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The Disc Structures of Commuting Involution Graphs for Certain Simple Groups

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Abstract: Suppose G is a finite group and X is a subset of G. The commuting graph on the set X, C (G, X) whose vertex set X with any two vertices connected by an edge, if and only if they commute. In this study, we consider as the Mathieu groups, symplectic groups, together with their automorphism groups and are conjugacy classes of involutions. Let $t \in X$, here, we investigate the orbits under the action of $C_G(t)$ from a fixed vertex t and describe the group theoretic structure of (t, x) where x is a $C_G(t)$ orbit representative.

Key words: Commuting graph, Mathieu group, symplectic group, conjugacy class, involution, automorphism

INTRODUCTION

The algebraic graph theory involves the use of group theory and the study of graph. Recently, mathematicians try to assign a graph to an algebraic structure, for examples, Bertram (1983), Camina and Camina (2011). They used the advantages of graph properties for the algebraic structures and vice versa.

Let G be any finite group and X is any G-conjugacy class. We form an undirected graph with vertex set X such that any distinct vertices $x, y \in G$ being joined whenever $x \neq y$ and $xy = yx$. Such a graph is known as a commuting graph of G on X denoted as C (G, X). Clearly, G embeds graph automorphism of C (G, X) and is transitive on the vertices of C (G, X). Without loss of generality, we choose $t \in G = t^G$ to be a fixed point to start off constructing C (G, X).

There is a large literature which is assigned to express groups as graphs for the purpose of investigating the properties of groups by using the structure of the graphs. Many literature has studied C (G, X) for various kind of G and X. To our best knowledge, the case when G has even order X = G/Z (G) and was first studied by Brauer and Fowler (1955). When X is specifically a G-conjugacy class of involution C (G, X) is called a commuting involution graph. This was happened by the research by Fischer (1971) which led to the construction of 3-transposition groups. Other resaerch of literatures on commuting involution graphs are stated by Bates *et al.* (2003, 2007), Everett (2011), Everett and Rowley (2010), Perkins (2006), Rowley and Taylor (2011).

Let $x \in X$ the group action of on induces an action by conjugation on the centralizer of x in G such that $C_G(x) = \{g \in G | xg = gx\}$. Let d (x, y) be the usual distance function on the commuting involution graphs. When C (G, X) is connected, the ith disc of C (G, X) around t is defined as $\Delta_i(t) = \{x \in X | d(t, x) = i\}$. We then define the diameter of the graph, Diam C (G, X) to be the maximal distance between two of its vertices.

By Bates *et al.* (2007), the commuting involution graphs is studied when G is one of the 26 sporadic finite groups and X is an involution conjugacy class of G as our main reference and the result is shown in theorem 1.1. Our approach, here is to compute diameter of C (G, X) and provide the elements at a given distance from a fixed involution in detail.

Theorem 1.1 (Bates *et al.*, 2007): Let G be any Mathieu groups and X be the conjugacy classes of involutions in G. Then, C (G, X) is connected with Diam C (G, X) = 3 excluding $G = M_{12}$ and X = 2A with Diam C (G, X) = 2. For convenience, the following lemma will be included here where this result plays an important role in the recent studies of this particular graph.

Lemma 1.1 (Bates *et al.* 2007): Let $x \in X$ where X consists of involutions and put z = tx and let m be the order of z:

- $x \in \Delta_1(t)$ if and only if m = 2
- If m is even $m \geq 4$ and $z^{m/2} \in X$ then $x \in \Delta_2(t)$
- If $C_{C_G(z)}(x) \cap X = \varnothing$, then $d(t, x) \geq 3$, if $C_{C_G(z)}$ has odd order
- Suppose that m is odd and assume that there do not exist any elements $g \in G$ of order 2 m such that $g^2 = z$ and $g^m \in X$. Then $d(t, x) \geq 3$

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Apart from that the group theoretic structure (t, x) was taken into consideration for the same group in the study by Bates *et al.* (2007). The outcomes of (t, x) motivate us to study the commuting involution graphs on one of the classical group -S_4 (2), S_4 (3) and S_6 (2) so called Symplectic group. We select some group among the Symplectic groups -S_4 (2)′, S_4 (3) and S_6 (2) to perform calculations as informations needed are already complete in ATLAS[13].

Besides that, we extend the study by Bates *et al.* (2007) to determine the number of C_G (t)-orbits together with their sizes at a given distance from a fixed involution t on the Mathieu groups -M_{11}, M_{12}, M_{22} and M_{24} and Symplectic group -S_4 (2)', S_4 (3) and S_6 (2). For the notations of all groups and conjugacy classes, we use standard conventions of the online ATLAS[13]. The Mathieu groups may be viewed as permutation groups on 11, 12, 22, 23 and 24 points, respectively. Meanwhile the Symplectic groups can be observed, respectively, on 10, 27 and 28 points of permutation groups.

The basic concept such as groups and finite simple groups can be found by Wilson (2009). We investigate the orbits under the action of C_G (t) and hence, determine the subgroups generated by elements of t and x∈X where x is known as a C_G (t)-orbit, representative. The results of this study are provided.

MATERIALS AND METHODS

Commuting involution graphs: In this study, we introduce our main results of this study which are catalogued into 3 subsections. Subsection 2.1 deals with the study of commuting involution graph in Mathieu group. Subsection 2.2 explains the results of commuting involution graph in symplectic group. Subsection 2.3 contains some observation on subgroup structures for both commuting involution graphs in two previous subsections and followed by some examples.

Commuting involution graphs in mathieu groups: In this subsection, we investigate the commuting involution graphs in five Mathieu groups. Table 1 is the disc sizes of C (G, X) first found by Bates *et al.* (2007). The majority of cases in Table 1 have the Diam C (G, X) = 3. The graphs C (M_{12}, 2A)and C (M_{22} 2B) have diameter 2. However, Diam C (M_{12}, 2C) = 4.

Although, C (G, X) brings the class of elements of the same order (involution) but every class has different cycle types. For instance, the graphs C (M_{24}, X) contains of 2 different conjugacy classes 2A and 2B with elements of cycle types 2^8 1^8 and 2^{12}, respectively (Anonymous, 2016).

Theorem 2.1 gives the main result of our investigation which is the disc structures of involution C(G, X) in Mathieu groups. The data are grouped according to the conjugacy class of tx for x∈Δ_i(t). We found the number and size of orbits in each Δ_i(t). Moreover, the subgroup H = (t, x) is identified whenever x is a C_G(t)-orbit representative.

Theorem 2.1: The disc structures of C (G, X) in Mathieu groups which determine the distance of t and x∈Δ_i (t) are given in Table 2.

Commuting involution graph in symplectic groups: This subsection focuses on the investigation of commuting involution graphs in three symplectic groups. The result in Table 3 demonstrates the disc sizes of C (G, X) for S_4 (2)′ and S_4 (3) that has been found by Everett and Rowley (2010) and we continue this research by obtaining the disc sizes of C (G, x) for S_6 (2). We can pin down Diam C (G, X) by using result concerning Δ_i(t) and Table 3 shows that 2≤ Diam C (G, X)≤5. As what has been covered in theorem 2.1, the next result is obtained and stated in theorem 2.2 for the disc structures of involution C (G, X) in symplectic groups.

Theorem 2.2: The disc structures of C (G, X) in symplectic groups which determine the distance of t and X∈Δ_i are given in Table 4.

RESULTS AND DISCUSSION

Observations on subgroup structures: This subsection starts with the result concerning the subgroup (t, x) where x is an element of C_G(t)-orbits representative (Table 1-4).

Theorem 2.1: Let x∈X and assume that m be the order of tx. Then, we have:

- If then x∈Δ_1 (t) then t≅K_4 Klein four-group of order 4
- If d (t, x)≥2 then t ≅D_{2m} dihedral group of order 2 m

Proof: Suppose that G is any Mathieu or symplectic groups and X are the G-conjugacy classes of involutions. If t∈X then t is an involution or:

$$t^2 = 1 \quad (1)$$

Since, x be the C_G(t)-orbits representative then x is also an element of X where x is an involution. Say that:

Table 1: Disc size and diametres of iovolution C (G, X) in Mathieu groups

G/t	|X|	|Δ_1 (t)|	|Δ_2 (t)|	|Δ_3 (t)|	|Δ_4 (t)|
M_{11}					
2A	165	12	104	48	-
M_{12}					
2A	396	35	360	-	-
2B	465	30	352	112	-
2C	792	31	360	360	40
M_{22}					
2A	1155	50	720	384	-
2B	330	49	280	-	-
2C	1386	25	400	960	-
M_{23}					
2A	3795	98	2800	896	-
M_{24}					
2A	11385	280	9184	1920	-
2B	31878	277	21680	9920	-

Table 2: Disc sizes and diameters of involution C(G, X) in Mathieu groups

G	t/Disc	Class of tx	No. of orbits	Orbit size	⟨t, x⟩
M_{11}	**2A**				
	Δ_1 (t)	2A	1	12	K_4
	Δ_2 (t)	3A	1	8	D_6
		3A	1	24	D_6
		4A	1	24	D_8
		6A	2	24	D_{12}
	Δ_3 (t)	5A	1	48	D_{10}
M_{12}	**2A**				
	Δ_1 (t)	2A	2	10	K_4
		2B	1	15	K_4
	Δ_2 (t)	3B	1	60	D_6
		4A	1	30	D_8
		5A	1	120	D_{10}
		6A	2	60	D_{12}
	2B				
	Δ_1 (t)	2B	1	6	K_4
		2B	1	24	K_4
	Δ_2 (t)	3A	2	32	D_6
		4A	1	48	D_8
		4B	1	48	D_8
		6B	2	96	D_{10}
	Δ_3 (t)	3B	1	16	D_6
		5A	1	96	D_{10}
	2C				
	Δ_1 (t)	2A	1	1	K_4
		2A	1	15	K_4
		2B	1	15	K_4
	Δ_2 (t)	3B	1	60	D_6
		5A	2	60	D_{10}
		6A	1	60	D_{12}
		10C	2	60	D_{20}
	Δ_3 (t)	6B	1	120	D_{12}
		11A	1	40	D_{22}
	Δ_4 (t)	3A	1	40	D_6
M_{22}	**2A**				
	Δ_1 (t)	2A	1	6	K_4
		2A	1	8	K_4
		2A	1	12	K_4
		2A	1	24	K_4
	Δ_2 (t)	3A	1	192	D_6
		4A	3	48	D_8
		4B	2	96	D_8
		6A	1	192	D_{12}
	Δ_3 (t)	5A	1	384	D_{10}
	2B				
	Δ_1 (t)	2B	1	7	K_4
		2B	1	42	K_4
	Δ_2 (t)	3A	1	112	D_6
		4B	1	168	D_8

Table 2:Continue

G	t/Disc	Class of tx	No. of orbits	Orbit size	⟨t, x⟩
	2C				
	Δ_1 (t)	2B	1	5	K_4
		2B	1	20	K_4
	Δ_2 (t)	4D	2	40	D_8
		5A	1	320	D_{10}
	Δ_3 (t)	3A	1	80	D_6
		4B	1	80	D_{83}
		6A	1	160	D_{12}
		11A	1	640	D_{22}
M_{23}	**2A**				
	Δ_1 (t)	2A	1	14	K_4
		2A	1	84	K_4
	Δ_2 (t)	3A	1	448	D_6
		4A	3	336	D_8
		6A	1	1344	D_{12}
	Δ_3 (t)	5A	1	896	D_{10}
M_{24}	**2A**				
	Δ_1 (t)	2A	2	14	K_4
		2A	1	168	K_4
		2B	1	84	K_4
	Δ_2 (t)	3A	1	896	D_6
		4A	2	112	D_8
		4B	2	672	D_8
		4B	1	1344	D_8
		6A	1	5376	D_{12}
	Δ_3 (t)	3B	1	128	D_6
	2B				
	Δ_1 (t)	2A	1	15	K_4
		2A	1	60	K_4
		2B	1	2	K_4
		2B	1	80	K_4
		2B	1	120	K_4
	Δ_2 (t)	3B	1	960	D_6
		4A	2	120	D_8
		4B	2	240	D_8
		4B	1	480	D_8
		4C	2	160	D_8
		4C	2	960	D_8
		5A	1	1920	D_{10}
		6B	2	1920	D_{12}
		10A	2	1920	D_{20}
		12B	2	3840	D_{24}
	Δ_3 (t)	3A	1	320	D_6
		6A	1	1920	D_{12}
		11A	1	7680	D_{22}

Table 3: Disc sizes and diameters of involution C (G, X) in symplectic groups

G/t	|X|	$\|\Delta_1$ (t)|	$\|\Delta_2$ (t)|	$\|\Delta_3$ (t)|	$\|\Delta_4$ (t)|	$\|\Delta_5$ (t)|
S_4 (2)'						
2A	45	4	8	16	16	-
2BC	30	9	20	-	-	-
2D	36	5	20	10	-	-
S_4 (3)						
2A	45	12	32	-	-	-
2B	270	21	136	112	-	-
2C	36	15	20	-	-	-
2D	540	15	104	228	184	8
S_6 (2)						
2A	63	30	32	-	-	-
2B	315	42	272	-	-	-
2C	945	64	496	384	-	-
2D	3780	51	560	2528	640	-

$$x^2 = 1 \quad (2)$$

Then, we construct a subgroup by t and x so-called (t, x). By considering (Eq. 1 and 2) yields the following statement. We note that x is a $C_G(t)$-orbits representative in $\Delta_1(t)$. The product of t and x which is tx has order 2 such that:

$$t^2x^2 = (tx)^2 = 1 \quad (3)$$

Table 4: Disc sizes and diameters of involution C(G, X) in symplectic groups

G	t/Disc	Class of tx	No. of orbits	Orbit size	(t, x)
$S_4(2)'$	**2A**				
	$\Delta_1(t)$	2A	2	2	K_4
	$\Delta_2(t)$	4A	2	4	D_8
	$\Delta_3(t)$	3A	1	8	D_6
		3B	1	8	D_6
	$\Delta_4(t)$	5A	1	8	D_{10}
		5B	1	8	D_{10}
	2BC				
	$\Delta_1(t)$	2A	1	3	K_4
		2A	1	6	K_4
	$\Delta_2(t)$	3AB	1	8	D_6
		4A	1	12	D_8
	2D				
	$\Delta_1(t)$	2A	1	5	K_4
	$\Delta_2(t)$	5AB	1	20	D_{10}
	$\Delta_3(t)$	4A	1	10	D_8
$S_4(3)$	**2A**				
	$\Delta_1(t)$	2B	1	12	K_4
	$\Delta_2(t)$	3D	1	32	D_6
	2B				
	$\Delta_1(t)$	2A	1	3	K_4
		2B	1	6	K_4
		2B	1	12	K_4
	$\Delta_2(t)$	3C	1	16	D_6
		4A	2	12	D_8
		4B	2	24	D_8
		6F	1	48	D_{12}
	$\Delta_3(t)$	3D	1	16	D_6
		5A	1	96	D_{10}
	2C				
	$\Delta_1(t)$	2B	1	15	K_4
	$\Delta_2(t)$	3C	1	20	D_6
	2D				
	$\Delta_1(t)$	2A	1	3	K_4
		2B	1	6	K_4
		2B	1	12	K_4
	$\Delta_2(t)$	3C	1	16	D_6
		4A	2	12	D_8
		4B	2	24	D_8
		6F	1	48	D_{12}
	$\Delta_3(t)$	3D	1	16	D_6
		5A	4	96	D_{10}
	$\Delta_4(t)$	3C	1	16	D_6
		4A	2	12	D_8
		4B	2	24	D_8
		6F	1	48	D_{12}
		3C	1	8	D_6
$S_6(2)$	**2A**				
	$\Delta_1(t)$	2A	1	30	K_4
	$\Delta_2(t)$	3A	1	32	D_6
	2B				
	$\Delta_1(t)$	2B	1	18	K_4
		2C	1	24	K_4
	$\Delta_2(t)$	3C	1	128	D_6
		4D	1	144	D_8
	2C				
	$\Delta_1(t)$	2B	1	6	K_4
		2C	1	2	K_4
		2C	1	8	K_4
		2C	1	24	K_4
		2D	2	12	K_4
	$\Delta_1(t)$	3A	1	32	D_6
		4A	1	48	D_8
		4B	2	16	D_8
		4E	2	96	D_8
		6D	1	192	D_{12}
	$\Delta_3(t)$	3C	1	128	D_6

Table 4:Continue

G	t/Disc	Class of tx	No. of orbits	Orbit size	(t, x)
	5A		1	256	D_{10}
	2D				
	$\Delta_1(t)$	2A	1	3	K_4
		2B	1	3	K_4
		2B	1	12	K_4
		2C	1	3	K_4
		2C	1	6	K_4
		2D	2	6	K_4
	$\Delta_2(t)$	3C	1	32	D_6
		3C	1	96	D_6
		4A	1	12	D_8
		4B	2	24	D_8
		4C	2	24	D_8
		4D	3	12	D_8
		4E	4	24	D_8
		6E	2	96	D_{12}
	$\Delta_3(t)$	3A	1	8	D_6
		3A	1	24	D_6
		4A	1	48	D_8
		4D	1	48	D_8
		5A	1	192	D_{10}
		6A	2	24	D_{12}
		6C	1	192	D_{12}
		6D	1	48	D_{12}
		7A	1	384	D_{14}
		8A	2	96	D_{16}
		8B	2	96	D_{16}
		9A	1	384	D_{18}
		12A	2	96	D_{24}
		12B	2	96	D_{24}
		15A	1	384	D_{30}
	$\Delta_4(t)$	3B	1	64	D_6
		6B	2	48	D_{12}
		6D	2	48	D_{12}
		12C	2	192	D_{24}

Hence:

$$\left\langle t,x \mid t^2 = x^2 = (tx)^2 = 1 \right\rangle \quad (4)$$

Which is clearly obtained that a presentation of the Klein four-group is an elementary abelian group of order 4. When is a C_G (t)-orbits representative in Δ_i (t) for $i \geq 2$, observe that the product of t and x which is tx has order m where m or:

$$t^m x^m = (tx)^m = 1 \quad (5)$$

Hence:

$$\left\langle t,x \right\rangle t^m = x^m = (tx)^m = 1 \quad (6)$$

Thus, Eq. 6 is a presentation of dihedral group D_{2m} of order 2 m. Given the subgroups K_4 and D_{2m} are distinguished based on their abelianization of subgroup structure. K_4 is an abelian subgroup while D_{2m} is a subgroup of non-abelian whenever $m \geq 3$.

The group theoretical computer algebra system (Groups, Algorithm and Programming) GAP[10] provides an access on the descriptions of small order groups, so called small groups library. All group are sorted by their orders and listed up to isomorphism.

Before, we illustrate some example on how the constituent C_G(t)-orbit representative in each Δ_i (t) can be broken down into the subgroup ⟨t, x⟩ the computational method are provided by using mathematical package MAGMA (Cannon and Playoust, 1997) mathematically to determine which known group it is isomorphic to.

Let t be a fixed conjugacy class of involution in one of the Mathieu or symplectic groups. Suppose that x be the C_G (t)-orbit representative in Δ_i(t) for $i \in \mathbb{N}$. The subgroup H of G is constructed by the elements of t and x. We find the composition factor series of H that provide an alternative to break up H into small pieces. By determining the order of H = ⟨t, x⟩ the factorizations of the order are resolved. This yields a sequence of two-element tuples with [⟨P_1, K_1, ..., ⟨$P_r K_r$⟩⟩] with P_1, P_2, ..., P_r distinct prime numbers and K_i positive which is employed to represent integers in factored form. For the particular order of H we search the candidates of known groups. By using the abstract properties, some candidate can be eliminated immediately once we know whether H is abelian or not. Thus, the following are the examples of some subgroup H.

Order of tx is 2: Let H = ⟨t, x⟩ be a group of order 4 and has the composition factor series 1 = $Z_2 \triangleleft Z_2$ = G. There are two distinct isomorphism types of groups of order 4 K_4 and Z_4 and both groups are abelian. To choose the known group, we find out the abstract properties of using MAGMA (Cannon and Playoust, 1997). Hence, H ≅ K_4, since, we know that H is an elementary abelian and non-cyclic subgroup.

Order of tx is 3: Let H = ⟨t, x⟩ be a group of order 6 and has the composition factor series 1 = $Z_3 \triangleleft Z_2$ = G. There are two distinct isomorphism types of groups of order 6 D_6 and Z_6. Based on their abelianization H is non-abelian. Since, we know Z_6 is abelian, thus, H ≅ D_6.

Order of tx is 4: Let H = ⟨t, x⟩ be a group of order 8 and has the composition factor series 1 = $Z_2 \triangleleft Z_2 \triangleleft Z_2$ = G. There are five distinct isomorphism types of groups of order 8, D^8, $Z^4 \times Z^2$, 2^3, Q_8 and Z_8. The properties of subgroup H are nilpotent and extraspecial groups. Groups Q_8 and D_8 satisfy these properties, thus by looking at the presentation of the product t and x which can admit $(tx)^4 = 1$ we can conclude that H ≅ D_8. The rest of the order of conjugacy class tx are having the same procedure to obtain the structure of H = ⟨t, x⟩ but the abstract properties make them different to find out the known group up to isomorphism.

CONCLUSION

Through, out this research, the structure of commuting graphs C (G, X) has been completely studied when considering G as the Mathieu or symplectic groups and involution conjugacy classes X. Having found the representatives x for the C_G (t)-orbits on each case of C (G, X) we wish to determine which disc of the graph lies in and hence, discover the diameter and disc sizes of C(G, X). We also, examine the order of the product tx in each C_G (t)-orbit to categorize the number and size of the orbits. Hence, the subgroup generated by elements t and x are obtained.

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