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Vibration Control of Structures using Frictional Force

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Abstract: This study investigates the effect of contact areas in the dynamic behavior of mechanical structures holding by bolts, highlight on the significant of nonlinear frictional force induced between the structural elements to control the structure vibration response. In this research, the nonlinear partial deferential equation describing the behavior of cantilever-layered beams has been derived. The equation has been formulated based on Euler-Bernoulli theory with coulomb friction model. Furthermore, an analytical solution of the derived equation for structures being in free vibration has been introduced. A structure includes two cantilever-layered beams connected by bolts has been assembled to validate the derived equation experimentally. A good agreement between the theoretical and experimental results has been obtained. Both results show that when the bolts of beams are loosed, then, the 2 beams will vibrate independently. But when the bolts tighten was increased, the stiffness and damping ratio of the beams were also increased until they reach to a certain maximum values at certain bolts tighten value. With continuing in increasing the bolts tighten, the value of stiffness was remained constant but the damping ratio was decreaseduntil it vanished when the bolts tighten becomes very large and the beams behave as 1 beam of double thick. Therefore, the bolts joints can be used to improve damping capacity of structure from adjusting each bolt tighten depending on the bolt location.

Key words: Dynamic behavior, cantilever-layered, Euler-Bernoulli theory, theoretical, vanished, location

INTRODUCTION

Generally, any structure has inherent damping capacity obtained from the type of material, boundary conditions, environment and joints. Beards (1992) proved that joints damping most considerable one. The joints damping is a frictional force induced at joint of a structure causing dissipation of energy when the structure is subjected to any vibrational excitation. Many parameters have effect on frictional damping between two contacted surfaces such as coefficients of friction (static and kinetic) ambient temperature, contacted material type and pressure at the interface.

In this research, the effect of pressure (bolts tighten) on the damping capacity of a structures will be investigated. Shin *et al*. (1991) showed that there is no damping in the joint that it is tighten very hard. The dampingis increased when the tightness of the joint is decreased. Thus, the vibration energy is dissipated. Popp *et al*. (2003) showed that the response of a structure includes a jointed layered cantilever is highly effected by the bolts tighten condition. They deduced that the jointed layered cantilever gives maximum dissipated energy at certain contact pressure distribution, i.e., each bolt has its own tighten depending on its location. Badrakhan (1994) modeled the energy dissipated with coulomb friction and optimum pressure for maximum energy for any number of layers. Goodman and Klumpp (1956) introduced the maximum amount of energy dissipation in two-layered. Beard (1992) found out the optimal normal force which causes the maximum energy dissipation.

The main task for this study is to model the effect of the frictional force on improving the damping capacity of a structure in order to control its response to vibrational excitation. This will help to design reliable structures can with minimal cost. In this research, the differential equation describing the behavior of two layered cantilever beam under the action of the frictional force is introduced. The equation was derived based on Euler-Bernoulli theory with Coulomb friction model. An analytical solution for the derived equation is addressed. Also, due to the impossibility of calculating surface stiffness of beam analytically with acceptable accuracy, empirical model was developed to calculate the surface stiffness based on some experimental measurements.

MATERIALS AND METHODS

Coulomb model: Although, Coulomb set his well-known theory about dry friction in 1780 but it still active at now

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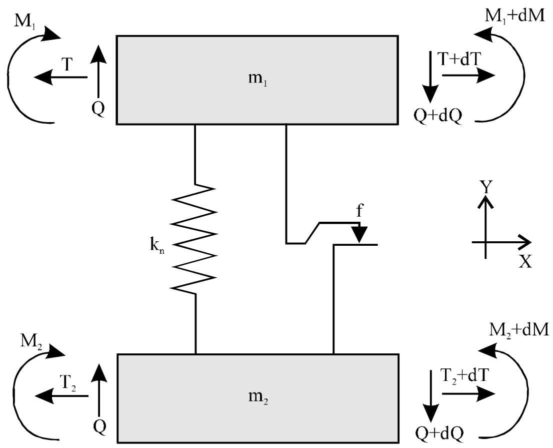

Fig. 1: System model

a days because of it characterized with simplicity and accuracyin solving complex systems (Dowson, 1988; Pust *et al*., 2011). This gives the motivation to consider the Coulomb model in this study.

In general, Coulomb model consist of two-part stick and slip. Stick stage happens when the induced friction force in the contact state less than the threshed force. Furthermore, the slip stage happens when the induced frictional force exceed the threshed one. Generally, to two kinds of motions are induced: micro-slip and macro-slip (Groper, 1985; Menq *et al*., 1986; Popp *et al*., 2003). Coulomb model can be expressed as Pust *et al*. (2011) and Bournine *et al*. (2011):

$$f = k_t u_r + \mu N \ \mathrm{sign}(\dot{u}_r) \tag{1}$$

where, $k_t, u_r, \dot{u}_r, \mu, N$ represented tangential stiffness, relative displacement and velocity in longitudinal direction, kinetic coefficient of friction and normal force. Depending on Coulomb model, the investigated system consist of layered cantilever beams holding together by bolts subjected to initial displacement can be modeled as two masses with vertical springs and passive damping in horizontal direction between them as shown in Fig. 1. When bolts are tightened, the peak asperities in two contact faces will pressto static approach (δ) where there normal stiffness (k_n) bolt tightened can expressed:

$$N = k_n \delta \tag{2}$$

When system subjected to free vibration, the normal force will be changed because of inducing relative displacement between two beams in transverse direction. Therefore, the normal force in the interface will express as:

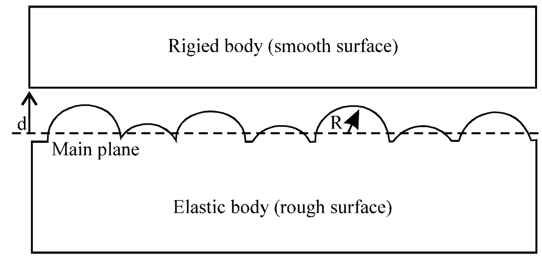

Fig. 2: Schematic of the real area of contact

$$N_1 = k_n[\delta\text{-}(\nu_1\text{-}\nu_2)] \tag{3}$$

And friction can be written as shown:

$$f_1 = \mu k_n [\delta\text{-}(\nu_1\text{-}\nu_2)]\mathrm{sign}(\dot{u}_r) \tag{4}$$

$$f_2 = -f_1 \tag{5}$$

where, N_1, ν_1, ν_2, f_1, f_2 represented normal force subjected by bolts on element one, the displacement for element (1 and 2) in transverse direction, frictional force induced in elements 1 and 2, respectively.

Surface effect: Unfortunately, the Coulomb model is unable to describe the behavior of surfaces (Pust *et al*., 2011). Therefore, the contact in this study is adopted from Greenwood and Williamson (G&W) in 1966. To simplify contact analysis between two rough surfaces it can be considered that one surface is smooth and the other is rough one as shown in Fig. 2. If z>d then approach:

$$\delta = z\text{-}d \tag{6}$$

by assuming function of roughness as shown, Tavares (2005), Malekan and Rouhani (2018) and Persson (2006):

$$\varnothing(z) = \sqrt{\frac{0.5}{\pi\sigma^2}}\exp\left(-\frac{z^2}{2\sigma^2}\right) \tag{7}$$

$$N_t = P_t A_t = \sqrt{\frac{8}{9\pi\sigma^2}} nE^* R_p^{1/2} \int_d^\infty (z\text{-}d)^{3/2} \exp\left(-\frac{z^2}{2\sigma^2}\right) dz \tag{8}$$

where, σ is the surface roughness (RMS of asperities) or the height standard deviation. Unfortunately, there is no analytical close-form solution for Eq. 8 to find contact force. Therefore, the contact stiffness can be find only numerically as Shi and Polycarpou (2005). However, Polycarpou and Etsion (1999) present an exponential asperities height distribution to give a good approximation for the Gaussian distribution:

$$\varnothing(z) = C\exp\left(-\frac{\lambda d}{R_q}\right) \quad (9)$$

$$N(d) = \frac{C\beta E^{*} A_n \sqrt{\pi}}{\lambda^{\frac{5}{2}}}\left(\frac{R_q}{R_s}\right)^{1/2}\exp\left(-\frac{\lambda d}{R_q}\right) \quad (10)$$

From above equations, there are some of necessary parameters which must be known to find the varying of surfaces stiffness in the contact area along the beams. Because of there is no available measurements able to measure these parameters such as approach, distance between two faces, etc., Bournine *et al*. (2011) showed that the normal contact stiffness can be calculated, only by experimental work as well as the coefficient of material damping. Thus, in this research, the experimental results will be considered to find the magnitude of stiffness.

RESULTS AND DISCUSSION

Static analysis

Theoretical stiffness calculation: To finding the theoretical stiffness it must finding moment function along the beams then by do one integration to find slop function the by multiplying it by thickness of one beam it can find out the relative slip in the interface. The moment at any section in upper beam or lower beam:

$$M = \frac{R}{2}(l\text{-}x) + \frac{w}{4}(l\text{-}x)^2 - \frac{f(2h)}{2}(l\text{-}x) \quad (11)$$

If assuming:

$$y_1 = y_2 \quad (12)$$

By applying boundary conditions:

- $x = 0$
- $y' = 0$
- $C_4 = 0$
- $x = 0$
- $y = 0$
- $C_5 = 0$

$$y(l) = \frac{\left(R + \frac{3wl}{8} - 2fh\right)l^3}{6El} \quad (13)$$

where, y (l), R, w, f, h, l, E, I represent the deflection at free end, the static load applied at free end, weight per unit length of two beam induced friction per unit length at

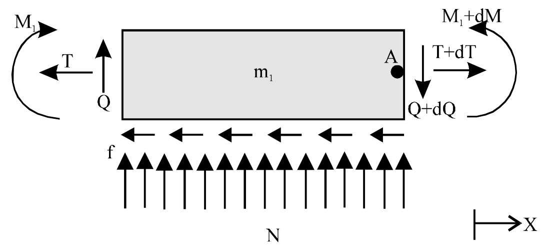

Fig. 3: An elements from upper and lower beams

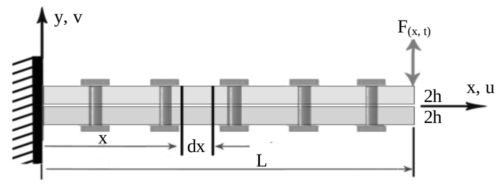

Fig. 4: Beam analysis

interface between two beams, half thick of one beam, length of beam, modulus of elasticity and moment of inertia for two beams, respectively. Now, taking moments around point (A) as shown in Fig. 3 will give the relationship between the shear force of upper beam of the hand and moment, friction force:

$$\sum M_A = 0 - M_1 + \left(M_1 + \frac{\partial M_1}{\partial x}dx\right) - Fh\,dx + N_b\frac{dx}{2} - k_n\delta\frac{dx}{2} - Q_1 dx = 0,\; Q_1 = \frac{\partial M_1}{\partial x} - Fh \quad (14)$$

Dynamic equations for free transverse vibration: Figure 4 consider two contact cantilever beams jointed together by a number of bolts. These beams own the same material and dimensions. They are infree vibration with transverse displacement v(x, t) and longitudinal displacement u(x, t).

Euler-Bernoulli beam theory is considering in the analysis to find out the dynamic governor equation of the beams. Therefore, the rotation of the element is negligible and the angular shear deformation is small in relation, if it compared with the bending deformation. This because of that the lengths to thicknesses of the investigated beams are more than ten times. By applying Newton's, second law:

$$Q_1 - \left(Q_1 + \frac{\partial Q_1}{\partial x}dx\right) - N_b dx + k_n\delta + N_1 dx + \frac{F(x,\,t)}{2}dx = (\rho_1 A_1)\frac{\partial^2 v_1}{\partial t^2}dx \quad (15)$$

where, N, N_b, δ represented normal forces in the interface, bolt forces that subjected on the out faces of beams and static displacement due to bolt tighten. In same way, analysis the lower element to get another equation sum it with Eq. 16 to get:

$$\begin{bmatrix} EI & 0 \\ 0 & EI \end{bmatrix}\begin{Bmatrix} \nu_1'''' \\ \nu_2'''' \end{Bmatrix}-\frac{h_\mu k_n}{1}\frac{\partial}{\partial x}\begin{bmatrix} (\delta\text{-}\nu_1)\text{sign}(u_r) & (\delta+\nu_2)\text{sign}(\dot{u}_r) \\ (\delta+\nu_1)\text{sign}(u_r) & (\delta\text{-}\nu_2)\text{sign}(\dot{u}_r) \end{bmatrix}+ \frac{1}{1}\begin{bmatrix} C_m & 0 \\ 0 & C_m \end{bmatrix}\begin{Bmatrix} \dot{\nu}_1 \\ \dot{\nu}_2 \end{Bmatrix}+\begin{bmatrix} k_n & -k_n \\ -k_n & k_n \end{bmatrix}\begin{Bmatrix} \nu_1 \\ \nu_2 \end{Bmatrix}+\begin{bmatrix} \rho A & 0 \\ 0 & \rho A \end{bmatrix}\begin{Bmatrix} \ddot{\nu}_1 \\ \ddot{\nu}_2 \end{Bmatrix}=0 \quad (16)$$

where, Eq. 16 described the nonlinear damping behavior of the two layers beam subjected to free vibration. The sign function represented nonlinear function. If the transverse displacements of two layered beams are equal, i.e., $\nu_1 = \nu_2 = \nu$, then, Eq. 16 can adjusted to be:

$$(EI)\nu''''-\frac{h_\mu k_n \delta}{1}\frac{\partial}{\partial x}\text{sign}(\dot{u}_r)+\frac{C_m}{21}\dot{\nu}+(\rho A)\ddot{\nu}=0 \quad (17)$$

where, C_m represented the constant damping coming from material. In addition, to solve Eq. 17 in general form and by use separation of variables method (modal method) as shown:

$$\nu_i = Y_i(x).q_i(t) \quad (18)$$

where, the bending slipping velocity in the interface can formulated:

$$\dot{u}_r = \dot{u}_1\text{-}\dot{u}_2 = h\frac{\partial}{\partial x}\dot{\nu}_1\text{-}\left(\text{-}h\frac{\partial}{\partial x}\dot{\nu}_2\right)=2h\frac{\partial\dot{\nu}}{\partial x} \quad (19)$$

Substitute, Eq. 18 and 19 in Eq. 20 and dividing new equation by m(x):

$$\frac{EI(x)}{m(x)}q(t)Y(x)''''-\frac{h_\mu k_n \delta}{m(x)}\frac{\partial}{\partial x}\text{sign}\left(2*h*\frac{\partial(Y(x)\dot{q}(t))}{\partial x}\right)+ \frac{C_m}{2m(x)l}Y(x)\dot{q}(t)+Y(x)\ddot{q}(t)=0 \quad (20)$$

From Eq. 20 and applying boundary condition, the natural frequency can formulated as shown, De Silva (2000) and Thomson and Dahleh (1997):

$$\beta I^2 = 3.52$$

Thus:

$$\omega_n = (\alpha\beta)^2\sqrt{\frac{EI}{ml^3}} \quad (21)$$

where, β, α represents the factor of characteristic of differential equation, the frictional stiffness factor which has the significant effect on the frequency of the structure. This stiffness depends on the normal force, roughness, type of material, temperature and friction coefficients (static and kinetic). To go further intothe solution, the mode shapeis assumed as Popp *et al*. (2003):

$$Y(x) = y_0\left[1\text{-cos}\left(\frac{\pi x}{21}\right)\right] \quad (22)$$

Based on Eq. 20, the solution of second-order differential equation can be obtained thus:

$$\zeta = \frac{\vartheta}{4}\left\{\frac{h_\mu N[\log\left(\tan\left(\frac{\pi x}{4l}\right)\right)+\frac{1}{2}\cot\left(\frac{\pi x}{4l}\right)^2]}{my_0^2\omega_n^2}+\frac{C_m}{m\omega_n}\right\} \quad (23)$$

where, ϑ represented frictional damping factor which caused from surfaces effect the constant damping ratio will become:

$$C_f = \frac{\vartheta h_\mu N}{2my_0|q(t)|}[\log\left(\tan\left(\frac{\pi x}{4l}\right)\right)+\frac{1}{2}\cot\left(\frac{\pi x}{4l}\right)^2] \quad (24)$$

By applying the following initial conditions:

- $t = 0$
- $\nu(1, 0) = y_0$
- $\nu(x, 0) = 0$

Then, the response ν(x, t) can be expressed as shown in Eq. 25:

$$\nu(x, t)=Y(x).e^{-\zeta\omega t}\left[\frac{\nu_0}{Y(1)}\cos(w_d t)+\frac{\zeta\nu_0}{Y(1)\sqrt{1-\zeta}}\sin(w_d t)\right] \quad (25)$$

Surface effect factor: As mentioned previously, the calculation of the surface stiffness cannot be achieved mathematically. It can be calculated numerically, Shi and Polycarpou (2005), the approximation for the Gaussian distribution presented by Polycarpou and Etsion (1999). This method is difficult to be used because of complexity of getting the required measurements sufficient accuracy.

While, Bournine *et al*. (2011) showed that surface stiffness can be calculated easily based on a simple experimental procedure which will be followed in this study. The experiments will be carried on to obtain an empirical expression simulates the effect of the surface on the frequency and damping capacity.

Surface effect on the stiffness: When bolts are used to joint structural elements, the stiffness of structure will be defining depending on the bolt tighten condition, thus:

$$k_{ex} = k_l + k_f \quad (26)$$

where, k_{ex}, k_l, k_f represent the experimental stiffness, linearly stiffness and frictional stiffness. The linearly stiffness can be calculate for one beam from Eq. 13 such as:

$$k_t = \frac{R}{\nu_p(l)} = \frac{6\ El\ R}{(R + \frac{wl}{2} - 2fh)l^3} \quad (27)$$

Surface effect on the damping capacity: According to, Coulomb model, the asperities research as passive dashpot in the longitudinal direction as in Fig 1. These asperities resist the slipping and hencethe dissipated energy will be increase for certain limit of bolts tightening. When this limit is exceeded, the asperities start to be squashed causing the damping capacity to be decreased. From above, the normal force has a significant effect on the damping capacity compared with other parameters such as friction coefficient (kinetic and static) or the type of material. Therefore, it is important to find a formula counting for the bolts normal force effect on the damping capacity at the interface, thus:

$$N^* = \frac{C_{f\ ex}}{C_{f\ 1}} * N = \vartheta * N \quad (28)$$

where, N^*, C_{f1} represented the effective normal force that simulation the behavior of contact area with increasing normal force and overlap asperities when contact areas converged, linearly frictional constant damping without surface effect considered from Eq. 24 when $\vartheta = 1$ whenever, C_{ex} represented experimental frictional constant damping which essay to calculated after known the material constant damping for one beam with double thick. The surface effect will be add to formula by replacing Eq. 28 and 24.

Case of study: This study investigates the effect of bolts joint on the damping capacity of structures. Therefore, a simple structure consists of two cantilever beams hold together by five bolts was used. This structure was studying under free transverse vibration condition as shown in Fig. 5. The rig containsfour major parts: platform, bench vise, measuring system and shooting mechanism. The three dimensions accelerometer type (GY 61) with UNO-Arduino used to obtain the required results. This accelerometer fixed on the free end of the cantilever and connectedwith very thin wires to reduce the disturbances in measurements.

The free end was draw to have 3 cm deformation and then, it released to vibrate freely. The cantilever was leave tooscillate until it stops. Meanwhile, the accelerometer records during the oscillation period.

Fig. 5: The rig

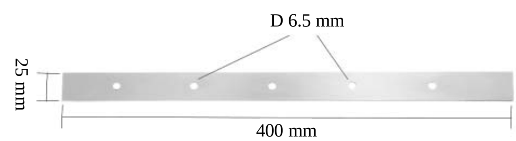

Fig. 6: Beam dimensions

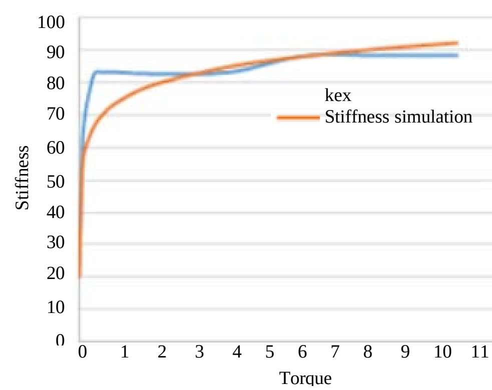

Fig. 7: Experimental stiffness diagram VS torque

The beams are stainless steel (304) of 193 GPa modulus of elasticity. They own the dimensions: 0.96 mm thickness, 25 mm width and 400 mm length. Each beam contains five bolts holes of 6.5 mm in diameter as shown in Fig. 6.

They arranged as layered cantilever beam when subjected to free vibration in transverse direction, from the results stiffness has been identified. Where the results showed that the system has 20 N/m when bolts tighten loses then increasing to maximum value 88.2 N/m as bolts tighten fastened as shown in Fig. 7.

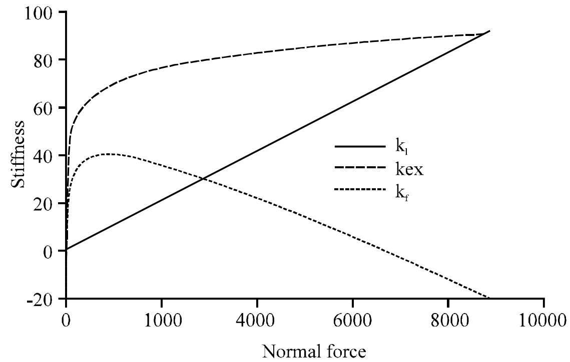

Fig. 8: Experimental, linearly, frictional stiffness VS normal force

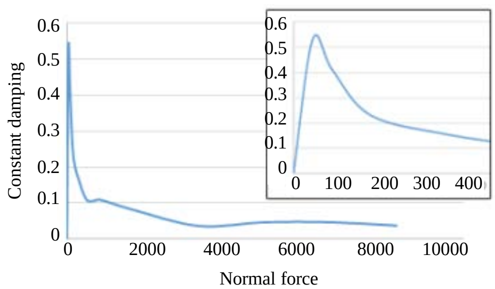

Fig. 9: Frictional constant damping VS normal force

In addition, to find the relationship between experimental stiffness and normal force or torque, the curve fitting techniques were applied on the results as shown in Fig. 7 where the relationship is found to be as:

$$k_{ex} = 20+8*\ln(N) \tag{29}$$

While the total experimental stiffness identified from experimental research. The frictional stiffness can calculated by Eq. 26. The results are shown in Fig. 8. To calculate natural frequency of the structurefrom Eq. 21, the value of $\propto$ must be defined:

$$k_{ideal} = \frac{3El}{l^3} \tag{30}$$

$$\propto = \sqrt[4]{\frac{k_{ex}}{k_{ideal}}} \tag{31}$$

From experimental research, the constant damping of the cantilever is measured and hence, the effective normal force can be calculated using Eq. 28 as shown in Fig. 9. In addition, the relationship between effective normal force and bolts normal force can be found out by using curve-fitting technic as shown in Fig. 10. This effective no can replacing in Eq. 23 and 24.

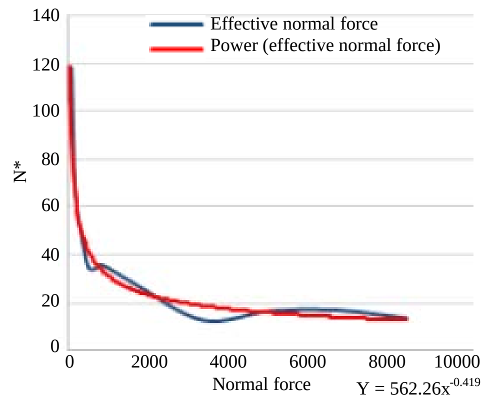

Fig. 10: Effective normal force VS normal force

As previously mentioned, the layered beams structure has nonlinear behaviour because of the non-unified shape of the asperities of the two contacted surfaces. When two contact surfaces are pressed toward against each other, the nonlinear contact stiffness will cause to increase the structure overall stiffness while the nonlinear frictional force will cause to increase the overall damping of the structure. In a classic cantilever beam, the stiffness is directly proportional tothe normal force. While in the layered cantilever beams, the stiffness is nonlinearity proportional to the normal force because of the frictional force as can be shown in Fig. 7 and 8.

Figure 8 shows that when the bolts were loosening, the structure has stiffness about 20 N/m. This stiffness is the same of that of one beam. This means when the layered beam excited and the bolts are loosening, then each layer will vibrate independently. That because the two layers are in full slipping. Therefore, there is no damping due to frictional whereas the structure will oscillate under the material damping only (Fig. 11-14).

With increasing the bolts tighten, the stiffness of structure is increasing until it reaches to its maximum value which is about 92 N/m when the normal force equals to 8330 N. In same time as the normal force is increasing, the frictional force is also increasing from zero at losing bolts tighten to maximum value at normal force (833 N). Further, increasing in normal force to about 8330 N will lead to unify the two layers and vibrate like one beam where there is no more frictional force between the layers.

The experiments also showed that the constant damping caused by the frictional force is increasing from zero at bolts losing case to its maximum value when the normal force equals to 41.5 N. With increasing the normal

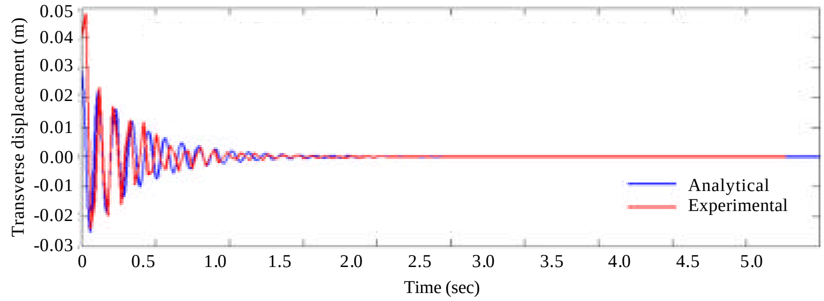

Fig. 11: Compared between experimental and theoratical results; Displacement diagram: compared between experimental and analytical result for layered system with 0.1 Nm bolts tighten

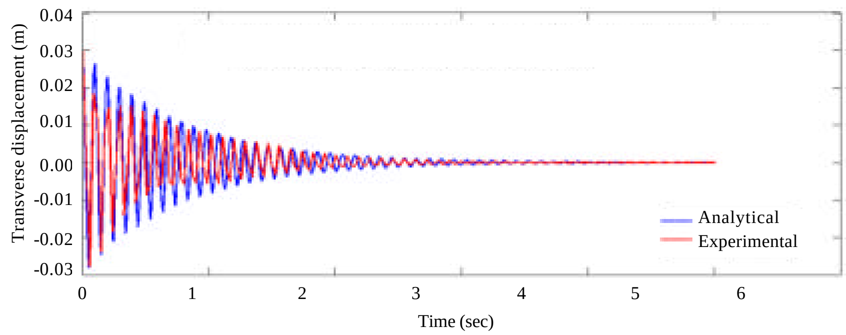

Fig. 12: Compared between experimental and theoratical results at 1 Nm; Displacement diagram: compared between analytical and experimental result for layered system at 0.1 Nm bolts tighten

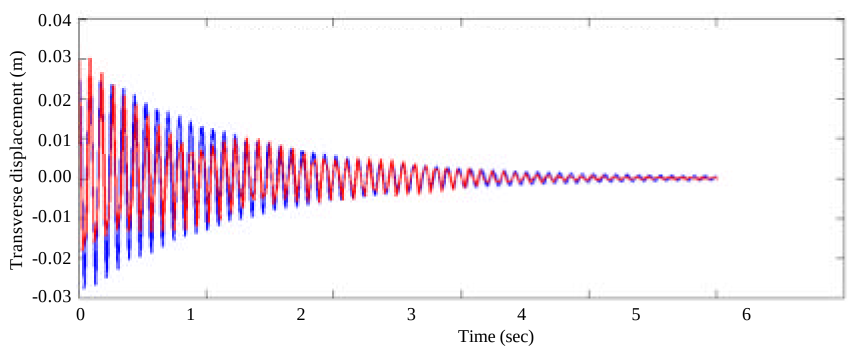

Fig. 13: Compared between experimental and theoratical results; Displacement diagram: compared between analytical and experimental at 10 Nm bolts tighten for layered cantilever beam

force further, the damping will be decreased until it becomes zero when the normal force is about 8330 as can be shown in Fig. 9.

Figure 11 show a comparison between the experimental and theoretical works at bolts tighten torques 0.1 Nm. where the two oscillations has same decay in vibrate, therefore their amplitudes at most time very closed. Where the damping ration defined 0.048, 0.05 in analytical and experimental result, respectively. Figure 12 also show a comparison between the experimental and theoretical works at bolts tighten torques 1 Nm. Where the two oscillations has same decay in vibrate, therefore, their amplitudes at most time very closed. Where the damping ration defined 0.0197, 0.02 in analytical and experimental result, respectively.

Figure 13 also show a comparison between the experimental and theoretical works at bolts tighten torques 1 N m. Where the two oscillations has same decay in vibrate therefore, their amplitudes at most time very closed. Where the damping ration defined 0.0116, 0.012 in analytical and experimental result, respectively.

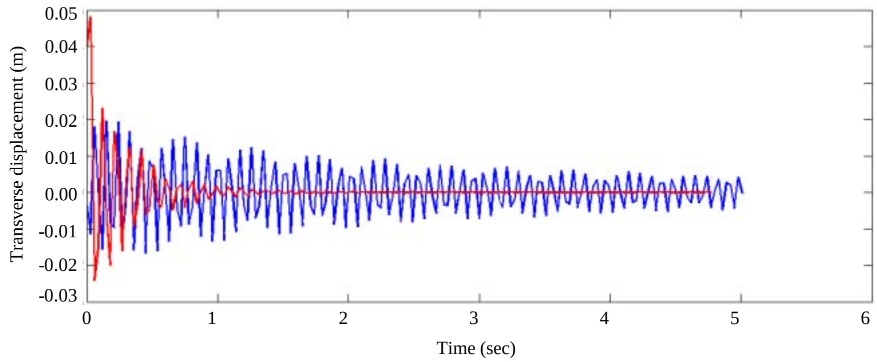

Fig. 14: Compared between layered cantilever and cantilever beam double thick; Displacement diagram: compared between layered system with cantilver beam has same thick

Figure 14 show an experimental comparison between the layered beams system at bolts tighten torques 0.1 Nm and cantilever beam has same thick. Where the frictional effect in the layered cantilever beams clear and damping ratio of system improving from 0.004-0.05.

CONCLUSION

The following remarks have been withdrawn from the present research. The damping capacity of a structure, e.g., cantilever beam can be adjusted if the structure is assembled from two layers connected together by bolts. The damping can be modified from tighten each bolt with certain torque depending on the bolt location. In layered beam structure, the increasing of bolts tighten causesan increasing in the structural stiffness until it reach to its maximum when the layers are firmly connected together.

The frictional damping is equal to zero at the full slipping case when the bolts are loosened. Also, the frictional damping is zero when the bolts are tightening very and the two layers are in stick case. Increasing bolts tighten causesto increase in the damping capacity of the structure until it reach to its maximum value because the overlap between asperities of two contact faces reach to optimal and gives maximum dissipated energy after this point the damping capacity decrease as bolts tighten increasing.

APPENDIX

Symbols	Definition/Units
m_1	An elements mass in upper beam (1) (kg)
m_2	An elements mass in lower beam (2) (kg)
k_n	Normal stiffness (N/m)
k_t	Tangential stiffness (N/m)
f	Friction force per unit length (N/m)
Q1	Shear force in upper beam 1 (N)
Q2	Shear force in lower beam 2 (N)
T1	Axial force in lower beam 1 (N)
T2	Axial force in lower beam 2 (N)
M1	Moment in upper beam 1 (Nm)
M2	Moment in lower beam 2 (Nm)
x	The distance in direction of length (x)
y	The distance from the center line of interface (y)
z	The distance from the surface of lower beam to of Opposite surface other beam at interface (z)
l	Length (l)
h	Half thickness of beam (h)
b	Width of beam (b)
A	Area (A)
I	Second moment of inertia (m^4)
ρ	Density (N/m^3)
ν_s	Static coefficient of friction (-)
ν_k	Kinetic coefficient of friction (-)
C_m	Material damping constant (Nm/sec)
C_f	Frictional damping constant (Nm/sec)
ν(x, t)	Transverse displacement (m)
u(x, t)	Longitudinal displacement (m)
F_{ex}	Excitation force (N/m)
N_b	Bolt force (N /m)
N1	Normal force per unit length in beam 1 (N/m)
P	Nominal pressure (N/m^2)
n	Number of asperities or summit
Ds	The summit density per unit area n/m^2
V	Poisson's ratio (-)
ϖ	Variance of summit height distribution (m)
τ	Shear stress (N/m^2)
R	Static force (N)
ϑ	Frictional damping factor (-)

REFERENCES

Badrakhan, F., 1994. Slip damping in vibrating layered beams and leaf springs: Energy dissipated and optimum considerations. J. Sound Vibr., 174: 91-103.

Beards, C.F., 1992. Damping in structural joints. Shock Vibr. Dig., 24: 3-7.

Bournine, H., D.J. Wagg and S.A. Neild, 2011. Vibration damping in bolted friction beam-columns. J. Sound Vibr., 330: 1665-1679.

De Silva, C.W., 2000. Vibration: Fundamentals and Practice. CRC Press, Boca Raton, Florida, USA., ISBN:9780849318085, Pages: 943.

Dowson, D., 1988. History of Tribology. 2nd Edn., Wiley Publishing Company, Hoboken, New Jersey, USA., ISBN:978-1-860-58070-3, Pages: 768.

Goodman, L.E. and J.H. Klumpp, 1956. Analysis of slip damping with reference to turbine blade vibration. J. Appl. Mech., 23: 421-429.

Groper, M., 1985. Microslip and macroslip in bolted joints. Exp. Mech., 25: 171-174.

Malekan, A. and S. Rouhani, 2018. Model of contact friction based on extreme value statistics. Frict., 1: 1-13.

Menq, C.H., J. Bielak and J.H. Griffin, 1986. The influence of m icroslip on vibratory response, part I: A new microslip model. J. Sound Vibr., 107: 279-293.

Persson, B.N.J., 2006. Contact mechanics for randomly rough surfaces. Surf. Sci. Rep., 61: 201-227.

Polycarpou, A.A. and I. Etsion, 1999. Analytical approximations in modeling contacting rough surfaces. J. Tribol., 121: 234-239.

Popp, K., L. Panning and W. Sextro, 2003. Vibration damping by friction forces: Theory and applications. Modal Anal., 9: 419-448.

Pust, L., L. Pesek and A. Radolfova, 2011. Various types of dry friction characteristics for vibration damping. Eng. Mech., 18: 203-224.

Shi, X. and A.A. Polycarpou, 2005. Measurement and modeling of normal contact stiffness and contact damping at the Meso scale. J. Vibr. Acoust., 127: 52-60.

Shin, Y.S., J.C. Iverson and K.S. Kim, 1991. Experimental studies on damping characteristics of bolted joints for plates and shells. J. Pressure Vessel Technol., 113: 402-408.

Tavares, S.M.O., 2005. Analysis of surface roughness and models of mechanical contacts. Ph.D Thesis, University of Pisa, Pisa, Italy.

Thomson, W.T. and M.D. Dahleh, 1997. Theory of Vibration with Applications. 5th Edn., Prentice Hall, UK.