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On Cyclic Butterfly k-Cycle Decomposition of the 2-Fold Complete Graph

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Abstract: In this study, we employ the near-two-factorization to develop a new type of simple k-cycle decomposition of the 2-fold complete graph $2K_v$, called a Butterfly k-cycle decomposition of $2K_v$. Especially, we focus on proving the existence of cyclic Butterfly (v-1/2)-cycle decomposition of $2K_v$ for the case $v \equiv 3$ (mod 12) using the difference method for constructing the starter cycles.

INTRODUCTION

Throughout this study, all graphs are considered undirected of odd order have vertices in Z_v. K_v will denote the complete graph of order v and λK_v will denote the λ-fold complete graph of order v which is obtained by replacing each edge of K_v by λ parallel edges.

A k-cycle decomposition of λK_v is a pair (V, C) where, V is the vertex set of λK_v and C is a multiset of k-cycles that partition the multiset $E(\lambda K_v)$. It is cyclic if $V = Z_v$ and for each k-cycle $C = (c_1, c_2, ..., c_k)$ in C we have $C+1 = (c_1+1, c_2+1, ..., c_k+1)$ (mod v) is also in C and it is simple if its cycles are all distinct. A multiset S of k-cycles that generates the multiset C by repeatedly adding 1 modulo v to S is called a starter of cyclic k-cycle decomposition of λK_v. A k-cycle decomposition of λK_v is also called a $(\lambda K_v, C_k)$-design. In general, a $(\lambda K_v, H)$-design is an edge-decomposition of λK_v into subgraphs each of which is isomorphic to H[1].

The existence problem of k-cycle decompositions of the λ-fold complete graph has received a prominent attention in recent years. The fundamental case $\lambda = 1$ has been completely solved by Alspach and Gavlas[2] and by Sajna[3] and for the case $\lambda = 2$ by Alspach *et al*.[4]. In particular, the existence of cyclic k-cycle decompositions of K_v has been solved when $v \equiv 1$ or k (mod 2k)[5-7], k is even with v>2k, k is a prime with the exception of (v, k) = (9, 3)[5], $k \leq 32$ or k is twice a prime power[8], k is thrice a prime[9]. Further results on cycle decompositions in the surveys[10, 11].

The necessary and sufficient conditions for the existence of cyclic v-cycle decomposition of λK_v and for the existence of simple cyclic v-cycle decomposition of λK_v in case of v prime have been proved by Buratti *et al*.[12]. The necessary and sufficient conditions for decomposing λK_v into λ-cycles and into cycles with prime length have been established by Smith[13]. Recently, Bryant *et al*.[14] proved that there exists a k-cycle decomposition of λK_v if and only if $3 \leq k \leq v$, $\lambda(v-1)$ is even and k divides the number of edges in λK_v. More general results for the existence of decomposition of λK_v into cycles of varying lengths have been very recently presented by Alqadri and Ibrahim[15] and Bryant *et al*.[16]. Nevertheless, the existence problem for cyclic k-cycle decomposition of λK_v is still open in general.

A path cover of a graph G is a collection of vertex-disjoint paths of G that covers the vertex set of G. For more details and developments regarding the path cover and the vertex cover problems, one may refer to Steiner[17] and Arumugam and Hamid[18]. A k-factor in a graph G is a spanning subgraph in which each vertex has degree k while a near-k-factor is a spanning subgraph in which exactly one isolated vertex (vertex of degree 0) and all remaining vertices have degree k. The edge decomposition of G into k-factors (respectively, near-k-factors) is called a k-factorization, (respectively, a near-k-factorization). A comprehensive background on factors and factorizations can be found by Wallis[19], Akiyama and Kano[20] and Horsley[21].

In this study, we define a new type of simple k-cycle decomposition of $2K_v$ whose k-cycles can be partitioned into near-two-factors, called a Butterfly k-cycle decomposition of $2K_v$. Some definitions, notations and introductory results are given in Section 2. Then, in Section 3, the difference method is used to construct a cyclic Butterfly (6n+1)-cycle decomposition of $2K_{12n+3}$. Finally, Section 4 discusses the conclusions and future work.

INTRODUCTORY RESULTS

This study provides some definitions, notations and results that will be required to prove our main results in the next section. First, we review the following definitions.

Definition 2.1; Buratti[22]: Let G be a graph and xy be an edge in G. The difference of an edge xy is defined as d(x, y) = ±|y-x|.

Definition 2.2; Buratti[22]: Let G = (V(G), E(G)) be a graph. The multiset:

$$\Delta G = \{\pm|y-x| \,|\, x, y \in V(G), xy \in E(G)\}$$

is called the list of differences from G. More generally, for a multiset g = {G_1, G_2, ..., G_n} of graphs, the list of differences from G is the multiset $\Delta g = \Delta G_1 \cup \Delta G_2 \cup, ..., \cup \Delta G_n$ which is obtained by linking together the (ΔG_i)'s.

Definition 2.3; Buratti *et al.*[12]: Let C be a k-cycle in λK_v. A cycle orbit of C, denoted Orb(C) is a set of distinct k-cycles in $\{C+i | i \in Z_v\}$. A cycle orbit of C is called full if its cardinality is v, otherwise the cycle orbit of C is short.

The next lemma is a particular consequence of the results developed by Buratti *et al*.[12]. It will be crucial for proving our main results.

Lemma 2.4: Let S be a multiset of k-cycles of λK_v. Then S is a starter of cyclic k-cycle decomposition of λK_v if and only if ΔS covers $Z^*_v = Z_v$-{0} exactly λ times.

In the following, we define the relative path, relative cycle and alternating arithmetic path and then we formulate some related results that will be the basis for constructing a starter of cyclic Butterfly (6n+1)-cycle decomposition of $2K_{12n+3}$.

Definition 2.5: Let G be a graph of order v, $P_n \equiv [x_1, x_2, ..., x_n]$ be an n-path of G and $Cn = (x_1, x_2, .., x_n)$ be an n-cycle of G:

- The n-path $\bar{P}_n = [v-x_1, v-x_2, ..., v-x_n]$ is called the relative path of P_n
- The n-cycle $\bar{C}_n = [v-x_1, v-x_2, ..., v-x_n]$ is called the relative cycle of C_n

Lemma 2.6: Let G be a graph of order v. If $\bar{C}$ is a k-cycle of G and $\bar{C}$ is the relative cycle of C, then $\Delta C = \Delta \bar{C}$.

Proof: Suppose $C = (x_1, x_2, .., x_k)$ and $\bar{C} = (y_1, y_2, ..., y_k)$ are k-cycle of G and its relative cycle, respectively. The list of differences from C and $\bar{C}$ can be defined as:

$$\Delta C = \{\pm|x_i - x_{i-1}| \,|\, i = 2,3,\ldots,k\} \cup \{\pm|x_1 - x_k|\} \quad (1)$$

$$\Delta \bar{C} = \{\pm|y_i - y_{i-1}| \,|\, i = 2,3,\ldots,k\} \cup \{\pm|y_1 - y_k|\} \quad (2)$$

Since, $\bar{C}$ is the relative path of C, then $y_i = v-x_i$ for all i = 1, 2, ..., k. Hence, substituting $y_i = v-x_i$ into (2), we obtain:

$$\begin{aligned}\Delta \bar{C} &= \{\pm|(v-x_i)-(v-x_{i-1})| \,|\, i=2,3,\ldots,k\} \cup \\ &\{\pm|(v-x_1)-(v-x_k)|\} = \{\pm|x_i - x_{i-1}| \,|\, i=2,3,\ldots,k\} \cup \\ &\{\pm|x_1 - x_k|\} = \Delta C\end{aligned}$$

Lemma 2.7: Let G be a graph of order v. If C_1 is a k-cycle of G and C_2 is the relative cycle of C_1, then $orb(C_1) \neq orb(C_2)$.

Proof: Let $_1 = (c_{1,1}, c_{1,2}, ..., c_{1,k})$ be a k-cycle of G and let $C_2 = (c_{2,1}, c_{2,2}, ..., c_{2,k})$ be the relative cycle of C_1. Assume by contrary that $orb(C_1) = orb(C_2)$, then there exists an integer $i \in Z_v$ such that $C_2 = i+C_1$. This implies that:

$$c_{2,j} = i + c_{1,j} \text{ for all } j = 1,2,\ldots,k. \quad (3)$$

Since, C_2 is the relative cycle of C_1, then:

$$c_{2,j} = v - c_{1,j} \text{ for all } j = 1, 2, \ldots, k \quad (4)$$

Solving (Eq.3) and (4) for $c_{1,j}$ and $c_{2,j}$ yields:

$$c_{1,j} = \frac{v-i}{2} \text{ and } c_{2,j} = \frac{v+i}{2} \text{ for all } j = 1, 2, \ldots, k$$

This contradicts with the fact that C_1 and C_2 are actually k-cycles. Thus, C_1 and C_2 must have different orbits, so orb(C_1)≠orb(C_2). An alternating arithmetic path is a path with two sets of vertics satisfying certain conditions as defined.

Defination 2.8: Let m and n be positive intergers with n≤m≤n+1. An (m+m)-alternating arithmetic path, denoted by AAP(m+n) is a path of length m+n with vertex set $V = \{x_1, x_2, ..., x_m\} \cup \{y^1, y_2, ..., y_n\}$ and edge set $E = \{\{x_i, y_i\}\}|I = 1, 2, ..., n\} \cup \{y_i, x_{i+1}\}|i = 1, 2, ..., m-1$ such that the following properties are satisfied:

- x_i-x_{i-1} is constant for all $2 \le i \le m$
- y_i-y_{i-1} is consatnt for all $2 \le i \le n$

Defiantion 2.9: Let AAP(m+n) be an (m+n)-alternating arithmetic path. The list of differences from AAP(m+n) is the multiset:

$$\Delta(\text{AAP}(m+n)) = \left\{ \begin{matrix} \pm|y_i\text{-}x_i| \\ |1 \le i \le n| \end{matrix} \right\} \cup \left\{ \begin{matrix} \pm \\ |x_{i+1}\text{-}y_i||1 \le i \le m\text{-}1 \end{matrix} \right\}$$

According to defination 2.8, the (m+n)-alternating arthimetic path either has odd order (2n+1) when m = n+1 or has even order (2n) when m = n. Throughout, we use the following notations for (m+n)-alternating arithmetic path of odd order and even order, respectively:

$$\Delta(\text{AAP}(2n+1)) = \begin{bmatrix} x_1, y_1, x_2, y_2, ..., x_n, \\ y_n, x_{n+1} \end{bmatrix} = [x_i, y_i]_{2n+1}$$
$$\text{AAP}(2n) = [x_1, y_1, x_2, y_2, ..., x_n, y_n] = [x_i, y_i]_{2n}$$

Next, we define a new way of writing the cycle as linked vertex-disjoint paths. This way will be used mainly to prove the existence results in the following section.

Defination 2.10: Let C_n be an n-cycle, k≥2 b a positive integer and let $P = \{P_1, P_2, ..., P_k\}$ be a path cover of C_n. The set of k edges in C_n taht links the end of P_i with the start of P_{i+1} for all i = 1, 2, ..., k where $P_{k+1} = P_1$ is called the link set of P.

Lemma 2.11: Let C_n be an n-cycle, $P = \{P_1, P_2, ..., P_k\}$ be a path cover of C_n and $E' = \{e_1, e_2, ..., e_k\}$ be a link set of P. Then, we have $\Delta_{cn} = \Delta P \uplus \Delta E'$.

Proof: Let $V(P) = U^r{}_{i=1} V(P_i)$ be the set of vertices of P and $E(P)$ $U^r{}_{i=1} V(P_i)$ the set of edges of P. Based on defination 2.2, the list of differences from C is defined as a multiset consisting of the difference for each edge in C as follows:

$$\Delta C = \{d(a,b) | a, b \in V(C), ab \in E(C)\} \quad (5)$$

Since, P is a path cover of C, then:

$$V(C) = V(P) \quad (6)$$

Also, from the defination of links set of P, we obatin:

$$E(C) = E(P) \cup E' \quad (7)$$

Subsituting (Eq. 6) and (7) into (5) yields:

$$\Delta C = \{d(a,b) | a, b \in V(P), ab \in E(P) \cup E'\}$$

$$= \{d(a,b) | a, b \in V(P), ab \in E(P)\} \uplus \left\{ \begin{matrix} d(e_i) \\ |e_i \in E' \end{matrix} \right\}$$
$$= \Delta P \uplus \Delta E'$$

Remark 2.12: Let C_n be an n-cycle, $P = \{P_1, P_2, ..., P_k\}$ be a path cover of C_n and $E' = \{e_1, e_2, ..., e_k\}$ be a link set of P. The cycle C_n can be expressed as linked vertex-disjoint paths as follows:

$$C_n = (P_1, P_2, ..., P_k)$$

Before closing this study, we provide an example which demonstrates the concepts discussed above.

Example 2.13: Let G - $2K_{11}$ and C = (1, 2, 10, 4, 9, 7, 5, 6, 3, 8) be a 10-cycle of G. Then C can be written as linked vertex-disjoint paths as follows:

$$C = (Q_1, \text{AAP}_1(4), Q_2, \text{AAP}_2(4))$$

Where, Q_1 = (1) and Q_2 = (7) are trival paths and $\text{AAP}_1(4)$ = (2, 10, 4, 9) = $(2i, 11\text{-}i)_4$ and AAP_2 (4) = (5, 6, 3, 8) = $(7\text{-}2i, 2i+4)_4$ are 4-alternating arithmetic paths. In addition, the set of four edges E' = {{1, 2}, {9, 7}, {7, 5} {8, 1}} that links the paths Q_1, $\text{AAP}_1(4)$, Q_2 and $\text{AAP}_2(4)$, respectively along the cycle C is considered the links set for the path cover P = {Q_1, $\text{AAP}_1(4)$}, Q_2 $\text{AAP}_2(4)$}.

Cyclic butterfly (6n+1)-cyclic decomposition of $2K_{12n+3}$: In this study, we define a butterfly k-cycle decomposition of $2K_v$. Then, the existence of cyclic butterfly (6n+1)-cycle decomposition of $2K_{12n+3}$ is proved using the difference method in constructing the starter cycles.

Defination 3.1: Let k and v be integer with 2<k<v. A butterfly k-cyclic decomposition of a graph $2K_v$, denoted by BkCD ($2K_v$) is an array of k-cycles which satisfies the following conditions:

- The cycles in row i from a near-two-factor with focus i
- The cycles associated with the rows contain no repetitions
- The cycles associated with the rows from a k-cycle decomposition of $2K_v$

A Butterfly k-cycle decomposition of a graph $2K_v$ with vertex set Z_v is cyclic if C = {C_1, C_2, ..., C_n} is a set of all k-cycles in BkCD($2K_v$), then we also have C = {C_1+1, C_2+1, ...,C_n+1} where, C_i+1 denotes the k-cycle obtained by adding 1 modulo v to each vertex of the cycle C_i. A set S of k-cycles which generates all the cycles of BkCD ($2K_v$) by repeatedly adding 1 modulo v is called a starter of cyclic BKCD ($2K_v$).

To construct a cyclic butterfly k-cyclic decomposition of $2K_v$ it is sufficient to exhibit a stater of cyclic k-cyclic decomposition of $2K_v$ which satisfies a near-two-factor and contains no two cycles in the same orbit. We now provide an example to illustrate the defination above.

Example 3.2: Let G = $2K_{15}$ and S = {C_1, C_2} be a set of 7-cycles of G such that C_1 = (13, 8, 9, 11, 5, 123, 1) and C_2 = (2, 7, 6, 4, 10, 3, 14).

Immediately, it can be noticed that the 7-cycles of S are vertex-disjoint and cover each nonzero element of Z_{15} exactly once. In other words S forms a near-two-factor with focus zero.

In order to show that S = {C_1, C_2} is a set of stater cycles for cyclic 7-cycle decomposition of G, we need to calculate the list of differences from S as illustrates in the Table 1.

Based on Table 1, since, $\Delta S = \Delta C_1 \cup \Delta C_2$ covers each element in Z_{15}-{0} exactly twice, then from Lemma 2.4 S = {C_1, C_2} is a stater of cyclic 7-cycle decomposition of G.

Since, the sum of each pair of corresponding vertices of C_1 and C_2 is equal to 15 (the order of G), then C_2 is the relative cycle of C_1 and so by Lemma 2.7 orb (C_1) ≠ or b (C_2). Therefore, all the generated cycles by repeatedly adding 1 modulo 15 to S = {C_1, C_2} contain no repetitions.

Table 1: The list of differences from S = {C_1, C_2 }

7-cycles	The list of differences
C_1 = (13, 8, 9, 11, 5, 12, 1)	{±5, ±1, ±2, ±6, ±7, ±11, ±12}
C_2 = (2, 7, 6, 4, 10, 3, 14)	{±5, ±1, ±2, ±6, ±7, ±11, ±12}

Table 2: A cyclic butterfly 7-cycle decomposition of $2K_{15}$

Focus	Orb(C_1)	Orb(C_2)
i = 0	(13, 8, 9, 11, 5, 12, 1)	(2, 7, 6, 4, 10, 3, 14)
i = 1	(14, 9, 10, 12, 6, 13, 2)	(3, 8, 7, 5, 11, 4, 0)
i = 2	(0, 10, 11, 13, 7, 14, 3)	(4, 9, 8, 6, 12, 5, 1)
⋮	⋮	⋮
i = 14	(12, 7, 8, 10, 4, 11, 0)	(1, 6, 5, 3, 9, 2, 13)

Now, S satisfies all the conditions to be a starter of cyclic Butterfly 7-cycle decomposition of G. Table 2 illustrates how the starter cycles generate all the cycles of cyclic B7CD.

In the following, we explicitly construct a cyclic Butterfly (6n+1)-cycle decomposition of $2K_{12n+3}$. Since, the construction is different depending on whether n is odd or even, we classify the construction into two cases: when n is odd and when is even.

Lemma 3.3: For any positive odd integer n, there exists a cyclic Butterfly (6n+1)-cycle decomposition of $2K_{12n+3}$.

Proof: Let n be a positive odd integer. Two cases are considered.

Case 1: n = 1. This case has been proved in Example 3.2.

Case 2: n>1. Let C_1 and C_2 be the (6n+1)-cycles of $2k_{12n+3}$ defined as linked vertex-disjoint paths as follows:

$$\begin{aligned} C_1 &= \left(AAP_1(4n), AAP_2(n+1), AAP_3(n)\right) \\ C_2 &= \left(\overline{AAP_1}(4n), \overline{AAP_2}(n+1), \overline{AAP_3}(n)\right) \end{aligned} \tag{8}$$

where:

$$\begin{aligned} AAP_1(4n) &= [2, 12n-1, 6, 12n-5, \ldots, 8n-2, 4n+3] \\ &= [4i-2, 12n-4i+3]_{4n} \end{aligned}$$

$$\begin{aligned} AAP_2(n+1) &= [12n+2, 3, 12n-2, 7, \ldots, 10n+4, 2n+1] \\ &= [12n-4i+6, 4i-1]_{n+1} \end{aligned}$$

$$\begin{aligned} AAP_3(n) &= [2n+3, 10n-2, 2n+7, 10n-6, \ldots, 4n-3, 8n+4, 4n+1] \\ &= [2n+4i-1, 10n-4i+2]_n \end{aligned}$$

$$\overline{AAP_1}(4n) = [v-(4i-2), v-(12n-4i+3)]_{4n} = [12n-4i+5, 4i]_{4n}$$

$$\begin{aligned} \overline{AAP_2}(n+1) &= [v-(12n-4i+6), v-(4i-1)]_{n+1} \\ &= [4i-3, 12n-4i+4]_{n+1} \end{aligned}$$

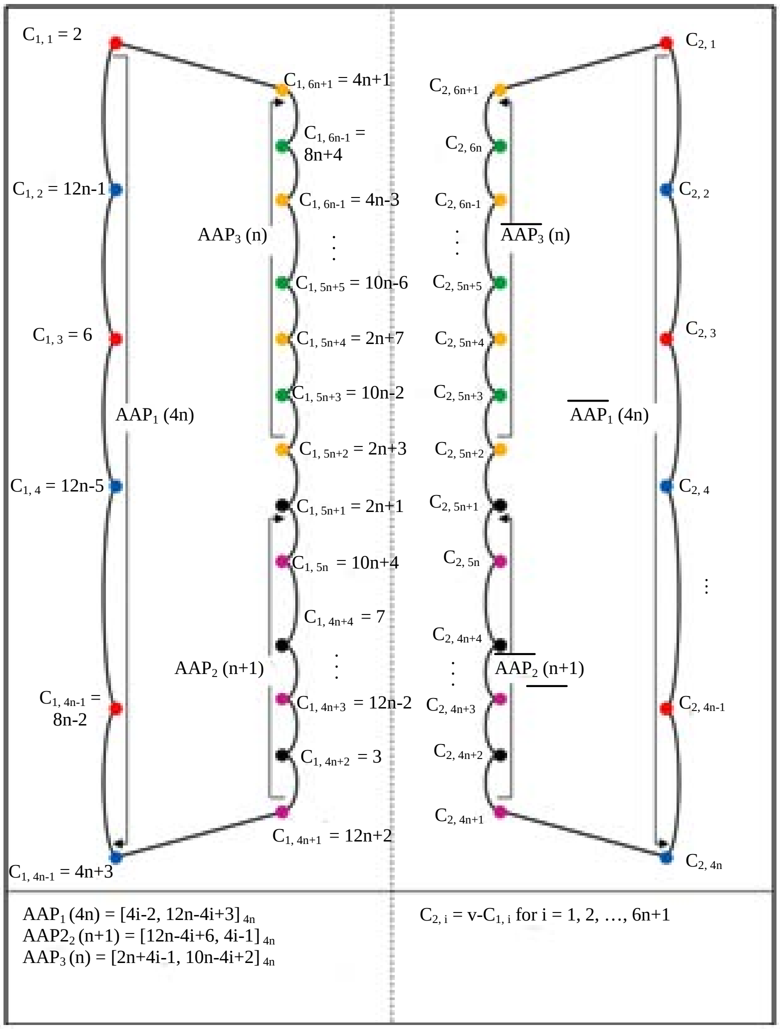

Fig. 1: The construction of C_1 and C_2 in $2K_{12n+3}$ when n>1 is an odd integer

$$\overline{AAP_3}(n) = \left[v-(2n+4i-1), v-(10n-4i+2)\right]_n$$
$$= \left[10n-4i+4, 2n+4i+1\right]_n$$

Since, n is a positive odd integer, then any 4n-alternating arithmetic path and any (n+1)-alternating arithmetic path have even order while any n-alternating arithmetic path has an odd order. As illustrated in Fig. 1, the construction of C_1 and C_2 can be described in terms of their vertices as $C_i = (c_{i,1}, c_{i,2}, ..., c_{i,6n+1})$ for i = 1, 2.

In the construction above, we note that $c_{1,i}$'S form the following increasing sequences:

- $C_{1,1}<C_{1,4n+2}<C_{1,3}<C_{1,4n+4}<\cdots<C_{1,n}<C_{1,5n+1}$ in the interval [2, 2n+1]
- $C_{1,5n+2}<C_{1,n+2}<C_{1,5n+4}<C_{1,n+4}<\cdots<C_{1,6n+1}<C_{1,2n+1}$ in the interval [2n+3, 4n+2]
- $C_{1,4n}<C_{1,2n+3}<C_{1,4n-2}<C_{1,2n+5}<\cdots<C_{1,2n+4}<C_{1,4n-1}<C_{1,2n+2}$ in the interval [4n+3, 8n-1]
- $C_{1,2n}<C_{1,6n}<C_{1,2n-4}<C_{1,6n-4}<\cdots<C_{1,n+3}<C_{1,5n+3}$ in the interval [8n+3, 10n-2]
- $C_{1,n+1}<C_{1,5n}<C_{1,n-1}<C_{1,5n-2}<\cdots<C_{1,2}<C_{1,4n+1}$ in the interval [10n+1, 12n+2]

The vertices of C_1 form increasing sequences in disjoint intervals, then we can say that the vertices of C_1 are pairwise distinct and then C_1 is actually a (6n+1)-cycle. In contrast, from (Eq. 8), we can deduce that $c_{2,i} = v-c_{1,i}$ for all i = 1, 2, …, 6n+1 and this implies that C_2 is the relative cycle of C_1 in $2K_{12n+3}$. Consequently, since, C_1 is actually a (6n+1)-cycle, it follows that C_2 is also actually a (6n+1)-cycle.

Now, we shall prove that the set of cycles S = {C_1, C_2} satisfies the conditions of cyclic Butterfly (6n+1)-cycle decomposition of $2K_{12n+3}$. To render this proof easier to follow, we shall divide this proof into three parts as follows:

Part 1: In this part, we prove that S = {C_1, C_2} forms a near-two-factor. This will be proved by showing that the union of vertex sets of C_1 and C_2 covers each nonzero element of Z_{12n+3} exactly once. The vertex sets of C_1 and C_2 can be calculated by the union of vertex sets of all linked paths in both C_1 and C_2, respectively:

$$V(C_1) = V(AAP_1(4n)) \cup V(AAP_2(n+1)) \cup V(AAP_3(n)) \quad (9)$$

$$V(C_2) = V(\overline{AAP_1}(4n)) \cup V(\overline{AAP_2}(n+1)) \cup V(\overline{AAP_3}(n)) \quad (10)$$

where:

$$V(AAP_1(4n)) = \bigcup_{i=1}^{\frac{4n}{2}}\{4i\text{-}2\} \,|\,|\, \bigcup_{i=1}^{\frac{4n}{2}}\{12n\text{-}4i+3\} =$$

$$\{2, 6, ..., 8n\text{-}2\} \cup \{12n\text{-}1, 12n\text{-}1, 12n\text{-}5, ..., 4n+3\}$$

$$V(AAP_2(4+1)) = \bigcup_{i=1}^{\frac{n+1}{2}}\{12n\text{-}4i+6\} \,|\,|\, \bigcup_{i=1}^{\frac{n+1}{2}}\{4i\text{-}1\} =$$

$$\{12n+2, 12n\text{-}2, ..., 10n+4\} \cup \{3, 7, ..., 2n+1\}$$

$$V(AAP_3(n)) = \bigcup_{i=1}^{\frac{n+1}{2}}\{2n+4i\text{-}1\} \,|\,|\, \bigcup_{i=1}^{\frac{n+1}{2}}\{10n\text{-}4i+2\} =$$

$$\{2n+3, 2n+7, ..., 4n+1\} \cup \{10n\text{-}2, 10n\text{-}6, ..., 8n+4\}$$

$$V(\overline{AAP_1}(4n)) = \bigcup_{i=1}^{\frac{4n}{2}}\{12n\text{-}4i+5\} \,|\,|\, \bigcup_{i=1}^{\frac{4n}{2}}\{4i\} =$$

$$\{12n+1, 12n\text{-}3, ..., 4n+5\} \cup \{4, 8, ..., 8n\}$$

$$V(\overline{AAP_2}(n+1)) = \bigcup_{i=1}^{\frac{n+1}{2}}\{4i\text{-}3\} \,|\,|\, \bigcup_{i=1}^{\frac{n+1}{2}}\{12n\text{-}4i+4\} =$$

$$\{1, 5, ..., 2n\text{-}1\} \cup \{12n, 12n\text{-}4, ..., 10n+2\}$$

$$V(\overline{AAP_3}(n)) = \bigcup_{i=1}^{\frac{n+1}{2}}\{10n\text{-}4i+4\} \,|\,|\, \bigcup_{i=1}^{\frac{n-1}{2}}\{2n+4i+1\} =$$

$$\{10n, 10n\text{-}4, ..., 8n+2\} \cup \{2n+5, 2n+9, ..., 4n\text{-}1\}$$

As shown above, each nonzero element of Z_{12n+3} occurs exactly once in $V(C_1) \underline{\cup} V(C_2)$. Since, any cycle is a 2-regular graph and $V(C_1) \underline{\cup} V(C_2) = Z^*_{12n+3}$, then the set of cycles S = {C_1, C_2} satisfies the near-two-factor with focus zero.

Part 2: This part shows that the set of cycles S = {C_1, C_2} is a starter of cyclic (6n+1)-cycle decomposition of $2K_{12n+3}$ (namely that the list of differences from S covers Z^*_{12n+3} exactly twice). The list of differences from S is defined as $\Delta S = \Delta(C_1) \underline{\cup} V(C_1)$ and from Lemma 2.11, the list of differences from C_1 is:

$$\Delta(C_1) = \Delta(AAP_1(4n)) \underline{\cup} \{d(4n+3, 12n+2)\} \underline{\cup} \Delta(AAP_2(n+1)) \underline{\cup}$$
$$\{d(2n+1, 2n+3)\} \underline{\cup} \Delta(AAP_3(n)) \underline{\cup} \{d(4n+1, 2)\}$$

$$\Delta AAP_1(4n) = \left\{\pm|y_i\text{-}x_i| \,\middle|\, 1 \le i \le \frac{4n}{2}\right\} \underline{\cup} \left\{\pm|x_{i+1}\text{-}y_i| \,\middle|\, 1 \le i \le \frac{4n\text{-}2}{2}\right\} =$$

$$\left\{\pm|12n\text{-}8i+5| \,\middle|\, 1 \le i \le \frac{4n}{2}\right\} \underline{\cup} \left\{\pm|12n\text{-}8i+1| \,\middle|\, 1 \le i \le \frac{4n\text{-}2}{2}\right\} =$$

$$\left\{\pm|12n\text{-}8i+5| \,\middle|\, 1 \le i \le \frac{3n}{2}\right\} \underline{\cup} \left\{\pm|12n\text{-}8i+5| \,\middle|\, \frac{3n+2}{2} \le i \le \frac{4n\text{-}2}{2}\right\} \underline{\cup}$$

$$\left\{\pm|12n\text{-}8i+1| \,\middle|\, 1 \le i \le \frac{3n}{2}\right\} \underline{\cup} \left\{\pm|12n\text{-}8i+1| \,\middle|\, \frac{3n+2}{2} \le i \le \frac{4n\text{-}2}{2}\right\} =$$

$$\{12n\text{-}3, 12n\text{-}11, ..., 5\} \underline{\cup} \{6, 14, ..., 12n\text{-}2\} \underline{\cup}$$
$$\{3, 11, ..., 4n\text{-}5\} \underline{\cup} \{12n, 12n\text{-}8, ..., 8n+8\} \underline{\cup}$$
$$\{12n\text{-}7, 12n\text{-}15, ..., 1\} \underline{\cup} \{10, 18, ..., 12n+2\} \underline{\cup}$$
$$\{7, 15, ..., 4n\text{-}9\} \underline{\cup} \{12n\text{-}4, 12n\text{-}12, ..., 8n+12\}$$

$$\Delta AAP_2(n+1) = \left\{\pm|y_i\text{-}x_i| \,\middle|\, 1 \le i \le \frac{n+1}{2}\right\} \underline{\cup} \left\{\pm|x_{i+1}\text{-}y_i| \,\middle|\, 1 \le i \le \frac{n\text{-}1}{2}\right\} =$$

$$\left\{\pm|12n\text{-}8i+7| \,\middle|\, 1 \le i \le \frac{n+1}{2}\right\} \underline{\cup} \left\{\pm|12n\text{-}8i+3| \,\middle|\, 1 \le i \le \frac{n\text{-}1}{2}\right\} =$$

$$\{12n\text{-}1, 12n\text{-}9, ..., 8n+3\} \underline{\cup} \{4, 12, ..., 4n\} \underline{\cup}$$
$$\{12n\text{-}5, 12n\text{-}13, ..., 8n+7\} \underline{\cup} \{8, 16, ..., 4n\text{-}4\}$$

$$\Delta AAP_3(n) = \left\{\pm|y_i\text{-}x_i| \,\middle|\, 1 \le i \le \frac{n\text{-}1}{2}\right\} \underline{\cup} \left\{\pm|x_{i+1}\text{-}y_i| \,\middle|\, 1 \le i \le \frac{n\text{-}1}{2}\right\} =$$

$$\left\{\pm|8n\text{-}8i+3| \,\middle|\, 1 \le i \le \frac{n\text{-}1}{2}\right\} \underline{\cup} \left\{\pm|8n\text{-}8i\text{-}1| \,\middle|\, 1 \le i \le \frac{n\text{-}1}{2}\right\} =$$

$$\{8n\text{-}5, 8n\text{-}13, ..., 4n+7\} \underline{\cup} \{4n+8, 4n+16, ..., 8n\text{-}4\} \underline{\cup}$$
$$\{8n\text{-}9, 8n\text{-}17, ..., 4n+3\} \underline{\cup} \{4n+12, 4n+20, ..., 8n\}$$

$$\{d(4n+3, 12n+2)\} = \{8n\text{-}1, 4n+4\}$$

$$\{d(2n+1, 2n+3)\} = \{2, 12n+1\}\{d(4n+1, 1, 2)\} = \{4n\text{-}1, 8n+4\}$$

Now, we observe that each nonzero element of Z_{12n+3} appears exactly once in (ΔC_1). Since, C_2 is the relative cycle of C_1, then by Lemma 2.6, we obtain $\Delta(C_1) = \Delta(C_2)$. Thus, we conclude that each nonzero element of Z_{12n+3} appears exactly twice in ΔS. According to Lemma 2.4, for all odd integer n>1, the set of cycles S = {C_1, C_2} is a starter of cyclic (6n+1)-cycle decomposition of $2K_{12n+3}$.

Part 3: We show that all the generated cycles from the starter S = {C_1, C_2} contain no repetitions by showing that all the cycles of have different orbit.

Clearly, since, C_2 is the relative cycle of C_1, then from Lemma 2.7, we have orb(C_1) ≠ orb(C_2). Thus, all the generated cycles by repeatedly adding 1 modulo 12n+3 to S = {C_1, C_2} contain no repetitions.

From the former three parts, all the conditions of cyclic Butterfly (6n+1)-cycle decomposition of $2K_{12n+3}$ are satisfied. Thus, for any odd integer n>1, the set of cycles S = {C_1, C_2} is a starter of cyclic Butterfly (6n+1)-cycle decomposition of $2K_{12n+3}$.

Lemma 3.4: For any positive even integer n, there exists a cyclic Butterfly (6n+1)-cycle decomposition of $2K_{12n+3}$.

Proof: Let n be a positive even integer. Let, C_1 and C_2 be the (6n+1)-cycles of $2K_{12n+3}$ defined as:

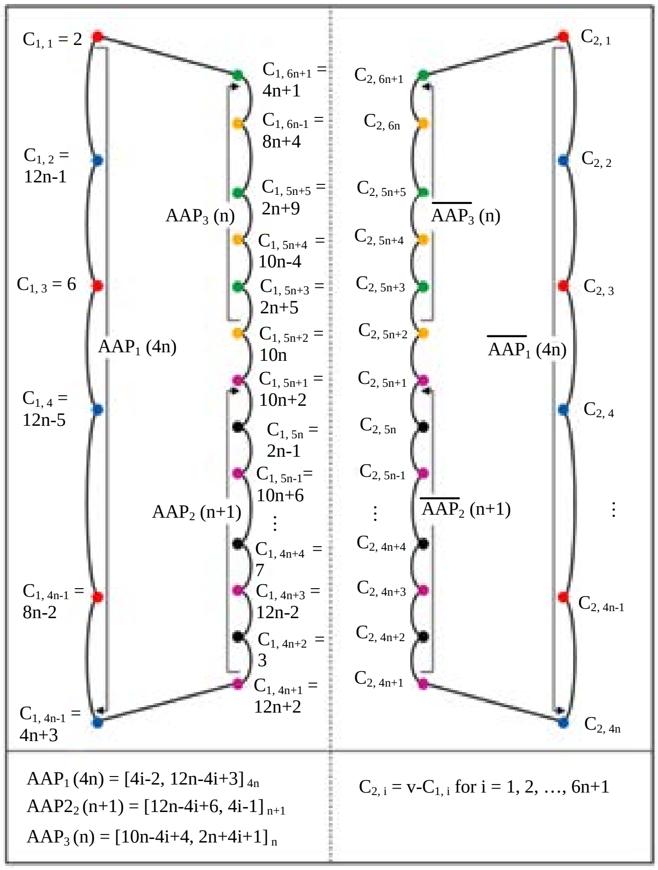

Fig. 2: The construction of C_1 and C_2 in $2K_{12n+3}$ when n is a positive even integer

$$
\begin{aligned}
C_1 &= (AAP_1(4n,\ AAP_2(n+1),\ AAP_3(n)) \\
C2 &= (\overline{AAP_1}(4n,\ \overline{AAP_2}(n+1),\ \overline{AAP_3}(n))
\end{aligned} \tag{11}
$$

Where:

$$
\begin{aligned}
AAP_1(4n) &= [2, 12n\text{-}1, 6, 12n\text{-}5, ..., 8n\text{-}2, 4n+3] \\
&= [4i\text{-}2,\ 12n\text{-}4i+3]_{4n} \\
AAP_2(n+1) &= [12n+2, 3, 12n\text{-}2, 7, ..., 10n+6,\ 2n\text{-}1, 10n+2] \\
&= [12n\text{-}4i+6,\ 4i\text{-}1]_{4n} \\
AAP_3(n) &= [10n, 2n+5, 10n\text{-}4, 2n+9, ..., 8n+4, 4n+1] \\
&= [10n\text{-}4i+4,\ 2n+4i+1]_n \\
\overline{AAP_1}(4n) &= [v-(4i\text{-}2),\ v\text{-}(12n\text{-}4i+3)]_{4n} \\
&= [12n\text{-}4i+5,\ 4i]_{4n} \\
\overline{AAP_2}(n+1) &= [v\text{-}(12n\text{-}4i+6),\ v\text{-}(4i\text{-}1)]_{n+1} \\
&\quad [4i\text{-}3, 12n\text{-}4i+4]_{n+1} \\
\overline{AAP_3}(n) &= [v\text{-}(10n\text{-}4i+4),\ v\text{-}((2n+4i+1)]_n \\
&\quad [2n+4i\text{-}1, 10n\text{-}4i+2]_n
\end{aligned}
$$

Since, n is a positive even integer, then any 4n-alternating arithmetic path and any n-alternating arithmetic path have even order while any (n+1)-alternating arithmetic path has an odd order. To make the construction in Eq. 11 easier to understand, Fig. 2 illustrates the construction of C_1 and C_2 in terms of their vertices as $C_i = (c_{i,\,1}, c_{i,\,2}, ..., c_{i,\,6n+1})$ for i = 1, 2.

This construction is similar to the construction of C_1 and C_2 in $2K_{(12n+3)}$, when n is an odd integer greater than one (that is proved in the previous lemma) with slight differences in the construction of $AAP_2(n+1)$, $AAP_3(n)$, $\overline{AAP_2}(n+1)$ and $\overline{AAP_3}(n)$. By applying the same strategy of proof as in Lemma 3.3, it can be proved that for any positive even integer n, the set of cycles $\{C_1, C_2\}$ is a starter of cyclic Butterfly (6n+1)-cycle decomposition of $2K_{(12n+3)}$.

Theorem 3.5: For every v = 3 (mod 12) with $v \geq 15$, there exists a cyclic Butterfly (v-1)/2)-cycle decomposition of $2K_v$.

Proof: Immediate from Lemma 3.3 and Lemma 3.4. By reviewing the construction of a starter of cyclic Butterfly (6n+1)-cycle decomposition of $2K_{(12n+3)}$ as shown in Fig. 1 and 2, the construction has a butterfly shape in which each cycle represents a side of symmetrical butterfly wings. If given one cycle C of the starter set, the other is the relative cycle of C.

CONCLUSION

This study has proposed the Butterfly k-cycle decomposition of $2K_v$ as an edge-decomposition of $2K_v$ into distinct k-cycles satisfy the near-two-factorization. In particular, the difference method has been exploited to construct cyclic Butterfly (v-1)/2)-cycle decomposition of $2K_v$ for the odd case v = 3 (mod 12) and this construction has been exemplified for the case v = 15. We expect this study can be developed and extended to construct cyclic Butterfly k-cycle decomposition of $2K_v$ for the case v odd.

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