|D\nu_i|)(y) \geq \omega_{d-1}\}\right|, \text{by applying}(1.7) \\
&\leq c\left(\int_{R^d}|D\nu_i(y)|dy\right)^{\frac{d}{d-1}}, \text{by applying}(1.15) \\
&\leq c\left(\int_{B_{i-1}}|D\nu_i(y)|dy\right)^{\frac{d}{d-1}} \\
&\leq c\left(\int_{B_{i-1}} g'\left(2^{1-i}|\nu(y)|-1\right)2^{1-i}|D\nu(y)|dy\right)^{\frac{d}{d-1}} \\
&\leq c2^{-i\frac{d}{d-1}}\left(\int_{B_{i-1}}|D\nu(y)|dy\right)^{\frac{d}{d-1}}
\end{aligned}$$

Applying summation for I ∈Z, we get:

$$\begin{aligned}
&\int_{R^d}|\nu(y)|^{\frac{d}{d-1}} = \sum_{i\in Z}\int_{B_i}|\nu(y)|^{\frac{d}{d-1}}dy \\
&\leq \sum_{i\in Z} 2^{(i+1)\frac{d}{d-1}}|B_i| \\
&\leq c\sum_{i\in Z}\left(\int_{i-1} \geq |D\nu(y)|dy\right)^{\frac{d}{d-1}} \\
&\leq c\left(\sum_{i\in Z}\int_{B_{i-1}}|D\nu(y)|dy\right)^{\frac{d}{d-1}}
\end{aligned}$$

Hence:

$$\int_{R^d}|\nu(y)|^{\frac{d}{d-1}} \leq c\left(\int_{R^d}|D\nu(y)|dy\right)^{\frac{d}{d-1}}$$

Remarks: By considering the fact that C^1_0 (R^d) is dense in $W^{1,p}$ (R^d) then for ν ∈$W^{1,p}$(R^d), the Sobolov-Gagliardo-Nirenberg inequality follows from the above theorem where 1≤p<n.

REFERENCES

01. Kalton, N.J., S. Mayboroda and M. Mitrea, 2007. Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin spaces and applications to problems in partial differential equations. Contemp. Math., 445: 121-178.
02. Hajlasz, P. and Z. Liu, 2013. Sobolev spaces, Lebesgue points and maximal functions. J. Fixed Point Theory Appl., 13: 259-269.
03. Mizuta, Y., 1996. Potential Theory in Euclidean Spaces. Tosho Printing Company Limited, Tokyo, Japan,.
04. Luiro, H., 2007. Continuity of the maximal operator in Sobolev spaces. Proc. Am. Math. Soc., 135: 243-251.
05. Gilbarg, D. and N.S. Trudinger, 1983. Elliptic Partial Differential Equations of Second Order. Springer, Berlin, Germany,.
06. Maz'ya, V., 2011. Sobolev Spaces with Applications to Elliptic Partial Differential Equations. Springer, Berlin, Germany, Pages: 866.
07. Hajlasz, P., 2003. A new characterization of the Sobolev space. Stud. Math., 159: 263-275.
08. Koskela, P. and E. Saksman, 2008. Pointwise characterizations of hardy-sobolev functions. Math. Res. Lett., 15: 727-744.
09. Leoni, G., 2009. A first course in Sobolev spaces. American Mathematical Society, Providence, Rhode Island.
10. Hajlasz, P. and J. Onninen, 2004. On boundedness of maximal functions in Sobolev spaces. Ann. Academiae Scientiarum Fennicae Math., 29: 167-176.
11. Aldaz, J. and P.J. Lazaro, 2007. Functions of bounded variation, the derivative of the one dimensional maximal function and applications to inequalities. Trans. Am. Math. Soc., 359: 2443-2461.
12. Kurka, O., 2015. On the variation of the hardy-littlewood maximal function. Ann. Academiae Scientiarum Fennicae Math., 40: 109-133.
13. Cinar, I., 2015. The Gagliardo-Nirenberg-Sobolev inequality for non-isotropic Riesz potentials. J. Math. Syst. Sci., 5: 83-85.
14. Tanaka, H., 2002. A remark on the derivative of the one-dimensional hardy-littlewood maximal function. Bull. Aust. Math. Soc., 65: 253-258.
15. Schikorra, A., 2017. Daniel spectory: An ^{1}-type estimate for Riesz potentials. Revista Matematica Iberoamericana, 33: 291-304.
16. Harjulehto, P. and R. Hurri-Syrjanen, 2018. Pointwise estimates to the modified Riesz potential. Manuscripta Math., 156: 521-543.