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Time-Variant System Reliability with Infinite Delay Based on Girsanov's Transformation

Hussein K. Asker
Department of Mathematics, Faculty of Computer Science and Mathematics, Kufa University, Al-Najaf, Iraq

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Corresponding Author:
Hussein K. Asker
Department of Mathematics, College of Science, Swansea University, Swansea, UK

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Abstract: This research addresses time-variant system reliability estimation models of dynamical systems where the governing equations formulated as a set of Stochastic Functional Differential Equations with Infinite Delay (SFDEwID) at state-space Cr. Reliability estimation forms of series and parallel systems tackled depending on Monte Carlo simulations based on extends Girsanov's transformation for infinite delay SFDEs.

INTRODUCTION

System reliability defines as the probability that a system (for one component or more) will execute properly under a given set of operating conditions for a particular time interval see[1]. The reliability of a system changes depending on time. In other words, reliability is a time-variant value. In the engineering practice, Time-variant reliability problems appear when the material properties of the structure deteriorate in time, like corrosion in steel structures, concrete shrinkage and creep phenomena. Also, when random loading modelled involved as random processes such as temperature, wave height, traffic loads[2]. Estimation the reliability consists of determining the probability of success and the probability of failure in a duration of time. Also, it depends on components configuration where system success (failure) described as combinations of unions or intersections of these component failure events[3, 4]. There are many kinds of component's settings available such as series, parallel and many others for details[5]. Any dynamic system represented as a system of equations but not all system of equations have specific solutions, especially, if the governing equations are Stochastic Differential Equations (SDEs). The Girsanov's transformations applied to such governing system of equations to achieve estimators for system reliability or failure probability[3, 4, 6, 7]. In this chapter, we extended and applied the Girsanov's transformations to a system of Stochastic Functional Differential Equations with Infinite Delay (SFDEwID) to get models of estimation of series and parallel system's reliability.

Theoretical background of system reliability: In this study, we outline the basic theory of system reliability, and the exposition follows mostly Barlow[1].

Reliability function: The reliability function related to the survival of a system in the specified interval of time (0, t) which is the probability that the system does not fail in the period (0, t) where t is the time at which the system is still operating and mathematically defined as follows:

$$Ps(t) = 1-P_F(t) = 1-\int_0^t f(s)\,ds = \int_t^\infty f(s)\,ds \qquad (1)$$

where, $P_F(t)$ is the probability of failure within the time interval (0, t) and f(s) is the probability density function (pdf) of time to failure, thus, from Eq. 2, we have $O \le P_S(t) \le 1$.

Failure rate function: The failure rate function (hazard rate function) is the probability that a system will fail within a specified time interval (t, t+h) by knowing the fact that the component or the system is functioning at time t. This probability defined as:

$$\Pr(t<T\le t+h \mid T>t) = \frac{\Pr(t<T\le t+h)}{\Pr(T>t)} = \frac{P_F(t+h)-P_F(t)}{P_s(t)} \qquad (2)$$

where, T is the failure time and one can obtain the failure rate function λ(t) by:

$$\lambda(t) = \lim_{h\to 0}\frac{P_r(t<T\le t+h \mid T>t)-P_F(t)}{h} = \lim_{h\to 0}\frac{P_F(t+h)-P_F(t)}{h}\frac{1}{Ps(t)} = \frac{f(t)}{Ps(t)} \qquad (3)$$

Modelling failure rate: In this research, the normal distribution used to describe the failure rate. The probability density function (pdf) of normal distribution with time to failure t is defined as follows:

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(t-\mu)^2}{2\sigma^2}} \qquad (4)$$

where, μ is the mean value (Mean Time to Failure (MTTF)), σ is the standard deviation and σ^2 is the variance of the normally distributed time to failure t. Cumulative distribution function (cdf) of the normal distribution is defined as:

$$P_F(t) = \phi\left(\frac{t-\mu}{\sigma}\right) = \int_{-\infty}^{t} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(s-\mu)^2}{2\sigma^2}} ds \qquad (5)$$

Thus, the reliability rate is:

$$P_s(t) = 1-\phi\left(\frac{t-\mu}{\sigma}\right) \qquad (6)$$

Consequently, the failure rate is:

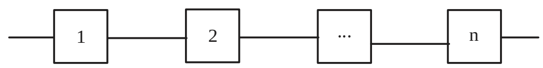

Fig. 1: Series system with n components

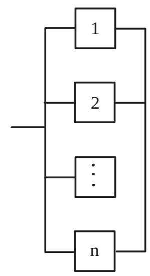

Fig. 2: Parallel system with n components

$$\lambda(t) = \frac{f(t)}{P_s(t)} = \frac{1}{\sigma\sqrt{2\pi\left[1-\phi\left(\frac{t-\mu}{\sigma}\right)\right]}} e^{\frac{-(t-\mu)^2}{2\sigma^2}} \qquad (7)$$

System structure function: Any system is a collection of components (subsystems), thus, the system structure function depends on the configuration of the elements (subsystems). For a system formed by n components, let a vector $\underline{x} = (x_1, x_2, ..., x_n) \in \{0, 1\}^n$ with $x_i = 1$ if the ith component is in working state and $x_i = 0$ if not, be the state vector which gives the state of each component in the system $\underline{x}$ and $\Phi = \Phi(x)$: $\{0; 1\}^n \to \{0; 1\}$ define the system structure function where $\Phi = 1$ if the system functions for its corresponding components state vector $\underline{x}$ and $\Phi = 0$ if not. To explain structural relationship between a system structure function and its components state vector we introduce two kind of systems as an examples, for more details[8].

Series system: A system that is working if and only if all the components are functioning is called a series system. Figure 1 the reliability block diagram of series system. The structure function for the system is given by:

$$\Phi(\underline{x}) = x_1 \times x_2,\ ...,\ x_n = \prod_{i=1}^{n} x_i \qquad (8)$$

Parallel system: A system that is working if and only if at least one component is functioning is called a parallel system. Figure 2 shows the corresponding block diagram for this system. The structure function for the system is given by:

$$\Phi(\underline{x}) = 1\text{-}(1\text{-}x_1)(1\text{-}x_2),\ ...,\ (1\text{-}x_n) = 1\text{-}\prod_{i=1}^{n}(1\text{-}x_i) \qquad (9)$$

System reliability function: The relationship within system's components is essential to determine the reliability of the system as a whole and thus once the system structure function $\Phi\underline{x}$ is known, the reliability can be calculated. For independently functional components if p_i the reliability of the component i and $R = P_S$ is the corresponding reliability (probability of success) of the system then the reliability of the series system, depending to Eq. 8 is:

$$\begin{aligned} P_s &= P_s(\Phi(\underline{x}) = 1) = P\left(\prod_{i=1}^{n} x_i = 1\right) = \\ P(x_1 = 1, x_2 = 1,\ ...,\ x_n = 1) &= \prod_{i=1}^{n} P(x_i = 1) = \prod_{i=1}^{n} p_i \end{aligned} \qquad (10)$$

And similarly, the reliability of a parallel system where it's structure function given in Eq. 9 is given by:

$$P_s = 1\text{-}\prod_{i=1}^{n}(1\text{-}p_i) \qquad (11)$$

In order to get an estimation of system reliability measure, we suppose that a dynamical system represented as an SFDEwID, then the probability measure will be changed to another absolutely continuous probability measure and this called the Girsanov's transformation where the change is accomplished by adding a control function (drift function) to the noise term[8] as we explain in the next section.

SFDEWID AND GIRSANOV'S TRANSFORMATION

Kanjilal[3, 6] and Sundar[4] considered a class of dynamical systems which are governed by SDE to estimate system reliability. In this section, we extended the governed equation to SFDEwID and applied the Girsanov's transformations to get models of estimation of series and parallel system's reliability. Consider a system of stochastic functional differential equations with infinite delay:

$$\begin{cases} d[x(t) = b(x_t)dt + \sigma(x_t)dw(t) \text{ for } t \geq 0] \\ x(0) = x_0 = \xi = \{\xi(\theta) : -\infty < \theta \leq 0\} \in C_r \end{cases} \qquad (12)$$

Here, x(t) is a n×1 vector and $x_t = x(t+\theta)$: -∞<0.b (.) is a non-negative n×1 drift vector, σ(.) is a n×m matrix of diffusion coefficients and w(t) is an m-dimensional Brownian motion. It is clear that the system (3.1) is a NSFDEwID with the neutral term D(.) = 0. Let (Ω, F, P) be a complete probability space with a filtration $\{Ft\}_{t\in[0,+\infty)}$ satisfying the usual conditions (i.e., it is right continuous and F_0 contains all P-null sets). The basic assumption is that performance and design requirements of a dynamical system restrain the acceptable values of the response to the safe domain and the dimension parameter n depends on the problem under study[9].

Let g(x(t)) be a scalar measure (limit state function) of a system and g* be its safe limit value on g(x(t)), so that, the failure event is given by:

$$F = \left\{\omega : g^* - \max_{-\infty\leq t\leq T} g(x(t,\omega)) \leq 0\right\} \qquad (13)$$

where, -∞<t≤T is the tome interval of interest. Let:

$$P_S(T) = P\left\{\omega : \max_{-\infty\leq t\leq T} g(x(t,\omega)) \leq g^*\right\}$$

for all t∈(-∞, T]. P{.} is a probability measure (reliability measure) that the system performance stays below the safe limit g* for all times during the time interval t∈(-∞, T]. Thus, PS denotes the reliability and:

$$P_F(T) = 1\text{-}P_s(T) = P\left\{\omega : \max_{-\infty\leq t\leq T} g(x(t,\omega)) > g^*\right\} \qquad (14)$$

the probability of failure. In other words the events $g^*\text{-}g_m<0$ and $g^*\text{-}g_m>0$ represent, respectively, failure and reliability of the system where, $g_m = \max_{-\infty\leq t\leq m\, g(x(t))}$. A direct Monte Carlo estimator for reliability of the system ($P_S(T)$) in terms of random draws $x^j(t)$, j = 1, ..., N of x(t) that, we can get by solving (Eq. 14) numerically is given by:

$$\hat{P}_S(T) = 1\text{-}\hat{P}_F(T) = 1\text{-}\frac{1}{N}\sum_{j=1}^{N} \{g^*\text{-}\max_{-\infty<t\leq T} g(x^j(t)) \leq 0\} \qquad (15)$$

It can be shown that:

$$\begin{aligned} E_p\left[\hat{P}_F(T)\right] &= E_P\left[\frac{1}{N}\sum_{j=1}^{N} \{g^*\text{-}\max_{-\infty<t\leq T} g(x^j(t)) \leq 0\}\right] \\ &= \frac{1}{N}\sum_{j=1}^{N}\left[P\left\{g^* - \max_{-\infty<t\leq T} g(x^j(t)) \leq 0\right\}\right] = P_F(T) \end{aligned} \qquad (16)$$

where $E_p[.]$ is the expectation under the measurement P and:

$$\begin{aligned} \mathrm{Var}\left(\hat{P}_F(T)\right) &= E_p\left[\hat{P}_F(T)^2\right] - \left(E_P\left[\hat{P}_F(T)\right]\right)^2 \\ &= E_p\left[\hat{P}_F(T)^2\right] - P_F^2(T) \end{aligned} \qquad (17)$$

$$
\begin{aligned}
E_p\left[\hat{P}_F(T)\right]^2 &= E_p\left[\frac{1}{N}\sum_{j=1}^{N} I_{\left\{g^*\text{-max-}\infty<t\le T g\left(x^j(t)\right)\le 0\right\}}\right] \\
&= \frac{1}{N^2}\sum_{j=i=1}^{N} E_p\left[I_{\left\{g^*\text{-max-}\infty<t\le T g\left(x^j(t)\right)\le 0\right\}} \times I_{\left\{g^*\text{-max-}\infty<t\le T g\left(x^j(t)\right)\le 0\right\}}\right] \\
&= \frac{1}{N^2}\sum_{j=i=1}^{N} P\left[I_{\left\{g^* - \max_{-\infty<t\le T} g\left(x^j(t)\right)\le 0\right\}}\right] \\
&\quad + \sum_{j\ne i=1}^{N} P\left[I_{\left\{g^* - \max_{-\infty<t\le T} g\left(x^j(t)\right)\le 0\right\}} P\left[I_{\left\{g^* - \max_{-\infty<t\le T} g\left(x^j(t)\right)\le 0\right\}}\right]\right] \\
&= \frac{1}{N^2}\left[NP_F(T) + \sum_{j\ne i=1}^{N} P_F^2(T)\right] = \frac{1}{N^2}\left[NP_F(T) + \left(N^2\text{-}N\right)P_F^2(T)\right]
\end{aligned} \tag{18}
$$

Thus:

$$
\begin{aligned}
\mathrm{Var}\left(\hat{P}_F(T)\right) &= E_p\left[\hat{P}_F(T)\right]^2 \text{-} \left(E_p\left[\hat{P}_F(T)\right]\right)^2 \\
&= E_p\left[\hat{P}_F(T)\right]^2 \text{-} P_F^2(T)
\end{aligned} \tag{19}
$$

And:

$$
\lim_{N\to\infty} \mathrm{Var}\left(\hat{P}_F\right) = \lim_{N\to\infty} \frac{P_F\left(1\text{-}P_F\right)}{N} \to 0 \tag{20}
$$

which means to obtain acceptable estimates of P_F, sample size needed would enormous[3, 4, 10]. So, to manage the sample size, the recourse is the method of Girsanovs transformation. By reconstructs the drift term in the system (Eq. 12) via. An additional control force $\mu(\tilde{x}_t)$ leads to obtain a modifield dynamical SFDEwID system governed by:

$$
\begin{cases}
d\left[\tilde{x}(t)\right] = b\left(\tilde{x}_t\right)dt + \sigma\left(\tilde{x}_t\right)\left(\mu\left(\tilde{x}_t\right)dt + d\tilde{\omega}(t)\right) \text{for } t \ge 0 \\
\tilde{x}(0) = x_0 = \xi = \left\{\xi(\theta) : \text{-}\infty<\theta\le 0\right\} \in C_r
\end{cases} \tag{21}
$$

where $\mu(\tilde{x}_t)$ is an additional drift term of dimension $n\times 1$ and $\tilde{\omega}(t)$ is an Itos process given by:

$$
d\tilde{\omega}(t) = \text{-}u\left(\tilde{x}t\right)dt + d\omega(t);\ \tilde{\omega}(0) = 0;\ t \ge 0 \tag{22}
$$

The transformation of Eq. 12, 21 essentially changes the underlying probability measure p to a new measure Q such that Q is absolutely continuous with respect to p(Q<<P). By the virtue of Girsanovs theorem $\tilde{\omega}(t)$ is a m-dimensional Brownian process with respect to the new probability measure Q defined on (Ω, F) an th associated Radeon-Nikodym's derivative that can compute explicit[11] is given by:

$$
\frac{y(t)}{y_0} = \frac{dP}{dQ}(t) = \exp\left(\text{-}\int_0^t u_j\left(\tilde{x}s\right)d\tilde{w}_j(s) \text{-} \frac{1}{2}\int_0^t \left(u_j\left(\tilde{x}_s\right)\right)^2 ds\right) \tag{23}
$$

Hence, once now has:

$$
\begin{aligned}
\tilde{P}_F &= \int_F dP = \int_F \left(\frac{dP}{dQ}(T)\right) dQ = \\
&\int_\Omega \left(\frac{dP}{dQ}(T)\right) I_F\, dQ = E_Q\left[\frac{y(t)}{y_0} I_{\left\{g^*\text{-max-}\infty\le t\le T\, g\left(\tilde{x}(t)\right)\le 0\right\}}\right]
\end{aligned} \tag{24}
$$

where, I{.} denotes the indicator function. From Eq. 23 it is clear that :

$$
Q\left[\frac{y(t)}{y_0} \ge 0\right] = 1 \text{and } E_Q\left[\frac{y(t)}{y_0}\right] = \int \frac{dP}{dQ}(t)\, dQ = 1
$$

By rewriting Eq. 23 as:

$$
y(t) = y_0 f\left(G(t)\right) = \exp\left(G(t) \text{-} \frac{1}{2}\int_0^t \left(u\left(\tilde{x}_s\right)^2 ds\right)\right)
$$

And:

$$
G(t) = \text{-}\int_0^t u\left(\tilde{x}_s\right) d\tilde{\omega}(s)
$$

And by differentiating using Ito's rule, it can be shown that:

$$
dy(t) = \text{-}y(t)u\left(\tilde{x}_t\right)d\tilde{x}(t);\ y(0) = y_0 \tag{25}
$$

An estimator for P_F, based on Eq. 25 can now be obtained as:

$$
\overline{P}_F = \frac{1}{N}\sum_{j=1}^{N} \frac{y^j(T)}{y_0^j} I_{\left\{g^* \max\text{-}\infty<t\le T h\left(\tilde{x}^j(t)\right)\le 0\right\}} \tag{26}
$$

where the realization $\tilde{x}^j(t)$ and $y^j(t)$ are obtained as sample solutions of Eq. 21 and 24, respectively with:

$$
y_0^j \ne 0 \text{ for all } j = 1, \ldots, N \tag{27}
$$

Estimation models of system's time-variant reliability based on component configuration: For highlight the purpose of determining the Girsanov control and the associated Radon-Nikodym derivative in the problem of time-variant reliability analysis, it is useful to introduce examples of estimation's models of series and parallel system reliability. For more configurations[3, 4].

Reliability of series system: If N number of failure components arranged in series then the system failure events, according of Eq. 12 is given by:

$$F_j = \left\{ g^*{}_j - \max_{-\infty \le t \le T} gj\left(x\left(t\right)\right) \le 0 \right\} \tag{28}$$

where j = 1, ..., N and the probability of failure will be:

$$P_F = \left(\bigcup_{j=1}^{N} \left\{ g^*{}_j - \max_{-\infty \le t \le T} g_j\left(x\left(t\right)\right) \right\} \le 0 \right) \tag{29}$$

Consequently, by the Eq. 14 and 15 the estimators for series system reliability given by:

$$\hat{P}_s = 1 - \frac{1}{N} \sum_{j=1}^{N} I_{\left(\bigcup_{j=1}^{N}\left\{g^*{}_j - \max_{-\infty \le t \le T} g_j\left(x\left(t\right)\right)\right\} \le 0\right)} \tag{30}$$

$$\bar{P}_s = 1 - \frac{1}{N} \sum_{j=1}^{N} \frac{y^j\left(T\right)}{y_0^j} I_{\left(\bigcup_{j=1}^{N}\left\{g^*{}_j - \max_{-\infty \le t \le T} g_j\left(x\left(t\right)\right)\right\} \le 0\right)} \tag{31}$$

Reliability of parallel system: By the same way above, If N number of failure components arranged in parallel then the system Probability failure is given by:

$$P_F = \left(\bigcap_{j=1}^{N} \left\{ g^*{}_j - \max_{-\infty \le t \le T} g^j\left(x\left(t\right)\right) \le 0 \right\} \right) \tag{32}$$

And the estimators for parallel system reliability are:

$$\hat{P}_S = 1 - \frac{1}{N} \sum_{j=1}^{N} I_{\left(\bigcap_{j=1}^{N}\left\{g^*{}_j - \max_{-\infty \le t \le T} g_j\left(x\left(t\right)\right)\right\} \le 0\right)} \tag{33}$$

And:

$$\bar{P}_S = 1 - \frac{1}{N} \sum_{j=1}^{N} \frac{y^j\left(T\right)}{y_0^j} I_{\left(\bigcap_{j=1}^{N}\left\{g^*{}_j - \max_{-\infty \le t \le T} g_j\left(x\left(t\right)\right)\right\} \le 0\right)} \tag{34}$$

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