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A Quantum Calculus Analogue of Dynamic Leontief Production Model with Quadratic Objective Function

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Abstract: In this study, we derive a new formulation for the dynamic Leontief production model with quadratic objective function using quantum calculus analogue. Our new formulation unifies discrete and continuous Leontief production models. Also, the classical Leontief production model is obtained by choosing q = 1. In addition, we give an introduction to quantum calculus. We formulate the primal and the dual continuous Leontief production models with quadratic objective function as well as quantum calculus models. Furthermore, we can get upper and lower bounds for the value of production at any production plan using the quadratic objective functions for the primal and the dual quantum calculus models. Moreover, the duality theorem for quantum calculus analogue are established. Finally, example is given in order to show the new results.

Key words: Leontief production model, quantum calculus, quadratic objective function, primal model, dual model, duality theorems

INTRODUCTION

The theory of continuous-time linear programming problems plays an important role in modelling various real world applications such as in operation research, economics, finance, production planning and transportation problem. For more details we refer to Bellman and Dreyfus (1962), Buie and Abrham (1973), Gale (1960), Grinold (1969), Hanson (1967), Hanson (1967), Levinson (1966), Tyndall (1965, 1967) and Wen *et al.* (2012). On the other hand, quantum calculus has been recently used to model many applications in number theory and physics such as conformal quantum mechanics nuclear and high energy physics and internal energy and specific heat. According to Adyvar and Bohner (2006), Adyvar and Koyuncuoglu (2016), Bohner and Chieochan (2013a, b), Bohner *et al.* (2007), Bohner *et al.* (2013), Bohner and Mesquita (2016), Bohner and Peterson (2001, 2003), Bohner and Wintz (2013). In this study, we present quantum calculus formulation for Leontief production model with quadratic objective function.

MATERIALS AND METHODS

Quantum calculus: In this study, we briefly give some basic concepts of the theory of quantum calculus. The material in this section can be found in monographs (Adyvar and Bohner, 2006; Adyvar and Koyuncuoglu, 2016; Bohner and Mesquita, 2016; Bohner and Peterson, 2001; Kac and Chebing, 2002) in which comprehensive details are given.

Definition 2.1: The q-derivative of a functionis f: $q^{\mathbb{N}_0}\to\mathbb{R}^n$ is defined as:

$$D_q f(t) = \frac{f(q(t))-f(t)}{(q-1)t}$$

The q-derivative is also called Jackson derivative.

Theorem 2.2: If f, g:$q^{\mathbb{N}_0}\to\mathbb{R}^n$ are q-differentiable then we have the following:

$$D_q\big(af(t)+bg(t)\big) = aD_q f(t)+bD_q g(t), t\in q^{\mathbb{N}_0}$$

$$D_q\big(f(t)g(t)\big) = f(qt)D_q g(t)+g(t)D_q f(t), t\in q^{\mathbb{N}_0}$$

$$D_q\frac{f(t)}{g(t)} = \frac{g(t)D_q f(t)-f(t)D_q g(t)}{g(t)g(qt)}, t\in q^{\mathbb{N}_0}$$

Definition 2.3: Assume f:$q^{\mathbb{N}_0}\to\mathbb{R}^n$ and a, b$\in q^{\mathbb{N}_0}$ with a<b. The definite integral of the function f is given by:

$$\int f(t)d_q(t) = (q-1)\sum_{(t\in[a,b)\cap q^{\mathbb{N}_0}} tf(t)$$

Definition 2.4: If f:$q^{\mathbb{N}_0}\to\mathbb{R}^n$ with q>1, m, n$\in\mathbb{N}_0$ and m<n, then:

$$\int_{q^m}^{q^n} f(t)d_q(t) = \sum_{k=m}^{n-1}(q-1)q^k f\big(q^k\big)$$

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Continuous-time Leontief production model: In this study, we formulate the continuous-time Leontief production model. First, we introduce the model with linear objective function and then with quadratic objective function. This model is a closed dynamic economy in which the production of goods with capital goods is limited at any time by the current accumulation of capital goods (Gale, 1960; Hanson, 1967). Consider the following: B: be an n×n matrix whose ijth entries represent the amount of the ith product consumed by the jth activity in producing one unit of jth product. S:be an n×n matrix whose ijth entries represent the amount of the ith product required as capital stock in order to produce the jth product at unit rate. The function z: [0, T]→$\mathbb{R}^n$ be a bounded measurable function represents the activity levels. Now, assume z(t) is positive then the net production at time t is (I-B)z(t) which is also summed positive. If there is an injection of external capital goods c(t) non negative into the system up to time t where, c:[0, T]→$\mathbb{R}^n$ is a bounded measurable function, then, we have the following inequality:

$$Sz(t)\le c(t)+\int_0^T (I-B)z(s)ds, t\in[0,T]$$

Assume the value of the unit goods vector at time t is a bounded measurable function u: [0, T]→$\mathbb{R}^n$ then the objective function is formulated as:

$$\text{Max } W(x)=\int_0^T u'(t)(I-B)z(t)dt$$

Since, (I-B)z represents the positive production of goods which can be achieved by some nonnegative z iff (I-B) has a nonnegative inverse (Gale, 1960). By E_n, we denote the space of bounded measurable function from [0, T] into $\mathbb{R}^n$. The Primal continuous-time Leontief Production Model (PLPM) is formulated as:

$$\begin{cases}\text{Max } W(x)=\int_0^T a'(t)x(t)dt\\ \text{s.t. } A(t)x(t)\le c(t)+\int_0^t Mx(s)ds, t\in[0,T]\\ \text{and } x\in E_n, x(t)\ge 0, t\in[0,T]\end{cases} \quad \text{(PLPM)}$$

where, x(t) = $(I-B)^{-1}Mz(t)$, M is an arbitrary positive n×n matrix, A = $S(I-B)^{-1}M$, $a'(t)=u'(t)\times M$, a∈E_n, c∈E_m and A and M are constants matrices of size m×n. The dual continuous-time Leontief production model is formulated as:

$$\begin{cases}\text{Min } G(y)=\int_0^T c(t)y(t)dt\\ \text{s.t. } A'y(t)\ge a(t)+\int_t^T M'y(s)ds, t\in[0,T]\\ \text{and } y\in E_m,\ y(t)\ge 0, t\in[0,T]\end{cases} \quad \text{(DLPM)}$$

Now, if we consider the value of a unit of each production decreases with increasing production, so that, that the total value is a non linear function of production then the objective function of the Leontief production model is described as follows.

Where D is a symmetric semi-definite matrix. Hence, we can formulate the Primal Quadratic Leontief Production Model (PQLPM) as follows:

$$\begin{cases}\text{Max } W(x)=\int_0^T \left(a'(t)x(t)\right)+\frac{1}{2}x'Dx(t)dt\\ \text{s.t. } A(t)x(t)\le c(t)+\int_0^t Mx(s)ds, t\in[0,T]\\ \text{and } x\in E_n,\ x(t)\ge 0, t\in[0,T]\end{cases} \quad \text{(PQLPM)}$$

where, x(t) = $(I-B)^{-1}Mz(t)$, M is an arbitrary positive n×n matrix, A = $S(I-B)^{-1}M$, $a'(t)=u'(t)\times M$, a∈E_n, c∈E_m and A and M are constants matrices of size m×n. The Dual Quadratic Leontief Production Model (DQLPM) as follows:

$$\begin{cases}\text{Min } G(y)=\int_0^T\left(-\frac{1}{2}\upsilon'(t)D\upsilon(t)+c(t)y(t)\right)dt\\ \text{s.t. } A'y(t)\ge a(t)+D\upsilon(t)+\int_t^T M'y(s)ds, t\in[0,T]\\ \text{and } y\in E_m,\ y(t)\ge 0, t\in[0,T]\end{cases} \quad \text{(DQLPM)}$$

RESULTS AND DISCUSSION

Quadratic Leontief models in quantum calculus: The researchers have been formulated the linear Leontief production models in quantum calculus. In this study, we present quantum calculus formulation for quadratic Leontief production model. Throughout this study, we use J to denote the quantum calculus interval:

$$J=[1,T]\cap q^{\mathbb{N}_0}$$

and by E_k, we denote the space of all rd-continuous functions from J into $\mathbb{R}^k$. The Primal Quantum Quadratic Leontief Production Model (PQQLPM) is formulated as:

$$\begin{cases} \text{Max } W(x) = \int_1^{q^{N+1}} \left(a'(t)x(t) + \frac{1}{2}x'(t)Dx(t) \right) d_q(t) \\ \text{s.t. } A(t)x(t) \le c(t) + \int_1^{q^n} Mx(s)d_q(s),\ q^n \in J \\ x \in E_n,\ x(t) \ge 0,\ t \in J \end{cases} \tag{PQQLPM}$$

where, $a \in E_n$, $c \in E_m$ and A and M are matrices of size $m \times n$. The Dual Quantum Quadratic Leontief Production Model (DQQLPM) is formulated as:

$$\begin{cases} \text{Min } G(y) = \int_1^{q^{N+1}} \left(-\frac{1}{2}\upsilon'(t)D\upsilon(t) + c(t)y(t) \right) d_q(t) \\ \text{s.t. } A'y(t) \ge a(t) + D\upsilon(t) + \int_{q^{n+1}}^{q^{N+1}} M'y(s)d_q(s),\ q^n \in J \\ \text{and } y \in E_m,\ y(t) \ge 0,\ t \in J \end{cases} \tag{DQQLPM}$$

Duality theoremes: In this study, we establish the weak duality theorem and strong duality theorem for quantum quadratic Leontief production models.

Theorem 5.1: (Weak duality theorem). If x and y are arbitrary feasible solutions of (PQQLPM) and (DQQLPM), respectively, then $W(\alpha) \le G(\gamma)$.

Theorem 5.2: (Strong duality theorem). If (PQQLPM) has an optimal solution x* then (DQQLPM) has an optimal solution y* such that W(x*) = G(y*).

Remark 5.3: Using definition 2.4, the proof of the weak duality theorem and the proof strong duality theorem are immediate from the proof of standard duality theorems (Dorn, 1960).

Example: In this study, example is given in order to illustrate our formulation.

Example 6.1: Let $T = q^{N0}$ and J = {1, 2, 4} with q = 2. Then we consider the primal quadratic Leontief production model in quantum calculus:

$$\begin{cases} \text{Max } W(x) = \int_1^{2^3} \left[-\frac{4}{2}x^2(t) + 100tx(t) \right] d_q(t) = \\ \sum_{k=0}^{2} 2^k \left[-\frac{4}{2}x^2(2^k) + 1002^k x(2^k) \right] \text{s.t } 600x(t) \le \\ 400t + 3\int_1^t x(s)d_q(s) = 400t + 3\sum_{k=0}^{\log_2 t-1} 2^k x(2^k), \\ t \in J \text{ and } x(t) \ge 0,\ t \in J \end{cases}$$

where, we have used σ and the integral given in definition 2.4. Using MATLAB command quadprog, we have:

$$x^*(1) = 0.67, x^*(2) = 1.34$$
$$x^*(4) = 2.18, W(x^*) = 4829.08$$

On the other hand, the dual quadratic Leontief production model in quantum calculus is:

$$\begin{cases} \text{Min } G(y,w) = \int_1^{2^3} \left[2y^2(t) + 400tz(t) \right] d_q(t) = \\ \sum_{k=0}^{2} 2^k \left[2y^2(2^k) + 400(2^k)z(2^k) \right] \text{s.t. } 600z(t) \ge \\ 100t - 4y(t) + 3\int_{\sigma(t)}^{\sigma(4)} z(s)d_q(s) = 100t - 4y(t) + \\ 3\sum_{k=1+\log_2 t}^{2} 2^k z(2^k),\ t \in J \text{ and } y(t), z(t) \ge 0,\ t \in J \end{cases}$$

where, we have used again definition 2.4. Using MATLAB command quadprog we have:

$$y^*(1) = 0.67,\ y^*(2) = 1.34,\ y^*(4) = 2.68$$
$$z^*(1) = 0.18,\ z^*(2) = 0.34,\ z^*(4) = 0.65$$

and the optimal value is G(y*, z*) = 48.2908, thus, U (x*) = G(y*, z*).

CONCLUSION

In this study, a quantum calculus analogue of non-linear dynamic production model function have been presented. This formulation is given for primal and dual quantum Leontief production models. Furthermore, a new version of some fundamental duality theorems is given for arbitrary quantum set. The new formulation provides the exact optimal solution for the production models by solving either the primal quadratic model or the dual quadratic model which reduced the large computation effort. Using the new formulation, less work has been devoted to reach the optimal solution of Leontief production models. Moreover, we can get upper and lower bounds for the value of production at any production plan using the quadratic objective functions for the primal and the dual quantum calculus models.

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