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New Types of Fixed Points in Operator Topological Space

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Abstract: In this study, we state and prove certain types of fixed point in operator topological spaces. We generalize, the concepts of orbit, periodic and fixed point property in operator topological space, also, we found new concepts in operator theory such as productive spaces and homeomorphism between these space.

Key words: ρ-fixed point, ρ-orbit, ρ-periodic, ρ_1-homeomorphism, composite operator function, topological space

INTRODUCTION

In dynamical system we are familiar with notions like orbit of point (Adams and Franzosa, 2008) fixed point (Adams and Franzosa, 2008), periodic or non-periodic fixed point (Ciric, 1975). We deals with these concepts in topological spaces using the theory of operators to obtain a new kinds of fixed point according to the sets con-taining these points and the act of an operators on these topological spaces. Some researchers has been discussed the properties of topological space caused by an operator γ: P(X)→P(X) associated with these spaces. Mustafa and Alhindawy (2010), Mustafa and Kasim, (2004), Mustafa and Hassan (2004), Colasante (2004), Rosas and Vielma (2000) and Ahmed and Hussain (2003), Vielma and Carpintero (2002) they are studied and defined basic notions such as operator topo-logical space, γ-open, γ-open cover, γ-semi connected, γ-locally connected spaces and γ-copmpact space. Ogata (1991) defined continuous function between two operator topological spaces X and Y with two operators γ and φ associated with these spaces, respectively, these functions named γφ-continuous functions, also he defined γ-connected space γ- T_2-space (Hausdorff operator topological space).

By means of operators, we added another notions concerning with these concepts. First of all, we recall in section two some basic concepts that mentioned above. In section three, we define the ρ-fixed point, finite intersection and an arbitrary union properties in operator topological spaces also productive between these spaces has been presented. The concept of ρ-fixed point in ρ-connected space discussed in that section beside that we present the idea of ρ-homeomorphism that is helpful in proving of the ρ-fixed point as a topological property.

A subset W of space X is said to be β-open (Mashhour *et al*., 1982) (resp. (α-open (Njastad, 1965), preopen (Velicko, 1968) if W⊆cl (int(cl(W))) (resp. W⊆int (cl(int(W)), W⊆int(cl(W))), We denote to the set of all sets of space X by P(X) or I^x. The set of indicator has been refered by the symbol Λ or J.

MATERIALS AND METHODS

Preliminaries

Definition 2.1; (Mustafa and Alhindawy, 2010; Mustafa and Kasim, 2004): Let, (X, T) be an topological space and ω: P(X)→P(X) be a function such that U⊆ω(U) for all U-open in X, then ω is called an operator associated with topology T on X. The triple (X, T, ω) is called an operator topological space (briefly O.T.S.).

Definition 2.2; (Ogata, 1991): Consider X be an operator topological space. X is said to be ρ-connected if it's satisfies one of the following conditions:

- There doesn't exist a pair of disjoint ρ-open sets whose union is X
- If A is ρ-clopen in X, then A = Ø or A = X

Otherwise, X is said to be ρ-disconnected.

Definition 2.3; (Ogata, 1991): Suppose f: (X, T, γ)→(y, σ, φ) be a function. f is called γφ-continuous if for every x∈X and for each N⊆y involving f(x), there exist M⊆X involving x such that $\gamma(M)\subset f^{-1}(\varphi(N))$.

RESULTS AND DISCUSSION

Types of ρ-fixed point: In the coming parts we show progressively the main results related with ρ-fixed point. Some theorems and propositions has been proved.

Definition: 3.1: Consider (X, T, ρ) be an operator topological space and f: (X, T, ρ)→(X, T, ρ) be ρ-continuous. Appoint $x_0 \in \rho(f^{-1}(x_0)) \subseteq X$ is said an ρ-fixed point if $f(x_0) = x_0$.

Example 3.2: Let, X= {1, 2, 3} with topology T = {Φ, X, {1}}. Define f: (X, T, ρ)→(X, T, ρ) as follows: f(1) = {1}, f(2) = {1, 3}, f(3) = {2, 3}. Also, defined the operator ρ: P(X)→P(X) as follows:

$$\rho(A) = int(cl(A)),\ A \in T$$

It's obviously that 1 is ρ-fixed point (preopen-fixed point), since, 1∈ρ(f(1)) = X and f(1) = 1 (a set K in space X is called to be preopen set if, K⊆int((cl(K)))) (Mashhour *et al*., 1982). In similar way we can see that the point 2 is preopen-fixed point.

Remark 3.3: Every ρ-fixed pointis is fixed point while the convers is not necessarily correct that is clearly appeared in example below.

If we have R to be the set of all real number associated by the usual topology T_u and operator ρ: P(X)→P(X) which is defined as follows:

$$\rho(A) = int(A),\ A \in T_u$$

Define f: (R, T_u, ρ)→(R, T_u, ρ) as follow:

$$f(x) = \begin{cases} \frac{1}{\sin x}, & x \in (0,1) \\ x, & x \notin (0,1) \end{cases}$$

it's clear that 1 is fixed point but not ρ-fixed point (since, 1∉ρ(0,1) = int(0,1)).

In other hand if we redefine the operator in the example above to be ρ(A) = cl(A), A∈T_u, then, 1 is fixed point and ρ-fixed point (θ-fixed point). (a subset F in space X is said to be θ-open set if, F⊆cl(F) (Velicko, 1968).

If we let ρ: P(X)→P(X) to be the identity operator in a space X, i.e., ρ(A) = id(A) = A, hence, the ρ-fixed point is the same as fixed point in the ordinary sense.

Definition 3.4: Let, h: (X, T, ρ)→(X, T, ρ) be a function, the ρ-orbit of a point x∈X under h is the sequence ρ({x, h(x), h2 (x), ...,}) and is denoted by ρ(O(x)).

Similarly, we define ρ(O(h(x))) = ρ({h(x), h^2(x), ...,}). In general, we have ρ (O(h^n (x))) = ρ({h^n (x), h^{n+1} (x), h^{n+2} (x), ...,}); n = 0, 1, 2, ..., Where:

$$h^1(x) = x,\ h^2(x) = h(h(x))$$

Remarks and example 3.5: If x is ρ-fixed point of f, then, ρ($_0$(x)) = ρ({x, x, ...,}). If ρ = id (the identity operator), then the ρ-orbit is the same as the orbit in ordinary sense. If x is ρ-fixed point of f and ρ: P(X)→P(X) defined as ρ(A) = int(cl(int(B))), hence, the ρ-orbit here is exactly the α-orbit. Since, O(x)⊆ρ(O(x)) = int(cl(int(O(x)))) which is analogous to α-open. We deals with O(x) as a set.

Definition 3.6: Let, (X, T, ρ) be an operator topological space and f: (X, T, ρ)→(X, T, ρ) be a function. A point x∈X is called ρ-periodic point with respect f, if there exist positive integer m≥2 such that x∈ρ (f^m (x)).

Remarks and examples 3.7: Let, f: (R-{0}, T_u, ρ)→(R-{0}, T_u, ρ) defined as f(x) = 1/x; ∀x∈R-{0}. Take the operator ρ: P(X)→P(X) to be the closure operator, i.e., ρ(B) = cl(B); ∀B∈T_u. Then, x∈R is cl-periodic for m≥2. Each ρ-fixed point is ρ-periodic but the convers is not necessarily true. We can prove that by example above, if x = 3, then, it's ρ-periodic point (cl-periodic) but not ρ-fixed point (cl-fixed point). If x is ρ-periodic then x∈ρ (O(f^2(x))).

Definition 3.8: Assume that f: (X, T, ρ)→(X, T, ρ) be a ρ-continuous function. Let, A = {ρ(A_i): i∈Λ} be a ρ-cover of X, then x_0∈X is called ρ-fixed point of the cover A (A-ρ-fixed point), if there exist A_{i_0}, i_0∈Λ contains x_0 and f(x_0).

Definition 3.9: Consider (X, T, ρ) be an operator topological space. We claim that: if the arbitrarily union of any ρ-open sets in (X, T, ρ) is also ρ-open set, then (X, T, ρ) is said to has property U. If the finite intersection of ρ-open sets is ρ-open set in (X, T, ρ), then (X, T, ρ) is said to has property I.

Theorem 3.10: Let, (X, T, ρ) be a Hausdorff operator topological space (ρ-T_2 space) has the property I. Let, A = {ρ(B_i): i∈Λ} be a ρ-cover of X and f: (X, T, ρ))→(X, T, ρ) is a ρ-continuous function, then f has an ρ-fixed point if and only if f has A-ρ-fixed point.

Proof: Let, f has ρ-fixed point and A = {ρ(A_i): i∈Λ} be an ρ-cover of X. From the definition 3.8 above, f has A-ρ-fixed point conversely,

Consider the ρ-continuous function f hasn't ρ-fixed point, then f(x)≠x for all x∈ρ(f(x))⊆X. By hypothesis f is ρ-continuous, X is ρ-T_2 space with property I, hence, there are ρ-open subsets of X, G_i and U_i, containing x and f(x), respectively such that ρ(G_i)∩ρ(U_i) = ∅ and f(ρ(G_i))⊆ρ(U_i). Now, g = {ρ(G_i): i∈Λ} is an ρ-cover of X, so there is $\rho(G_{i_0}) \in g$ containing x_0∈X and f(x_0). Since, $x_0 \in \rho(G_{i_0})$, so, $f(x_0) \in f(\rho(G_{i_0})) \subseteq \rho(U_{i_0})$ on the other hand $f(x_0) \in \rho(G_{i_0})$, therefore, $f(x_0) \in \rho(G_{i_0}) \cap \rho(U_{i_0})$ which is contradiction, hence, f has ρ-fixed point.

Definition 3.11: Two operator topological space (X_1, T_1, ρ_1) and (X_2, T_2, ρ_2) are said to be $\rho_1\ \rho_2$-homeo-morphic if there is a bijection f: $(X_1, T_1, \rho_1)\to(X_2, T_2, \rho_2)$ such that f is $\rho_1\ \rho_2$-continuous and f^{1} is $\rho_2\ \rho_1$-continuous. If $\rho_1 = \rho_2$ then, the spaces becomes ρ_1-homeo- morphic and f is called ρ_1-homeomorphism.

Remark and examples 3.12: Let, f : $(R, T_1, \rho_1)\to((a, \infty), T_2, \rho_2)$ be a function while T_1 and T_2 are stand for ordinary and relative ordinary topology on set of real numbers R and subset (a, ∞), respectively. Let, the operators ρ_1 and ρ_2 defined as follows:

- ρ_1 = id: $P(X)\to P(X)$ to be the identity operator in R i.e., $\rho_1(A) = id(A) = A$
- ρ_2 = cl: $P(X)\to P(X)$ to be the closure operator i.e., $\rho_2(A) = cl(A)$

Define f to be $f(x) = e^x+a$. Hence, f is $\rho_1\rho_2$-homeomorphism, since, f and f^{-1} are contin-uous function. If we take $\rho_1 = \rho_2$ = id in the example ahead then $\rho_1\rho_2$-homeomorphism is exactly the homeomorphism in ordinary sense.

Definition 3.13: Suppose that (X_i, T_i, ρ_i), be an operator topological space while i = 1, 2, 3 and let f: $(X_1, T_1, \rho_1)\to(X_2, T_2, \rho_2)$ and g: $(X_2, T_2, \rho_2)\to(X_3, T_3, \rho_3)$ are two functions. The function $g^\circ f$: $(X_1, T_1, \rho_1)\to(X_3, T_3, \rho_3)$ is called a composite operator function which is defined by $g^\circ f(x) = f(g(x))$.

Proposition 3.14: Consider that (X_i, T_i, ρ_i), i = 1, 2, 3 be operator topological space and let f: $(X_1, T_1, \rho_1)\to(X_2, T_2, \rho_2)$ and g: $(X_2, T_2, \rho_2)\to(X_3, X_3, \rho_3)$ be $\rho_1\rho_2$-continuous and $\rho_2\rho_3$-continuous functions, respectively then the composition $g^\circ f$: $(X_1, T_1, \rho_1)\to(X_3, T_3, \rho_3)$ is $\rho_1\rho_3$-continuous function.

Proof: Assume $x\epsilon X_1$ and $H\subseteq X_3$ be a ρ_3-open set such that $x\epsilon(g^\circ f)^{-1}(H)$. From the defini-tion of $g^\circ f$, we have $x\epsilon(g^\circ f)^{-1}(H) = f^{1}(g^{-1}(H))$. Since, $x\epsilon f^{-1}(g^{-1}(H))$ it follows that there exist a $y\epsilon g^{-1}(H)$ such that $x\epsilon f^{1}(y)$. Since, g is $\rho_2\rho_3$-continuous function, this due to that there exist a ρ_2-open set $G\subseteq X_2$ in which $\rho_2(G)\subseteq g^{-1}(\rho_3(H))$. From here we obtain that $x\epsilon f^{-1}(G)$ and $f^{-1}(\rho_2(G))\subseteq f^{1}(g^{-1}(\rho_3(H))) = (g^\circ f)^{-1}(\rho_3(H))$. Since, $x\epsilon f^{1}(G)$ and f is $\rho_1\rho_2$-continuous, it follows that there exist a ρ_1-open set $A\subseteq X_1$ such that $\rho_1(A)\subseteq f^{1}(\rho_2(G))$. Thus, from $f^{1}(\rho_2(G))\subseteq(g^\circ f)^{-1}(\rho_3(H))$ we obtain that $\rho_1(A)\subseteq(g^\circ f)^{-1}(\rho_3(H))$ this is implies that $g^\circ f$ is $\rho_1\rho_3$-continuous function.

Definition 3.15: Let, (X, T, ρ) is an operator topological space. Then, (X, T, ρ) is said to has the ρ-fixed point property if each ρ-continuous function f: $(X, T, \rho)\to(X, T, \rho)$ has a ρ-fixed point.

Definition 3.16: Let, (X, T, ρ) be an operator topological space has the property P. We say that P is ρ-topological property, if P preserved by every ρ-homeomorphism.

Proposition 3.17: ρ-fixed point property is an ρ-topological property.

Proof: Consider h: $(X, T, \rho)\to(y\ T, \rho)$ be a ρ-homeomorphism and let (X, T, ρ) has the ρ-fixed point property. We must show that (y, T, ρ) has the ρ-fixed point property too. Let, g: $(y, T, \rho)\to(y, T, \rho)$ is a ρ-continuous function, hence, $h^{-1}{}^\circ g^\circ h : (X, T, \rho)\to(X, T, \rho)$ is ρ-continuous function (by Proposition3.14). Since, (X, T, ρ) has the ρ-fixed point property, then, there is a point $x_0\epsilon X$ such that $h^{-1}{}^\circ g^\circ h(x_0) = x_0$. Set $y_0 = h(x_0)$, then $h^{-1}{}^\circ g(y_0) = x_0$. This due to $g(y_0) = h(x_0) = y_0$. Hence, g has ρ-fixed point. Therefore (y, T, ρ) has the ρ-fixed point property.

Theorem 3.18: If the operator topological space (X, T, ρ)has property u and ρ-fixed point property, then, X is a ρ-connected space.

Proof: Consider X is ρ-disconnected space, hence, there exist nonempty ρ-open subset of X, U and V such that $X = U\cup V$. Then, there are two elements $a\in U$ and $b\in V$. Define ρ-continuous function f: $(X, T, \rho)\to(y, T, \rho)$ by:

$$f(x) = \begin{cases} a\ if\ x\in V \\ b\ if\ x\in U \end{cases}$$

Since, $U\cap V = \varnothing$ and $U\cup V = X$, the function is well-define. Also $a\notin V$ and $b\notin U$, so has no ρ-fixed point, therefore, X does not have the ρ-fixed point property and this is contra-diction, hence, X is an ρ-connected.

Theorem 3.19: ρ-connectedness is an ρ-topological property.

Proof: Let X be a ρ-connected and h: $(X, T, \rho)\to(y, T, \rho)$ be a ρ-homeomorphism. If B is an ρ-clopen of y, $h^{-1}(B)$ is an ρ-clopen in X. By hypothesis X is ρ-connected, hence, $h^{-1}(B)$ is either empty or X. This due to $h(h^{-1}(B))$ is either empty or y. Therefore, y is ρ-connected.

Corollary 3.20: Let, (X, T, ρ) and (y, T, ρ) be two operator topological spaces and h: $(X, T, \rho)\to(X, T, \rho)$ be an ρ-homeomorphism. If, X be ρ-connected with ρ-fixed point property, then, y is a ρ-connected.

Definition 3.21: Let, $X = \prod_{i\in J} X_i$. Let, $(\prod_{i\in J} X_i, T)$ be a product space and let ρ be an operation on I^X and I^{X_i} (the class of sets on X and X_i for all $i\in J$). ρ is called productive operation if $\rho(\prod_{i\in J} G_i) \subseteq \prod_{i\in J} \rho(G_i)$ for all $(\prod_{i\in J} G_i) \subseteq \prod_{i\in J} X_i$, $G_i \subseteq X_i$, for all $i\in J$. The triple $(\prod_{i\in J} X_i, T, \rho)$ is called productive operation space.

Corollary 3.22: Suppose that (X_1, T_1, ρ_1), (X_2, T_2, ρ_2) and (X_3, T_3, ρ_3) are operator topological spaces an f: $(X_1, T_1, \rho_1) \rightarrow (X_2, T_2, \rho_2)$ and g: $(X_1, T_1, \rho_1) \rightarrow (X_3, T_3, \rho_3)$ are functions between these spaces. Let, $f\times g$: $(X_1, T_1, \rho_1) \rightarrow (X_2\times X_3, T, \rho)$ be a function defined by $(f\times g)(x) = f(x)\times g(x)$. Then, $f\times g$ is ρ_1 ρ-continuous iff f is ρ_1 ρ_2-continuous and g is ρ_1 ρ_3-continuous.

Proof: $\Rightarrow$ Let, $x\in X_1$ and let, $B\subseteq X_2$, $G\subseteq X_3$ be ρ_2-open and ρ_3-open sets, respectively, such that $x\in f^{-1}(B)$ and $x\in g^{-1}(G)$. Then, we obtain that $f(x)\in B$ and $g(x)\in G$, hence, $f(x)\times g(x) = (f\times g)(x)\in B\times G$. We $x\in(f\times g)^{-1}(B\times G)$, since, $f\times g$ is ρ_1 ρ-continuous it im-plies that there exist ρ_1-open set U in X_1 having the point x such that $U\subseteq(f\times g)^{-1}(B\times G)$ this implies that $U\subseteq f^{-1}(B)$ and $U\subseteq g$-1 (G) and $U\subseteq f^{-1}(B)$. Thus, we prove that f is ρ_1 ρ_2-continuous and g is ρ_1 ρ_3-continuous.

$\Rightarrow$Let, $x\in X_1$ and let, $B\times G \subseteq X_2\times X_3$ be a ρ-open set such that $x\in(f\times g)^{-1}(B\times G)$. We abtain that $(f\times g)(x) = f(x)\times g(x)\in B\times G$, hence, we have $f(x)\in B$ and $f(x)\in G$ and then, $x\in g^{-1}(G)$ and $x\in f^{-1}(B)$. The function f is ρ_1 ρ_2-continuous, hence, there exist ρ_1-open set A such that $x\in A$ and $A\subseteq f^{-1}(B)$ also, since, g is ρ_1 ρ_3-continuous this implies that there exist ρ1-open set D containing x such that $D\subseteq g^{-1}(G)$. We take $x\in A\cap D$ then, we obtain $A\cap D\subseteq f^{-1}(B)$ and $A\cap D\subseteq g^{-1}(G)$, hence, $A\cap D\subseteq(f\times g)^{-1}(B\times G)$. Thus, we prove that $f\times g$ is ρ_1 ρ-continuous.

Theorem 3.23: If X does not have ρ-fixed point property. Then, for any operator topological space y, the operator product space $x\times y$ does not have the ρ-fixed point property.

Proof: Let, y operator topological space and $x\times y$ has the ρ-fixed point property. Let, f: $(X, T, \rho) \rightarrow (X, T, \rho)$ be ρ-continuous and let, Id_y $(y, T, \rho) \rightarrow (y, T, \rho)$ be the iden-tity function.

Let, $f\times Id_y$: $(X\times y, T, \rho) \rightarrow (X\times y, T, \rho)$ be ρ-continuous, by corollary 3.22, since, $x\times y$ has ρ-fixed point property, there exist $(x_0, y_0)\in x\times y$ such that $(f\times Id_y)(x_0, y_0) = (x_0, y_0)$ this implies $(f\times Id_y)(x_0, y_0) = (f(x_0), y_0) = (x_0, y_0)$ implies $f(x_0) = x_0$, that is X has the ρ-fixed point property, this contradiction, therefore, for all y; $x\times y$ does not have ρ-fixed point property.

Corollary 3.24: Let, (X, T, ρ) and (y, T, ρ) be two operator topological spaces if $x\times y$ has ρ-fixed point property, then both spaces have the ρ-fixed point property.

Proof: We can prove that by contradiction. Let, X has not ρ-fixed point property by theorem (3.23) $x\times y$ does not have ρ-fixed point property and this contradiction.

Now, suppose y does not have the ρ-fixed point property. Let f: $(y, T, \rho) \rightarrow (y, T, \rho)$ be a ρ-continuous and assume for all $v\in y$, $f(v) \neq v$, and Id_x: $(X, T, \rho) \rightarrow (X, T, \rho)$ be a ρ-continuous function by corollary 3.22. Since, $x\times y$ has ρ-fixed point property, there exist $(u_0, v_0)\in x\times y$ such that $(Id_x\times g)(u_0, v_0) = (u_0, v_0)$ this implies $f\times Id_y (u_0, v_0) = (u_0, g(v_0)) = (u_0, v_0)$, therefore, $g(v_0) = v_0$ and this contradiction. Hence, y has the ρ-fixed point property.

CONCLUSION

As we state before, some basic notions has been showed that we can find the fixed points in more precisely way, according to the sets contained them but not according to the whole space that located in. depending on that we can say the same things with respect periodic and non-periodic fixed points. On the other hand we prove that the fixed point property in

O.T.S. can be moved between these spaces as a topological property if there is a ρ-homeomorphism tied these spaces.

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