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A Simple Expression for the Center of Mass of a System of Particles in a Two-Dimensional Space with Constant Positive Curvature

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Abstract: In this research is given a simple expression for the center of mass of a system of material points in a two-dimensional surface of Gaussian constant positive curvature. Using basic techniques of geometry, an expression in intrinsic coordinates is obtained and it is showed how this extend the definition for the Euclidean case. The argument is constructive and also serves for defining center of mass of a system of particles on the one-dimensional Sphere S^1_R.

Key words: Center of masses, conformal metric, spherical rule of the lever, geodesic, Euclidean case, one-dimensional sphere

INTRODUCTION

Center of mass (center of gravity or centroid) is a fundamental concept and its geometrical and mechanics properties are very important in the comprehension of a great variety of physical problems. Their definition for Euclidean spaces is elemental, nevertheless a definition for curved spaces is few frequent. Galperin (1993) makes an extensive explanation showing the possibilities of construction this concept in more general spaces and it is signalized the difficulties to defined in spaces of non zero curvature because this lack of linear structure. While it is true that the researcher synthesizes the basic properties of the center of mass in his approach appear some ones lacking of physical meaning such as the non conservation of total mass of system, in normal conditions or the presence of infinities velocities. Diacu *et al*. (2012) makes mention about of the difficulty for defining of center of mass in curved spaces. He provides a class of orbits in the curved n-body problem for which apparently “no point that could play the role of the center of mass is fixed or moves uniformly along a geodesic”. This proves that the equations of motion lack center-of-mass and linear-momentum integrals. But nevertheless, it is not provide a way to calculate or determinate this element. Applications in fields like Chemistry by Berrio-Guzman *et al*. (2015). By Aarseth (2003), it can found gravitational n-body simulations. The gravitational million-body problem has a multidisciplinary approach to star cluster dynamics (Heggie and Hut, 2003). Barnes and Hut (1986) was obtained a hierarchical O(N log N) force-calculation algorithm. Initial conditions for star clusters was taken for the dynamics of the n body problem (Kroupa, 2008). A good book for an introduction to celestial mechanics (Moulton, 1970). An algorithm for finding best matches in logarithmic expected time it is useful for codes in n-body problems (Friedman *et al*., 1977). Steinbach and Brooks (1994) obtain new spherical-cutoff methods for long-range forces in macromolecular simulation in this case the potential is similar to the potential for n-body problems. Multifocus image fusion using improved dual tree complex wavelet transform and discrete optimization method, this method of dual tree is useful for simulation in n-body problems (Srilatha, 2014). Location of collinear equilibrium points in the generalized photogravitational elliptic restricted three body problem is a difficult problem (Kumar and Ishwar, 2011). Artificial satellites, center of mass and the three body problem are related. For artificial satellites (Hillier and Balyan, 2019; Si Mohammed *et al*., 2007; Cooksley *et al*., 2007;

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Adediji *et al*., 2007; Ahmed *et al*., 2010; Emetere *et al*., 2016; Singh, 2017; Anand, 2017; Buliali *et al*., 2017; Phonphan, 2017; Basha and Vijayakumar, 2018; Nathan, 2017; Parthasarathy, 2017; Kaur and Singh, 2017; Rajab *et al*., 2018; Ramakrishnan, 2018; Nathan, 2018; Manivannan, 2018; Ik-Soo and Myung-Jin Bae, 2019; Sergey *et al*., 2019; Ashraf *et al*., 2019).

Garcia-Naranjo *et al*. (2016) in which they refer to a center of masses in a curved space, in a problem of celestial mechanics but they do not give a formula or a procedure to calculate it. Diacu (2012) study the n-body problem in spaces of constant curvature. Ortega *et al*. (2019) find the hyperbolic center of mass for a system of particles on the Poincare upper half-plane. Diacu *et al*. (2018) investigate the stability of fixed points and associated relative equilibria of the 3-body problem on S^1 and S^2. Shchepetilov (2006) is a book of mechanics on curved spaces.

In this research, the problem of gives a mathematical expression for computing the center of mass of a system of n particles sited on the two-dimensional sphere with Ratio R, S_R^2 is considered. Through stereographic projection of S_R^2 on the extend Complex plane $\hat{C}$, endowed with the conformal metric (Perez-Chavela and Reyes-Victoria, 2012):

$$ds^2 = \frac{4R^4\, dwd\bar{w}}{\left(R^2+|w|^2\right)^{2'}} \quad (1)$$

Both, $\hat{C}$ with the metric (Eq. 1) and S_R^2 with the Euclidean metric have the same Gaussian curvature $K = 1/R^2$ and for the Minding's Theorem belong to the isometric differentiable class (Do Carmo, 1976; Dubrovin *et al*., 1984; Perez-Chavela and Reyes-Victoria, 2012) is proved the equivalence of the n-body problem for both models, the one on the sphere S_R^2 with the Euclidean metric of ambient space R^3 and the other on the extend Complex plane $\hat{C}$ with the metric (Eq. 1). Following the basic methods of the geometry, we obtain, here, the expression for the center of mass for a system of n particles sited in the sphere S_R^2 of arbitrary Radio R.

This study is organized as follow: in section 1 are introduced some concepts relative to center of mass in the euclidean spaces. In section 2, are remembered some properties of stereographical projection and it is proceeded to deduce the expression for the center of mass, for two particles on the sphere from the "spherical rule of the lever" (Galperin, 1993) extended to surface of S_R^2, this case can be reduced to the Sphere S_R^1, under a suitable two rotations and using the invariance of the relative position of center of mass under isometrics of sphere (Galperin, 1993). Once obtained the expression for the center of mass for two particles in S_R^1, it can be extended naturally to a system of n particles in S_R^1 and the same way, to a system of particles in S_R^2. The expression obtained, here, satisfies the five axioms for the "Axiomatic Centroid" established by Galperin (1993).

MATERIALS AND METHODS

One-dimensional Euclidean case: Let consider two particles with positive masses sited in the real line at the points x_1 and x_2. The center of mass of system is defined be the point x_c:

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \quad (2)$$

A direct calculation shows that $m_1|x_c\text{-}x_1| = m_2|x_c\text{-}x_2|$ (Euclidean rule of the lever). It is easy to prove that x_c is the unique point in the segment (geodesic) joining x_1 and x_2 with this property. This definition can be extend to more dimensions in Euclidean spaces. This definition can not be extended to spaces in general because is possible that in such spaces is not defined a linear structure. But with the "rule of the lever" in mine is possible carries this definition to Riemannian surfaces as we shall see later.

In Fig. 1 showed that: $s_1 = \theta_1 R$, $s_2 = \theta_2 R$ $\tan\theta_1 = d_1/R$ and $\tan\theta_2 = d_2/R$. Moreover, when $R\to\infty, \theta_1, \theta_2 \to 0$ and $s_1 \to d_1$ y $s_2 \to d_2$ then,

$$1 = \lim_{R\to\infty} \frac{\tan\theta_1}{\theta_1} = \lim_{R\to\infty} \frac{d_1/R}{s_1/R} = \lim_{R\to\infty} \frac{d_1}{s_1}$$

Hence, $s_1 \to d_1$ and in similar way $s_2 \to d_2$.

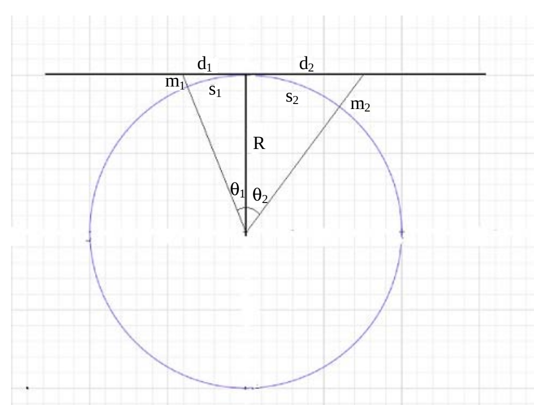

Fig. 1: Center of masses in the circumference and the real line

RESULTS AND DISCUSSION

Center of masses in a two-dimensional spherical space Some observations about the stereographic projection: Let $P:S_R^2 \to \hat{C}$ the stereographic projection, then for $(x, y, z) \in S_R^2$, we have P(x, y, z) = w = u+iv where u = Rx/R-z and v = Ry/R-z and P(0, 0, R) = ∞ and moreover the inverse projection is $P^{-1}:\hat{C} \to S_R^2$ and:

$$P^{-1}(u+iv) = \frac{2R^2u}{u^2+v^2+R^2}, \frac{2R^2v}{u^2+v^2+R^2}, \frac{R(u^2+v^2-R^2)}{u^2+v^2+R^2}$$

P leaves invariant the set $\{(x, y, z) \in S_R^2 : z = 0\}$ and it is the same set and will called equator. For analogy, the set $\{W \in C : |W| < R\}$ will called South Hemisphere and the set $\{W \in C : |W| > R\}$ will called North Hemisphere, w = 0 is called South Pole and w = ∞ will be called North Pole. P^{-1} transform lines through the origin in meridians and the circles with center in origin, $\{W \in C : |w| < \text{const.}\}$ in parallels. Finally, for two complex numbers w_1, w_2, if $|w_1||w_2| = R^2$, then each point is sited in two parallels, the same radio and symmetric respect to Equator.

If consider the stereographic projection of the one-dimensional sphere S_R^2 on the real line, then the above equation is reduced to:
P(x, y) = u where u = Rx/R- y and P(0, R) = ∞ and moreover the inverse projection is:

$$P^{-1}(u) = \left(\frac{2R^2u}{u^2+R^2}, \frac{R(u^2-R^2)}{u^2+R^2}\right)$$

In this last case, the length of arc from the South Pole P_s to arbitrary point (x, y) is:

$$s = |_0^u \frac{2R^2dt}{t^2+R^2} = 2R\arctan\left(\frac{u}{R}\right)$$

More general, the length of arc s from the point $Q_1(x_1, y_1)$ to $Q_2(x_2, y_2)$, if their stereographical projections are u_1 and u_2 is:

$$s = 2R\left(\arctan\left(\frac{u_2}{R}\right) - \arctan\left(\frac{u_1}{R}\right)\right)$$

Consider now two masses m_1, m_2 sited in the points Q_1, Q_2, respectively and let $Q_c(x_c, y_c)$ the coordinates of center of mass and s_1 the length of arc from Q_1 to Q_2 and s_2 the length of arc from Q_c to Q_2. Then, from the relation (spherical rule of the lever) $m_1s_1 = m_2s_2$ it follows:

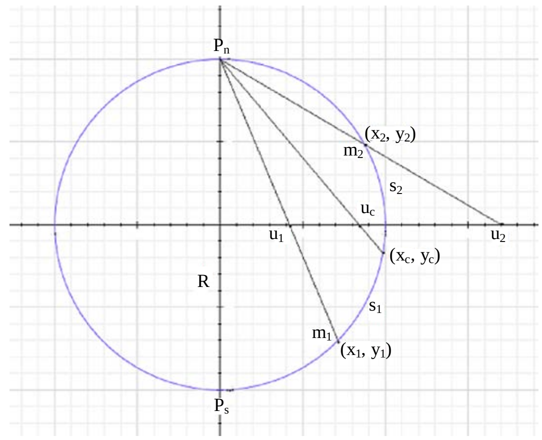

Fig. 2: Center of mass on the one-dimensional Sphere S_R^2

$$2Rm_1\left(\arctan\left(\frac{u_c}{R}\right) - \arctan\left(\frac{u_1}{R}\right)\right) = 2Rm_2\left(\arctan\left(\frac{u_2}{R}\right) - \arctan\left(\frac{u_c}{R}\right)\right)$$

Therefore:

$$\text{Arc}\tan\left(\frac{u_c}{R}\right) = \frac{1}{m_1+m_2}\left(m_1\arctan\left(\frac{u_1}{R}\right) + m_2\arctan\left(\frac{u_2}{R}\right)\right) \quad (3)$$

Figure 2 illustrates the situation. This concept can be extended a n particles with masses m_1, m_2, ..., m_n sited in the points (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) on S_R^2 and with stereographical projections u_1, u_2, ..., u_n in the real line the following way:

$$\text{Arctan}\left(\frac{u_c}{R}\right) = \frac{1}{m}\sum_{k=1}^{n} m_k \arctan\left(\frac{u_k}{R}\right) \quad (4)$$

where, $m = \sum_{k=1}^{n} m_k$.

Center of masses for a system of two particles in S_R^2: Now, we extend the "rule of the lever" to context more general: let a Riemannian surface T and two particles with masses m_1, m_2 sited in the points s_1, $s_2 \in T$, respectively, then the center of mass is defined the point s_c in the geodesic joining s_1 to s_2 such that is verified the following relation:

$$m_1d(s_1, s_c) = m_2d(s_2, s_c)$$

where, d is the metric in T. For the case of S_R^2, geodesics are great circles and distances are measure take the shorter arc.

Let m_1, m_2, ..., m_n, n masses sited, respectively in the points (x_1, y_1, z_1), (x_2, y_2, z_2), ..., (x_n, y_n, z_n) in S_R^2 with stereographical projections w_1, w_2, ..., w_n in the extend complex plane and let w_c their center of mass, then it is satisfy the next relation:

$$\text{Arctan}\left(\frac{w_c}{R}\right) = \frac{1}{m}\sum_{k=1}^{n} m_k \arctan\left(\frac{w_k}{R}\right)$$

where, $m = \sum_{k=1}^{n} m_k$. Note that in 5, if multiplied by R in both sides and we take the limit when $R \to \infty$ then result:

$$w_c = \frac{1}{m}\sum_{k=1}^{n} m_k w_k$$

And this correspond to the equation for the center of mass in the Euclidean complex plane, that is the complex plane (or R^2), with Euclidean metric and zero curvature.

CONCLUSION

There is a formula to calculate centers of mass in curved spaces that extends this concept of Euclidean spaces (of null curvature).

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STATEMENT OF SIGNIFICANCE

In this research we present for the first time a formula to calculate centers of mass in curved spaces.

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