

On Chromatic Polynomial of Elementary h-Uniform Hyper Cycles

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Abstract: One of the most popular and useful areas of graph theory is graph colorings. Frequently, we are concerned with determining the least number of colors with which we can achieve a proper coloring on a hypergraph. Furthermore, we want to count the possible number of different proper colorings on a hypergraph with a given number of colors which is called the chromatic polynomial.

Key words: Graph theory, graph colorings, hypergraph, polynomial, chromatic, h-uniform, hypercycle

INTRODUCTION

Let $P(G, \lambda)$ denote the chromatic polynomial of a graph G . Two graphs G and H are chromatically equivalent, written $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique, written X -unique, if for any graph H , $G \sim H$ implies that G is isomorphic with H . Many papers have been published about chromaticity of graphs which study the chromatically equivalent and the chromatically unique of graphs, see, for example (Khalaf and Peng, 2009, 2010). For more details about chromatic polynomials and chromaticity of graphs, we refer to the monograph (Dong *et al.*, 2005). Hypergraph theory is a part of general study of combinatorial properties of finite families of finite sets. A hypergraph is defined to be a family of hypergraphs which are sets of vertices of cardinality not necessarily 2 (as for graphs).

The concept of a hypergraph was first introduced by C. Berge at Tihany colloquium in 1960 to extend, simplify and unify the results about graph and he developed many properties of vertex coloring of a hypergraph (Berge, 1973, 1989; Barchadt and Lazuka, 2007).

MATERIALS AND METHODS

Definition 1.1; Berge (1973, 1989): The couple $H = (V, \epsilon)$ where $\{v_1, v_2, \dots, v_n\}$ ($n \in \mathbb{N}$) is a finite set of vertices. And $\epsilon(H) = \{e_1, e_2, \dots, e_m\}$ ($m \in \mathbb{N}$) is an edge-sets with $e_i \subseteq V(H)$ and $|e_i| \geq 1$ ($1 \leq i \leq m$), then we say that H is a hypergraph of order $|V| = n$ and size $|\epsilon| = m$ where e_i is called an edge or hyperedge.

Definition 1.2; Zhao (2009): Let $H = (V, \epsilon)$ be a hypergraph. If no hyperedge of $\epsilon(H)$ is subset of any other hyperedge of $\epsilon(H)$, then we say that H is sperner.

Definition 1.3; Allagan (2014): Let $H = (V, \epsilon)$ be a hypergraph then we say that H is a simple hypergraph, if all hyperedges are different.

Definition 1.4; Borowiecki and Ewa (2000): Let $H = (V, \epsilon)$ be a hypergraph of size $m \in \mathbb{N}$ and Let $e_i, e_j \in \epsilon(H)$ ($1 \leq i, j \leq m$), if $e_i \cap e_j = \emptyset$ or $e_i \cap e_j = \{v\}$ where, $v \in V(H)$ then we say that H is a linear hypergraph.

Definition 1.5; Walter (2009): Let $H = (V, \epsilon)$ be a hypergraph of size $m \in \mathbb{N}$ and Let $e_i \in \epsilon(H)$ ($1 \leq i \leq m$) be a hyperedge if $|e_i| = h$ for each i , then we say that H is h -uniform hypergraph, notice that 2-uniform hypergraph is just a graph. The number of hyperedges containing a vertex $v \in V(H)$ is its degree $d_H(v)$.

Definition 1.6: Let $H = (V, \epsilon)$ be a hypergraph and let $u, v \in V(H)$ where $u \neq v$, then u, v are lying in the same component, if there are vertices $u = v_0, v_1, \dots, v_k = v \in V(H)$ and hyperedges $e_1, \dots, e_k \in \epsilon(H)$ such that $v_{i-1}, v_i \in e_i$ for each i ($1 \leq i \leq k$). Thus, if H has only one component then H is connected, otherwise H has $r \geq 2$ ($r \in \mathbb{N}$) connected components.

Definition 1.7; West (2005): Let $H = (V, \epsilon)$ be a hypergraph. A hyperpath (or path) of length L joining two vertices u and v in H is a subhypergraph consisting of $L+1$ distinct vertices $u = v_0, v_1, \dots, v_L = v$ and L distinct hyperedges e_1, e_2, \dots, e_L of and such that $v_{j-1}, v_j \in e_j$ for each j ($1 \leq j \leq L$). An example is given in Fig. 1.



Fig. 1: A hyperpath

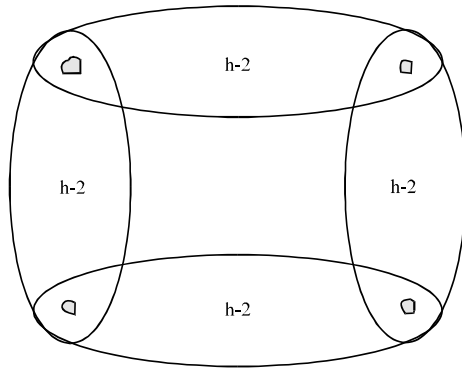


Fig. 2: C_4^h an elementary hypercycle

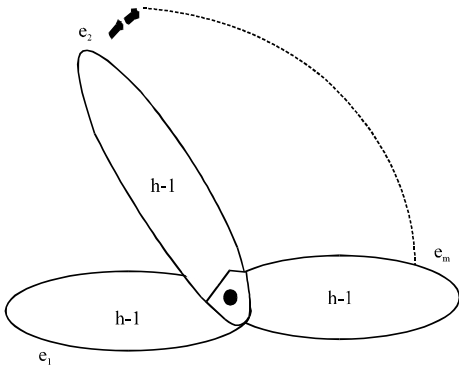


Fig. 3: A hyperstar

Definition 1.8: Let $H = (V, \mathcal{E})$ be a hypergraph. A hypercycle (cycle) C of Length L in H is a subhypergraph connecting L different vertices v_1, \dots, v_L and L different hyperedges e_1, e_2, \dots, e_L such that $v_j, v_{j+1} \in e_j$ for each j ($1 \leq j \leq L-1$) and $v_1, v_L \in e_L$.

If $d_c(v_j) = 2$ for each j and $d_c(y) = 1$ for each other vertex $y \in \bigcup_{j=1}^L e_j$, then we say that C is an elementary hypercycle. An elementary h -uniform hypercycle with $m \geq 3$ is denoted by C_m^h and with 2 hyperedges by C_2^h or $C(2, h)$. If we join the end vertices of an hyperedge, then we get a hypercycle of length 1 denoted by C_1^h . Elementary hypercycle C_4^h is shown in Fig. 2.

Definition 1.9; Donmen (1993): Let $H = (V, \mathcal{E})$ be a connected h -uniform hypergraph without hypercycles. Then, H is called h -uniform hypertree and denoted T_m^h where, m is the number of its hyperedges. If all hyperedge of T_m^h are intersecting in only one vertex then we say that T_m^h is a hyperstar as shown in Fig. 3.

Definition 1.10; Borowiecki and Lazuka (2000) and Drgas-Burchardt and Lazuka (2006): Let $H = (V(H), \mathcal{E}(H))$ be a hypergraph and $\lambda \in \mathbb{N}$, a λ -coloring of H is a function $f: V(H) \rightarrow \{1, \dots, \lambda\}$ such that for each $e \in \mathcal{E}(H)$, here exists x, y in e for which $f(x) \neq f(y)$.

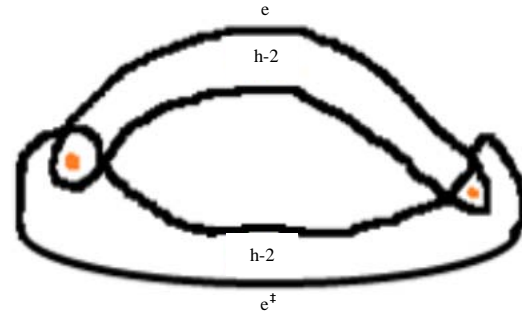


Fig. 4: e^* inactive hyperedge

The number of different λ -coloring of H is given by a polynomial $P(H, \lambda)$ of degree $|V(H)|$ in λ called the chromatic polynomial of H .

Definition 1.11; Barchadt and Lazuka (2007): Two hypergraphs H_1 and H_2 are said to be chromatically equivalent, denoted by $(H_1 \sim H_2)$, if $P(H_1, \lambda) = P(H_2, \lambda)$. A simple hypergraph H_1 is chromatically unique, if $H_1 \sim H_2$ for every simple hypergraph H_2 such that $H_1 \cong H_2$; that is the structure of H_1 is uniquely determined up to isomorphism by its chromatic polynomial.

Definition 1.12; Barchadt and Lazuka (2007): Let H be a hypergraph. If we drop an edge $e \in \mathcal{E}(H)$ we get a hypergraph $H - e$ which is chromatically equivalent to H , then e is called chromatically inactive. Otherwise, e is said to be chromatically active. The edge e^* in Fig. 4 is chromatically inactive H .

RESULTS AND DISCUSSION

Some known results on enumeration of $P(H, \lambda)$

Theorem 2.1; Borowiecki and Lazuka (2000): Let H be a hypergraph with order n and size m , then:

$$P(H, \lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_2\lambda^2 + a_1\lambda$$

where, $a_i = \sum_{j=0}^i (-1)^j N(i, j)$ for $1 \leq i \leq n-1$ and $N(i, j)$ denoted the number of subhypergraphs of H with $|V(H)|$ vertices, i components and j edges.

Theorem 2.2; Kashif (2011): Let $H = (V, \mathcal{E})$ be a hypergraph and let $H - e$ be the hypergraph obtained by deleting some edge $e \in \mathcal{E}(H)$, $H \cdot e$ be the hypergraph obtained by contracting all vertices in e to a common vertex v and dropping all chromatically inactive edges then:

$$P(H, \lambda) = P(H - e, \lambda) - P(H \cdot e, \lambda)$$

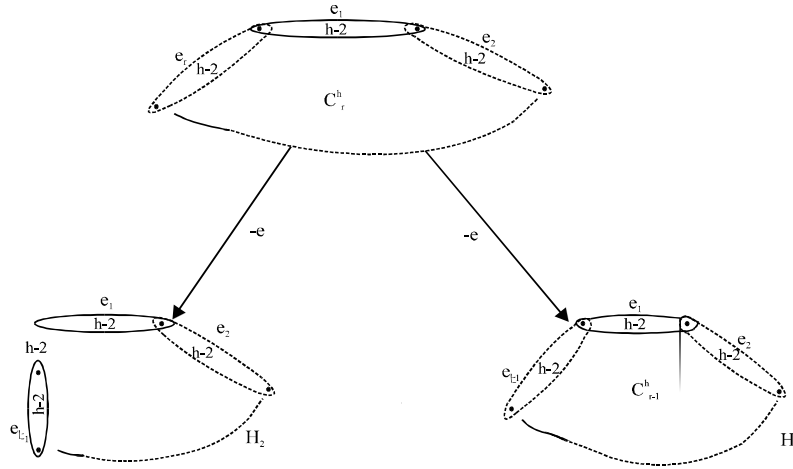


Fig. 5: Hypercycle

Theorem 2.3; West (2005): Assume that H_1, H_2, \dots, H_m be a connected components of hypergraph H , then:

$$P(H, \lambda) = \prod_{i=1}^m P(H_i, \lambda)$$

Theorem 2.4; Bokhary et al. (2009): Let H be a hypergraph such that:

$$H = H_1 \cup H_2, \dots, \cup H_m \text{ for } m \geq 2$$

Where:

$$H_i \cap H_j = K_p, i \neq j \text{ and } \bigcap_{i=1}^m H_i = K_p$$

where, K_p is a complete graph with $P \geq 1$ vertices, then:

$$P(H, \lambda) = [P(K_p, \lambda)]^{1-m} \prod_{i=1}^m P(H_i, \lambda)$$

Theorem 2.5; Borowiecki and Ewa (2007): Let T_m^h be a linear h -uniform hypertree with size m and $h \geq 2$ then:

$$P(T_m^h, \lambda) = \lambda(\lambda^{h-1} - 1)^m$$

Theorem 2.6; Tomescu (1998): Let C_m^h is elementary h -uniform hypertree with size m then:

$$P(C_m^h, \lambda) = (\lambda^{h-1} - 1)^m + (-1)^m (\lambda - 1)$$

Elementary h -uniform hypercycle: In this section, we give the recurrence relation for the chromatic polynomial of an elementary h -uniform hypercycle.

Example 3.1: Let C_3^3, C_4^3, C_5^3 be three elementary $h = 3$ uniform hypercycles. Apply the theorem (3.3.6) on C_3^3, C_4^3, C_5^3 resp:

$$\begin{aligned} P(C_3^3, \lambda) &= (\lambda^2 - 1)^3 + (-1)^3 (\lambda - 1) = \\ &\lambda^6 - 3\lambda^4 + 3\lambda^2 - 1 - (\lambda - 1) = \lambda^6 - 3\lambda^4 + 3\lambda^2 - 1 - \lambda + 1 = \\ &\lambda^6 - 3\lambda^4 + 3\lambda^2 - \lambda \end{aligned} \quad (1)$$

And:

$$\begin{aligned} P(C_4^3, \lambda) &= (\lambda^2 - 1)^4 + (-1)^4 (\lambda - 1) = \\ &\lambda^8 - 4\lambda^6 + 6\lambda^4 - 4\lambda^2 + 1 + \lambda - 1 = \lambda^8 - 4\lambda^6 + 4\lambda^4 - 4\lambda^2 + \lambda \end{aligned} \quad (2)$$

Also:

$$\begin{aligned} P(C_5^3, \lambda) &= (\lambda^2 - 1)^5 + (-1)^5 (\lambda - 1) = \\ &\lambda^{10} - 5\lambda^8 + 10\lambda^6 - 10\lambda^4 + 5\lambda^2 - 1 - \lambda + 1 = \\ &\lambda^{10} - 5\lambda^8 + 10\lambda^6 - 10\lambda^4 + 5\lambda^2 - \lambda \end{aligned} \quad (3)$$

To introduce the relation, we need this proposition.

Proposition 3.2: For $r \geq 2, h \geq 3$ we have:

$$P(C_r^h, \lambda) = \lambda^{h-1} (\lambda^{h-1} - 1)^{r-1} - P(C_{r-1}^h, \lambda)$$

Proof: Let C_r^h be an elementary h -uniform hypercycle as shown in Fig. 5. If we delete any hyperedge and contract it we obtain two hypergraphs H_1 and H_2 , thus by the theorem (2.2) we get:

$$P(C_r^h, \lambda) = P(H_1, \lambda) - P(H_2, \lambda) \quad (4)$$

where, H_1 composed from $h-2$ isolated vertices and hypertree T_{r-1}^h , thus, by the theorems (2.3 and 2.5):

$$P(H_1, \lambda) = \lambda^{h-2} \lambda (\lambda^{h-1} - 1)^{r-1} = \lambda^{h-1} (\lambda^{h-1} - 1)^{r-1} \quad (5)$$

And H_2 is just hypercycle C_{r-1}^h , thus:

$$P(H_2, \lambda) = P(C_{r-1}^h, \lambda) \quad (6)$$

Substitute 5 and 6 in 4 we get:

$$P(C_r^h, \lambda) = \lambda^{h-1}(\lambda^{h-1}-1)^{r-1} - P(C_{r-1}^h, \lambda)$$

As required:

Example 3.3: Consider C_3^3, C_2^3, C_1^3 . By the theorem (2.6):

$$C_3^3 = (\lambda^2-1)^3 - (\lambda-1) = \lambda^6 - 3\lambda^4 + 3\lambda^2 - \lambda \quad (7)$$

$$C_2^3 = (\lambda^2-1)^2 + (\lambda-1) = \lambda^4 - 2\lambda^2 + \lambda \quad (8)$$

Now by applying our result (3.2) and the theorem (2.6):

$$\begin{aligned} P(C_3^3, \lambda) &= \lambda^2(\lambda^2-1)^2 - P(C_2^3, \lambda) = \\ &= \lambda^2(\lambda^4 - 2\lambda^2 + 1) - [\lambda^2(\lambda^2-1)^2 + (\lambda-1)] = \lambda^6 - 3\lambda^4 + 3\lambda^2 - \lambda \end{aligned}$$

Which is equal to 7. Also:

$$P(C_1^3, \lambda) = (\lambda^2-1) - (\lambda-1) = \lambda^2 - \lambda \quad (9)$$

Then:

$$\begin{aligned} P(C_2^3, \lambda) &= \lambda^2(\lambda^2-1) - P(C_1^3, \lambda) = \\ &= \lambda^4 - \lambda^2 - ((\lambda^2-1) - (\lambda-1)) = \lambda^4 - \lambda^2 - (\lambda^2 - \lambda) = \lambda^4 - 2\lambda^2 + \lambda \end{aligned}$$

Which is equal to 7. Now, we can introduce the general recurrence formula for any elementary h-uniform hypercycle.

Theorem 3.4: For $r \geq 2, h \geq 3$, the recurrence relation for the chromatic polynomial of an elementary h-uniform hypercycle C_r^h is given by:

$$\begin{aligned} P(C_r^h, \lambda) &= \lambda^{h-1} P(C_{r-1}^h, \lambda) - (\lambda^{h-1}-1) P(C_{r-2}^h, \lambda) - \\ &= \lambda^{h-1}(\lambda-1) [(-1)^{r-1} - (-1)^{r-2}] \end{aligned}$$

Notice: If r is odd, then:

$$P(C_r^h, \lambda) = \lambda^{h-1} P(C_{r-1}^h, \lambda) - (\lambda^{h-1}-1) P(C_{r-2}^h, \lambda) - 2\lambda^{h-1}(\lambda-1)$$

If r is even, then:

$$P(C_r^h, \lambda) = \lambda^{h-1} P(C_{r-1}^h, \lambda) - (\lambda^{h-1}-1) P(C_{r-2}^h, \lambda) - 2\lambda^{h-1}(\lambda-1)$$

Proof: Let C_r^h be an elementary h-uniform hypercycle. By applying proposition (3.2) on C_r^h and C_{r-1}^h then we get:

$$P(C_r^h, \lambda) = \lambda^{h-1}(\lambda^{h-1}-1)^{r-1} - P(C_{r-1}^h, \lambda)$$

But:

$$P(C_{r-1}^h, \lambda) = [\lambda^{h-1}(\lambda^{h-1}-1)^{r-2} - P(C_{r-2}^h, \lambda)]$$

Substitute above result we get:

$$\begin{aligned} &= \lambda^{h-1}(\lambda^{h-1}-1)^{r-1} - [\lambda^{h-1}(\lambda^{h-1}-1)^{r-2} - P(C_{r-2}^h, \lambda)] = \\ &= \lambda^{h-1}(\lambda^{h-1}-1)^{r-1} - \lambda^{h-1}(\lambda^{h-1}-1)^{r-2} + P(C_{r-2}^h, \lambda) \end{aligned}$$

By adding and subtracting the terms $(-1)^{r-1}\lambda^{h-1}(\lambda-1)$ and $(-1)^{r-2}\lambda^{h-1}(\lambda-1)$ we get:

$$\begin{aligned} P(C_r^h, \lambda) &= \lambda^{h-1}(\lambda^{h-1}-1)^{r-1} - \lambda^{h-1}(\lambda^{h-1}-1)^{r-2} + P(C_{r-2}^h, \lambda) + \\ &+ (-1)^{r-1}\lambda^{h-1}(\lambda-1) - (-1)^{r-1}\lambda^{h-1}(\lambda-1) + (-1)^{r-2}\lambda^{h-1}(\lambda-1) - (-1)^{r-2}\lambda^{h-1}(\lambda-1) = \\ &= [\lambda^{h-1}(\lambda^{h-1}-1)^{r-1} + (-1)^{r-1}\lambda^{h-1}(\lambda-1)] - [\lambda^{h-1}(\lambda^{h-1}-1)^{r-2} + (-1)^{r-2}\lambda^{h-1}(\lambda-1)] + \\ &+ P(C_{r-2}^h, \lambda) - [(-1)^{r-1}\lambda^{h-1}(\lambda-1) - (-1)^{r-2}\lambda^{h-1}(\lambda-1)] = \\ &= \lambda^{h-1}[(\lambda^{h-1}-1)^{r-1} + (-1)^{r-1}(\lambda-1)] - \lambda^{h-1}[(\lambda^{h-1}-1)^{r-2} + (-1)^{r-2}(\lambda-1)] + \\ &+ P(C_{r-2}^h, \lambda) - \lambda^{h-1}(\lambda-1)[(-1)^{r-1} - (-1)^{r-2}] = \lambda^{h-1}P(C_{r-1}^h, \lambda) - \lambda^{h-1}P(C_{r-2}^h, \lambda) + \\ &+ P(C_{r-2}^h, \lambda) - \lambda^{h-1}(\lambda-1)[(-1)^{r-1} - (-1)^{r-2}] = \lambda^{h-1}P(C_{r-1}^h, \lambda) - \\ &+ \lambda^{h-1}(\lambda-1)P(C_{r-2}^h, \lambda) - \lambda^{h-1}(\lambda-1)[(-1)^{r-1} - (-1)^{r-2}] \end{aligned}$$

As required:

Example 3.5: From example (3.1) we know that:

$$P(C_3^3, \lambda) = \lambda^6 - 3\lambda^4 + 3\lambda^2 - \lambda \quad (10)$$

$$P(C_4^3, \lambda) = \lambda^8 - 4\lambda^6 + 6\lambda^4 - 4\lambda^2 + \lambda \quad (11)$$

$$P(C_5^3, \lambda) = \lambda^{10} - 5\lambda^8 + 10\lambda^6 - 10\lambda^4 + 5\lambda^2 - \lambda \quad (12)$$

Now, if we apply our result (3.4) on C_5^3 we get:

$$\begin{aligned} P(C_5^3, \lambda) &= \lambda^2 P(C_4^3, \lambda) - (\lambda^2-1) P(C_3^3, \lambda) - \lambda^2(\lambda-1)[(-1)^4 - (-1)^3] = \\ &= \lambda^2[\lambda^8 - 4\lambda^6 + 6\lambda^4 - 4\lambda^2 + \lambda] - (\lambda^2-1)[\lambda^6 - 3\lambda^4 + 3\lambda^2 - \lambda] - 2\lambda^2(\lambda-1) = \\ &= \lambda^2[\lambda^8 - 4\lambda^6 + 6\lambda^4 - 4\lambda^2 + \lambda] - (\lambda^2-1)[\lambda^6 - 3\lambda^4 + 3\lambda^2 - \lambda] - 2\lambda^2(\lambda-1) = \\ &= \lambda^{10} - 4\lambda^8 + 6\lambda^6 - 4\lambda^4 + \lambda^3 - \lambda^8 + 3\lambda^6 - 3\lambda^4 + \lambda^3 + \lambda^6 - 3\lambda^4 + 3\lambda^2 - 2\lambda^3 + \\ &+ 2\lambda^2 = \lambda^{10} - 5\lambda^8 + 10\lambda^6 - 10\lambda^4 + 5\lambda^2 - \lambda \end{aligned}$$

Which is equal to 12.

Example 3.6: Consider C_2^4, C_3^4, C_4^4 . By applying the theorem (2.6):

$$P(C_2^4, \lambda) = (\lambda^3 - 1)^2 + (\lambda - 1) = \lambda^6 - 2\lambda^3 + \lambda \quad (13)$$

$$P(C_3^4, \lambda) = (\lambda^3 - 1)^3 + (\lambda - 1) = \lambda^9 - 3\lambda^6 + 3\lambda^3 - \lambda \quad (14)$$

$$P(C_4^4, \lambda) = (\lambda^3 - 1)^4 + (\lambda - 1) = \lambda^{12} - 4\lambda^9 + 6\lambda^6 - 4\lambda^3 + \lambda \quad (15)$$

By applying our result (3.4):

$$\begin{aligned} P(C_4^4, \lambda) &= \lambda^3 P(C_3^4, \lambda) - (\lambda^3 - 1) P(C_2^4, \lambda) - \lambda^3 (\lambda - 1) [(-1)^3 - (-1)^2] = \\ &= \lambda^3 [\lambda^9 - 3\lambda^6 + 3\lambda^3 - \lambda] - (\lambda^3 - 1) [\lambda^6 - 2\lambda^3 + \lambda] + 2\lambda^4 - 2\lambda^3 = \lambda^{12} - 3\lambda^9 + 3\lambda^6 - \lambda^4 + \\ &+ 2\lambda^6 - \lambda^4 + \lambda^6 - 2\lambda^3 + \lambda + 2\lambda^4 - 2\lambda^3 = \lambda^{12} - 4\lambda^9 + 6\lambda^6 - 4\lambda^3 + \lambda \end{aligned}$$

Which is equal to 15.

CONCLUSION

In this study we introduce the recurrence relation for the chromatic polynomial of an elementary h-uniform hypercycle.

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