Journal of Engineering and Applied Sciences 14 (23): 8812-8816, 2019
ISSN: 1816-949X

Asymptotic Solution for a Temperature Problem with the Fluid of Poiseuille into the Navier-Stokes Equations Applying the Boundary Layer Method

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Abstract: In this study, we study the variation of temperature in a fluid with uniform velocity between two parallel planes located at a constant distance that contains certain boundary layer. We study the region outside of boundary layer considering it in one dimension. After this, we study the inner of the boundary layer using certain transformation and a method called combination of variables and it arises in the main term of the expansion, after this we apply the boundary conditions to find the approximate solution of the temperature.

Key words: Boundary layer, inner expansion, outer expansion, Prandtl's matching condition, parallel planes, solution

INTRODUCTION

In physics the problem of the transfer of energy between two bodies has resulted in many lines of research around the world. In this study, we study the temperature of a fluid near a solid wall where the temperature shall be obtained from a partial differential equation which will be resolved by the method of the matched asymptotic expansions with some boundary conditions. By Diaz-Salgado *et al.* (2014), it was treated the fluid of Coutte and the boundary layer. By Bush (1992), we can encounter the boundary layer method. By Murray (1984), we can find the background of asymptotic analysis. By Ayala-Hernandez and Hijar (2016), they saw the method multiparticle collision dynamics is reliable to simulate cylindrical Poiseuille flow for a wide range of system sizes, applied pressure gradients and viscosities and densities of the simulated fluids. We studied the similarity of the partial differential equation by Strauss (2007). By Ma and Wang (2009), they derive a rigorous characterization of the boundary-layer and interior separations in the Taylor-Couette-Poiseuille flow. By McKernan (2006), they have that from the Navier-Stokes and continuity equations, a spatial-state spectral model of Poiseuille flow with transpiration action of the wall. By McKernan *et al.* (2006), a method for the incorporation of wall transpiration in a Poiseuille flow model in a linearized plane was presented.

MATERIALS AND METHODS

Preliminaries: In this study, we present some preliminary data that are used throughout the document. The temperature field can be obtained for the next equation:

$$u\frac{\partial T}{\partial x}+\nu\frac{\partial T}{\partial y}=\frac{1}{Pe}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \quad (1)$$

Where:
u = U/U_0
• = V/U_0
U_0 = The characteristic velocity

Consider fluid flow inside two parallel walls of length L and the walls are separated by a distance H and the temperatures are held constant $T = T_w$. Suppose a fluid flows between the two walls with a velocity flow profile $U = 1.5U_0y(2\text{-}y)$ and $V = 0$. Replace values of u and •, respectively, in Eq. 1. We obtain that:

$$u=\frac{U}{U_0}=1.5y(2\text{-}y)$$

$$\nu=\frac{V}{U_0}=0$$

$$\frac{1.5y(2\text{-}y)\partial T}{\partial x}=\frac{1}{Pe}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

The boundary conditions for the problem are:

$$T(0,y)=T_0 \quad 0<y<h$$
$$T(x,0)=T_w \quad 0<x<1$$
$$T(x,h)=T_w \quad 0<x<1$$

Here, T_0 is the fluid inlet temperature which is a constant value and we suppose that $T_0>T_w$. Temperature profile at the exit is T_w:

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$$T(1, y) = T_w \quad 0<y<h$$

RESULTS AND DISCUSSION

Technical development

Theorem 3.1: The composite one-term approximation $\left(T_0^{comp}\right)$ of the partial differential equation:

$$\frac{1.5y(2\text{-}y)\partial T}{\partial x} = \frac{1}{Pe}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \quad (2)$$

With the following boundary conditions:

$$T(0, y) = T_0 \quad 0<y<h$$
$$T(x, 0) = T_w \quad 0<x<1$$
$$T(x, h) = T_w \quad 0<x<1$$

is:

$$T_0^{comp} = T_0+(T_w\text{-}T_0)e^{-\frac{(1\text{-}x)y}{\varepsilon}}+\frac{(T_w\text{-}T_0)}{\Gamma\left(\frac{1}{3}\right)}\Gamma\left(\frac{1}{3}, \frac{y^3}{9\varepsilon x}\right)+\frac{(T_w\text{-}T_0)}{\Gamma\left(\frac{1}{3}\right)}\Gamma\left(\frac{1}{3}, \frac{(h\text{-}y^3)}{9\varepsilon x}\right)$$

Proof: For finding T_0^{comp}, we must have T_0^{in}, T_0^{out} and T_0^{match} because T_0^{comp} is formed as follows:

$$T_0^{comp} = T_0^{out}+T_0^{in}(x=1 \text{ boundary layer})\text{-}T_0^{match}+T_0^{in}(y=0 \text{ boundary layer})\text{-}T_0^{match}+T_0^{in}(y=h \text{ boundary layer})\text{-}T_0^{match}$$

Replacing the small parameter 1/Pe by • into Eq. 2, we obtain:

$$\frac{1.5y(2\text{-}y)\partial T}{\partial x} = \varepsilon\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \quad (3)$$

We analyze the boundary layer in one dimension:

$$\frac{1.5y(2\text{-}y)\partial T}{\partial x} = \varepsilon\left(\frac{\partial^2 T}{\partial x^2}\right) \quad 0<x<1$$
$$T(0) = T_0 \text{ and } T(1) = T_w$$

from this problem, we have the following general solution:

$$T = A+\frac{Be^{\frac{1.5y(2\text{-}y)x}{\varepsilon}}}{y}$$

where, A and B are constants and by the initial conditions, we have:

$$A+B = T_0 \text{ and } A+Be^{\frac{1.5y(2\text{-}y)}{\varepsilon}} = T_w$$

and solving this system, we obtain:

$$A = \frac{T_0\text{-}T_w e^{-\frac{1.5y(2\text{-}y)}{\varepsilon}}}{1\text{-}e^{-\frac{1.5y(2\text{-}y)}{\varepsilon}}}$$

$$B = \frac{e^{-\frac{1.5y(2\text{-}y)}{\varepsilon}}(T_w\text{-}T_0)}{1\text{-}e^{-\frac{1.5y(2\text{-}y)}{\varepsilon}}}$$

and replacing A and B:

$$T = \frac{T_0\text{-}T_w e^{-\frac{1.5y(2\text{-}y)}{\varepsilon}}+(T_w\text{-}T_0)e^{-\frac{(1\text{-}x)1.5y(2\text{-}y)}{\varepsilon}}}{1\text{-}e^{-\frac{1.5y(2\text{-}y)}{\varepsilon}}}$$

as $e^{-1.5y(2\text{-}y)}$/•• 0, we obtain:

$$T = T_0+(T_w\text{-}T_0)e^{-\frac{(1\text{-}x)1.5y(2\text{-}y)}{\varepsilon}}+, ...,$$

by analyzing one-dimensional case, O(•) is the boundary layer thickness occurs at x = 1, we now return to the two-dimensional problem. The walls y = 0 and y = h will have a parabolic profile boundary layer with thickness $O(3\sqrt{\varepsilon})$ and the temperature in the outer region is the same as is the one-dimensional case namely T_0. The outer expansion is:

$$T^{out}(x, y, \varepsilon) = T_0^{out}(x, y)+\varepsilon T_1^{out}(x, y)+\varepsilon^2 T_2^{out}(x, y)+, ...,$$

And replacing in Eq. 3, we obtain:

$$O(1): \frac{1.5y(2\text{-}y)\left(\partial T_0^{out}(x, y)\right)}{\partial x} = 0$$

$$O(\varepsilon): \frac{1.5y(2\text{-}y)\left(\partial T_1^{out}(x, y)\right)}{\partial x} = \frac{\partial^2 T_0^{out}(x, y)}{\partial x^2}+\frac{\partial^2 T_0^{out}(x, y)}{\partial y^2}$$

$$O(\varepsilon^2): \frac{1.5y(2-y)(\partial T_2^{out}(x,y))}{\partial x} = \frac{\partial^2 T_1^{out}(x,y)}{\partial x^2} + \frac{\partial^2 T_1^{out}(x,y)}{\partial y^2}$$

$$O(\varepsilon^n): \frac{1.5y(2-y)(\partial T_n^{out}(x,y))}{\partial x} = \frac{\partial^2 T_{n-1}^{out}(x,y)}{\partial x^2} + \frac{\partial^2 T_{n-1}^{out}(x,y)}{\partial y^2}$$

for n• 1. And $T^{out}_0(x, y) = g(y)$ where, g is a function that satisfies:

$$T_0^{out}(x,y) = T_0$$

It is only necessary to consider the boundary layer at y = 0, the y = h boundary layer will be a reflection of it. Suppose we have the transformation of the form s = y/•^{p}. Replacing in Eq. 3, we obtain that:

$$\left(\frac{6}{5}s-\frac{3}{5}\varepsilon^p s^2\right)\frac{\partial T}{\partial x} = \frac{1}{\varepsilon^{p-1}}\frac{\partial^2 T}{\partial x^2} + \frac{1}{\varepsilon^{3p-1}}\frac{\partial^2 T}{\partial s^2}$$

The left-hand side of equation is O(1) and the principle of least degeneracy requires that p = 1/3, then, the one-term inner expansion $T^{in}_0(x, s)$ holds that:

$$\frac{6s}{5}\frac{\partial T_0^{in}}{\partial x} = \frac{\partial^2 T_0^{in}}{\partial s^2} \tag{4}$$

with $T_0^{in}(s = 0) = T_w$. We will find the general solution of this partial differential equation by combination of variables (Strauss, 2007). Assume:

$$T_0^{in}(x,y) = T_0\phi(\eta)$$

where:

$$\eta = \frac{s}{\delta(x)}$$

and •(x) function that will be determined. Now, we obtain the derivatives with respect to x and s:

$$\frac{\partial T_0^{in}}{\partial x} = T_0\frac{\partial(\phi(\eta))}{\partial x} = -T_0\frac{\eta}{\delta}\frac{\partial\phi}{\partial\eta}\frac{\eta\delta(x)}{\partial x}$$

and:

$$\frac{\partial^2 T_0^{in}}{\partial s^2} = T_0\frac{1}{\delta^2(x)}\frac{\partial^2\phi}{\eta^2}$$

Substituting these results in Eq. 4 leads the following differential Eq. 5:

$$\phi''+\frac{6}{5}\eta^2\delta^2(x)\delta'(x)\phi' = 0 \tag{5}$$

We suppose that:

$$\delta^2(x)\delta'(x) = C \tag{6}$$

where, C is an arbitrary constant. We set C = 5/3, then, we write Eq. 5 as:

$$\phi''+2\eta^2\phi' = 0 \tag{7}$$

The general solution of Eq. 7 can be written as:

$$\phi(\eta) = a_1+a_2\Gamma\left(\frac{1}{3},\frac{2\eta^3}{3}\right) \tag{8}$$

where, a_1 and a_2 are constants and:

$$\Gamma\left(\frac{1}{3},\frac{2\eta^3}{3}\right) = \int_{\frac{2\eta^3}{3}}^{\infty} t^{-\frac{2}{3}}\exp(-t)dt$$

The solution of Eq. 6 with initial condition • = 0 is:

$$\delta(x) = \sqrt[3]{5x}$$

Then:

$$\eta = \frac{s}{\delta} = \frac{s}{\sqrt[3]{5x}}$$

From Eq. 8, we obtain:

$$\phi(x,s) = a_1+a_2\Gamma\left(\frac{1}{3},\frac{2s^3}{15x}\right)$$

Therefore:

$$T_0^{in} = T_w - a_2\Gamma\left(\frac{1}{3}\right)+a_2\Gamma\left(\frac{\frac{1}{3,2s^3}}{15x}\right) \tag{9}$$

with $T_0^{in}(s = 0) = T_w$. We can check that it is a solution for Eq. 4. Therefore, the boundary conditions are:

$$T_0^{in}(x,s=0) = T_w \text{ for } x>0$$

$$T_0^{in}(x,s\to\infty) = T_w - a_2\Gamma\left(\frac{1}{3}\right) \text{ for } x>0$$

$$T_0^{in}(x\to 0,s) = T_w - a_2\Gamma\left(\frac{1}{3}\right) \text{ for } s>0$$

And the Prandtl's matching condition is:

$$\lim_{s\to\infty} T_0^{in}(x,s) = \lim_{y\to 0} T_0^{out}(x,y)$$

$$T_w - a_2\Gamma\left(\frac{1}{3}\right) = T_0$$

so that:

$$a_2 = \frac{T_w - T_0}{\Gamma\left(\frac{1}{3}\right)}$$

Now, T^{in}_0 is:

$$T_0^{in}(x,y) = T_0 + \frac{T_w - T_0}{\Gamma\left(\frac{1}{3}\right)}\Gamma\left(\frac{1}{3}, \frac{2y^3}{15\varepsilon x}\right) \quad (10)$$

T_0^{match} is T_0. And thus, we can write the composite one-term approximation as:

$$T_0^{comp} = T_0 + (T_w - T_0)e^{-\frac{(1-x)y}{\varepsilon}} + \frac{(T_w - T_0)}{\Gamma\left(\frac{1}{3}\right)}\Gamma\left(\frac{1}{3}, \frac{2y^3}{15\varepsilon x}\right) + \frac{(T_w - T_0)}{\Gamma\left(\frac{1}{3}\right)}\Gamma\left(\frac{1}{3}, \frac{2(h-y)^3}{15\varepsilon x}\right)$$

Now, we show a program for a numerical solution to the heat equation:

- clear all
- clc
- EPS = 0.1
- N = 9
- M = 9
- dx = 1/N
- dy = 1/M:

$$CO = 2*EPS*\left(\frac{1}{(dx)^2} + \frac{1}{(dy)^2}\right)$$

$$CN = \frac{EPS}{(dy)^2}$$

- CS = CN

$$CE = \frac{EPS}{(dx)^2} - \frac{0.5}{dx}$$

$$CW = \frac{EPS}{(dx)^2} - \frac{0.5}{dx}$$

- for j = 2: M-1
- T(1, j) = 1
- end
- for i = 1: N
- T(i, 1) = 0
- end
- for j = 2: M-1
- T(N, j) = 0
- end
- for i = 1: N
- T(i, M) = 0
- end
- j = 2: M-1
- i = 2: N-1
- foriter = 1:6:

$$T(i, j) = \frac{CN*T(i, j+1) + CS*T(i, j-1) + CE*T(i+1, j) + CW*T(i-1, j)}{CO}$$

- end
- contourf(T)

CONCLUSION

When analyzing the data of the temperature problem, a great practical importance is the speed of heat transfer on the surface of the walls. When applying the proposed method with which we develop the problem, the approximations are consistent with the phenomenon of temperature and its transfer.

This result is a very good approximation, since, the solution can be expressed as an incomplete gamma function when using Asymptotic methods being more specific boundary layer.

ACKNOWLEDGEMENT

The researchers express their deep gratitude to Universidad de Cartagena for financial support.

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