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A Unique Fixed Point Using Fuzzy Cone and ξ-Fuzzy Cone Integrable Functions

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Abstract: The purpose of this study is to present the idea of an f-partition for a closed interval. Also, we introduce a new notion to a notion of fuzzy cone metric space that is called fuzzy cone function as well we prove fundamental fixed-point theorems. Moreover, we obtain a concept of ξ-fuzzy cone integrable function. Finally, we conclude new results for fixed point in the ξ-fuzzy cone integrable functions.

Key words: Fuzzy cone metric space, f-partition for (u, v), fuzzy cone integrable function, ξ-h function, ξ-fuzzy cone integrable function and fixed-point theorems, fundamental

INTRODUCTION

The idea of the cone metric space and the fixed point in the cone metric space was presented by Huang and Zhang (2007). They replaced the set of the real numbers by an ordered Banach space in the definition of the metric and it is a generalized to the notion of the metric space. Further, there are many researchers discussed the generalization of the fixed point in the cone of these spaces (Abbas and Jungck, 2008; Huang and Zhang, 2007; Vetro, 2007). Di Bari and Vetro (2008) introduced a new concept of the fixed point in the field of φ-mapping and concluded some of the results and examples achieved.

One of their interested on the applications in the fuzzy cone metric space was introduced by Bag (2013) where he studied many subjects that are related to it. After that many researchers touched on the concept of the fixed point in this space. Oner *et al.* (2015) provided another definition of the concept of fuzzy cone metric space and concluded some of the results related to it and fuzzy metric space is studied by many researchers (AL-Mayahi and Hadi, 2015).

In this study, the notion of fuzzy cone metric space is investigated and some fundamental definitions are given. Also, we define a new notion of integration to the notion of fuzzy cone metric space that is called fuzzy cone integrable function. We prove fundamental fixed-point theorems. Moreover, we introduce a concept of ξ-fuzzy cone integrable function. Finally, we conclude new results for the fixed point in the ξ-fuzzy cone integrable function. The range of fuzzy cone metric is considered as such where such is defined by the set of all non-negative fuzzy real number that are defined on E*(I) where E*(I) is a given real Banach space (Bag, 2013).

MATERIALS AND METHODS

Background materials: The definitions of the fuzzy number, the convex fuzzy real number and the normal fuzzy number are mentioned by Mizumoto and Tanaka (1979) as well as for the arithmetic operations ⊕, ⊖, ⊙ on E×E were presented by Bag (2015). There are also important definitions such as fuzzy real Banach space, interior point and fuzzy closed subset of such are mentioned by Bag (2013). A subset P of such is called fuzzy cone if:

- P is fuzzy closed, nonempty and $P \neq \{0\}$
- $a, b \in R$, $a, b \geq 0$, $v, w \in P \Rightarrow av \oplus bw \in P$
- $v \in P$ and $\ominus v \in P \Rightarrow v = 0$

Given a fuzzy cone $P \subset E^*(I)$ such, we define a partial ordering $\preceq$ with respect to P by $v \preceq w$ if $w \ominus v \in P$. On the other hand $v \prec w$ such that $v \neq w$ while $v \preceq w$ will stand $\ominus v \in \text{Int } P$.

A fuzzy cone P is said to be normal if there is a number $k > 0$ such that for all $x, y \in E^*(I)$ such with $0 \preceq v \preceq w \Rightarrow v \preceq kw$.

A fuzzy cone P is said to be regular if $\{x_n\}$ is an increasing sequence which is bounded above is convergent that is $x_1 \preceq x_2 \preceq, ..., x_n \preceq w$ for some $w \in E^*(I)$ such, then there is $x \in E^*(I)$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

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Equivalently, P is called regular if $\{x_n\}$ is decreasing sequence which is bounded below is convergent (Bag, 2013 , 2015).

Definition 2.1; Bag (2013): The function D_*: $X \times X \to E^*(I)$ such is satisfies:

- $D_*(x, y) \geq 0$ for all $x, y \in X$
- $D_*(x, y) = 0 \Leftrightarrow x = y$
- $D_*(x, y) = D_*(y, x)$ For all $x, y \in X$
- $D_*(x, y) \preceq D_*(x, y) \oplus D_*(y, z)$ for all $x, y, z \in X$

where, X a nonempty set, then (X, D_*) is called fuzzy cone metric space.

Definition 2.2; Bag (2013): Let (X, D_*) be a fuzzy cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\|c\| \geq 0$, there is a positive $k \in N$ such that $D_*(x_n, x) \prec \|c\|$ for all $n \succ k$, then $\{x_n\}$ is said to be convergent we denote it by $\lim_{n \to \infty} x_n = x$.

Definition 2.3; Bag (2013): Let (X, D_*) be a fuzzy cone metric space and let $\{x_n\}$ be a sequence in x and $x \in X$. If for every $c \in E$ with $0 \preceq \|c\|$ there is a positive $k \in N$ such that $D_*(x_n, x_m)$ for all $m \succ k$, then $\{x_n\}$ is called Cauchy sequence in X.

RESULTS AND DISCUSSION

Definition 3.1: Suppose that P is a normal fuzzy cone in $E^*(I)$. Let $u, v \in E$ and $u \prec v$. Define $(u, v) = \{s \in E: s = \zeta u \oplus (1-\zeta)v$, for some $\zeta \in (0, 1)\}$.

Definition 3.2: The set $P = \{u = s_0 \prec s_1 \prec s_2 \prec, ..., \prec s_n = v\}$ is called a f-partition for (u, v) if and only if the sets $\{(s_{i-1}, s_i)\}_{i=1}^{n}$ are pair wise disjoint and $(u, v) = \{V_{i=1}^{n}[s_{i-1}, s_i]\} V\{v\}$.

Where the fuzzy sets $M_1, ..., M_n$ identified with their membership functions $M_1(s), ..., M_n(s)$ defined on (u, v), form a f-partition of [u, v] if the following conditions for j = 1, ..., n:

- M_j: $[u, v] \to 1$, $M_j(s_j) = 1$
- $M_j(s) = 0$ if $s \notin (s_{j-1}, s_j)$ as mentioned in the definition we will input $u = s_0$, $s_n = v$
- $M_j(s)$ is continuous
- $M_j(s)$, j = 2, ..., n monotone increasing on $[s_{j-1}, s_j]$ and $M_j(s)$, j = 1, ..., n-1 Monotone decreasing on $[s_j, s_{j+1}]$
- For all $s \in [u, v]$:

$$\sum_{j=1}^{n} M_j(s) = 1$$

Definition 3.3: Let F: $[u, v] \to P$ be an increasing bounded function and P is a f-partition of [u, v], we define fuzzy cone upper summation and fuzzy cone lower summation as:

$$\bar{U}^{fcone}(F, P) = \sum_{i=0}^{n-1} F(s_{i+1}) \|s_i - s_{i+1}\|$$

$$\underline{U}^{fcone}(F, P) = \sum_{i=0}^{n-1} F(s_i) \|s_i - s_{i+1}\|$$

Definition 3.4: Let F: $[u, v] \to P$ be an increasing bounded function. The function f is said to be integrable on [u, v] with respect to fuzzy cone [u, v] if and only if P is a f-partition on [u, v]:

$$\lim_{n \to \infty} \bar{U}^{fcone}(F, P) = \lim_{n \to \infty} \underline{U}^{fcone}(F, P) = S^{fcone}$$

We show the common value S^{fcone} $\int_u^v F(s) d_P(s)$ is simply by $\int_u^v F(s) d_P$. Let, $C^I([u, v], P)$ denote the set of all fuzzy cone integrable function.

Lemma 3.5:

- If $[u, v] \subseteq [u, x]$, then $\int_u^v M(s) d_P \preceq \int_u^x M(s) d_P$ for $M \in C^I([u, v], P)$
- $\int_u^v (\alpha M + \beta N)(s) d_P = \alpha \int_u^v M(s) d_P + \beta \int_u N(s) d_P$ for $\alpha, \beta \in R$

Proof: Suppose that the f-partitions P and Q of [u, v] and [v, x], respectively, then $P = \{u = s_0 \prec s_1 \prec s_2 \prec, ..., \prec s_n = v\}$, $Q = \{s_n \prec s_{n+1} \prec, ..., \prec s_{m-1} \prec s_m = x\}$. Let, $0 = QVP$. The 0 is f-partition of [u, x]. Therefore:

$$\underline{U}^{fcone}(M, P) = \sum_{i=0}^{n-1} M(s_i) \|s_i - s_{i+1}\| \preceq \sum_{i=0}^{n-1} M(s_i) \|s_i - s_{i+1}\| + \sum_{i=0}^{m-1} M(s_i) \|s_i - s_{i+1}\| = \underline{U}^{fcone}(M, P) + \underline{U}^{fcone}(M, Q) = \underline{U}^{fcone}(M, O)$$

Hence, $\int_u^v M(s) d_P \preceq \int_u^x M(s) d_P$ (Eq. 2). Suppose that P is a f-partition for [u, v] that is:

$$P = \{u = s_0 \prec s_1 \prec s_2 \prec, ..., \prec s_n = v\}$$

Then:

$$\underline{U}^{fcone}(M, P) = \sum_{i=0}^{n-1} (\alpha M(s_i) + \beta N(s_i)) \|s_i - s_{i+1}\| = \alpha \sum_{i=0}^{n-1} M(s_i) \|s_i - s_{i+1}\| + \beta \sum_{i=0}^{n-1} N(s_i) \|s_i - s_{i+1}\| = \alpha \underline{U}^{fcone}(M, P) + \beta \underline{U}^{fcone}(N, P)$$

Thus:

$$\int_u^v (\alpha M + \beta N)(s) d_P = \alpha \int_u^v M(s) d_P + \int_u^v N(s) d_P$$

Definition 3.6: The function ψ: P→P is called subadditive fuzzy cone integrable function if and only if for each u, v∈P:

$$\int_0^{u\oplus v}\Psi d_P \preceq \int_0^{u}\Psi d_P + \int_0^{v}\Psi d_P$$

Example 3.7: Let X = E = R, then P = $\{\eta_1\in E^*(I):\eta_1\geq 0\}\subset E$ is a normal fuzzy cone, $D_*(x, y) = \|x-y\|$ and $\psi(\tau) = 1/\tau+1$ for all τ>0 then for all ω, υ∈P.

Solution:

$$\int_0^{\omega\oplus\upsilon}\frac{1}{\tau+1}d\tau = \text{In}(\omega\oplus\upsilon\oplus 1), \int_0^{\omega}\frac{1}{\tau+1}d\tau = \text{In}(\omega\oplus 1)$$
$$\int_0^{v}\frac{1}{\tau+1}d\tau = \text{In}(\upsilon\oplus 1)$$

Since, ωυ≥0, thus, ω⊕υ⊕1 ≤ω⊕υ⊕1⊕ωυ = (ω⊕1)(υ⊕1). Therefore, In(ω⊕υ⊕1)≤In(ω⊕υ⊕1⊕ωυ) = In (ω⊕1)+In (υ⊕1). So, we get that ψ is a subadditive fuzzy cone integrable function.

Theorem 3.8: Let (X, D*) be a complete fuzzy cone metric space and P a normal fuzzy cone. Suppose that ψ: P→P is a subadditive fuzzy cone integrable on each [u, v]⊂P such that for all ε≥0, $\int_0^{\varepsilon}\psi d_P\geq 0$. If h: X→X is a function such that for all x, y∈X:

$$\int_0^{D_*(hx,\ hy)}\Psi d_P \preceq \delta\int_0^{D_*(x,\ y)}\Psi d_P$$

for some δ∈(0, 1), then h has a unique fixed point in X.

Proof: Let x_0∈X and let $x_{n+1} = hx_n$. We have:

$$\int_0^{D_*(x_{n+1},\ x_n)}\Psi d_P = \int_0^{D_*(hx_n,\ hx_{n-1})}\Psi d_P \preceq \delta\int_0^{D_*(x_n,\ x_{n-1})}\Psi d_P,\ldots,\preceq \delta^{n-1}\int_0^{D_*(x_1,\ x_0)}\Psi d_P$$

Since, δ∈(0, 1), thus:

$$\lim_{n\to\infty}\int_0^{D_*(x_{n+1},\ x_n)}\Psi d_P = 0$$

If $\lim_{n\to\infty} D_*(x_{n+1}, x_n)\neq 0$, then $\lim_{n\to 0} D^{(x_{n+1},\ x_n)}\psi d_P \neq 0$ which is contradiction. So, a $\lim_{n\to\infty} D_*(x_{n+1}, x_n) = 0$. Now, we have to prove $\{x_n\}$ is a Cauchy sequence. By triangle inequality:

$$\int_0^{D_*(hx_n,\ hx_m)}\psi d_P \preceq \int_0^{D_*(hx_n,\ hx_{n+1})\oplus D_*(hx_{n+1},\ hx_{n+2})\oplus,\ldots,\oplus D_*(hx_{m-1},\ hx_m)}\psi d_P$$

and by sub additive to ψ, we get:

$$\int_0^{D_*(hx_n,\ hx_m)}\psi d_P \preceq \int_0^{D_*(hx_n,\ hx_{n+1})}\psi d_P+,\ldots,+\int_0^{D_*(hx_{m-1},\ hx_m)}\psi d_P \preceq$$
$$\left(\delta^n + \delta^{n+1}+,\ldots,+\delta^{m-1}\right)\int_0^{D_*(x_1,\ x_0)}\psi d_P \preceq \frac{\delta^n}{\delta-1}\int_0^{D_*(x_1,\ x_0)}\psi d_P \to 0$$

Thus:

$$\lim_{m,n\to\infty} D_*(hx_n, hx_m) = 0$$

This means that $\{x_n\}$ be a Cauchy sequence and since, X is a complete fuzzy cone metric space, therefore, $\{x_n\}$ is a converge to x. Since:

$$\int_0^{D_*(x_{n+1},\ hx)}\psi d_P = \int_0^{D_*(hx_n,\ hx)}\psi d_P \preceq \delta\int_0^{D_*(x_n,\ x)}\psi d_P$$

Thus, $\lim_{n\to\infty} D_*(x_{n+1}, h_x) = 0 \Rightarrow hx = x$. Now, if w is another fixed point of h:

$$\int_0^{D_*(x,w)}\psi d_P = \int_0^{D_*(hx,\ hw)}\psi d_P \preceq \delta\int_0^{D(x,w)}\psi d_P$$

which is contradiction ⇒x = w. Therefore, h has a unique fixed point.

Corollary 3.9: Let (X, D*) be a complete fuzzy cone metric space and P a normal fuzzy cone. Suppose that ψ: P→P is a subadditive fuzzy cone integrable on each [u, v]⊂P. If h: X→X is a function satisfy for all x, y∈X:

$$\int_0^{D_*(hx,\ hy)}\psi d_P \preceq q\int_0^{D_*(x,\ hy)\oplus D_*(hx,\ y)}\psi d_P$$

for some q∈(0, 1), the h has a unique fixed point in X.

Definition 3.10: Let P be a fuzzy cone. A non-decreasing function ξ: P→P is called ξ- function if:

- ξ(0) = 0 and 0<ξ(υ)<υ for υ∈P/{0}
- υ⊖ξ(υ)∈Int P for every υ∈Int P
- $\lim_{n\to\infty}\xi^n(\upsilon) = 0$ for every υ∈P/{0}

Definition 3.11: Let P be a fuzzy cone and take $\{w_n\}$ be a sequence in P. We say that $w_n\to 0$ if for every ε∈P and ε>0 there exist m∈N such that $w_n<\varepsilon$ for all n≥N.

For a non-decreasing function H: P→P, we define the following condition which will used in definition above:

- (H_1) H(w) = 0 if and only if w = 0
- (H_2) for every w_n∈P, $w_n\to 0$ if and only if $H(w_n)\to 0$
- (H_3) for every w_1, w_2∈P, $H(w_1\oplus w_2)\leq H(w_1)\oplus h(w_2)$

Definition 3.12: The function h: X→X is called ξ-h function if there exist a ξ-function and a function H satisfy the condition (H_1-H_3) such that:

$$H(D_*(hx, hy)) \preceq H(D_*(x, y)) \text{ for all } x, y\in X, \ldots, (2.15a)$$

Now, using the condition, we can define ξ-f cone integrable:

$$\int_0^{H(D_*(hx,hy))}\psi d_p \preceq \xi\int_0^{H(D_*(x,y))}\psi d_p, \ldots, (2.15b)$$

Theorem 3.13: Let (X, D_*) be a complete fuzzy cone metric space and let h: X→X be a ξ-f cone integrable, then the function h has a unique fixed point.

Proof: Let x_0∈X and let $x_1 = hx_0$. Choosing $\{x_n\}$∈X, since, h is ξ-h function by definition (2.14) there exist a ξ-function and a function H satisfy the condition (H_1-H_3) by (2.15.b), we deduce:

$$\int_0^{H(D_*(x_{n+1},x_n))}\psi dp = \int_0^{H(D_*(hx_n, hx_{n-1}))}\psi dp \preceq \xi\int_0^{H(D_*(x_n, x_{n-1}))}\psi dp$$
$$\preceq \xi^2\int_0^{H(D_*(x_{n-1}, x_{n-2}))}\psi dp \preceq, \ldots, \preceq \xi^n\int_0^{H(D_*(x_1, x_0))}\psi dp$$

Let υ∈Int P, then by (2), υ⊖ξ(υ)∈Int P. By (3):

$$\lim_{n\to\infty}\xi^n\int_0^{H(D_*(x_1,x_0))}\psi d_p = 0$$

Hence:

$$\lim_{n\to\infty} H(D_*(x_1, x_0)) = 0$$

Therefore, we can find that n∈N such that, $H(D_*(hx_n, hx_{n+1}))\prec υ\ominus\xi(υ)$ for all m≥n, we show that:

$$H(D_*(x_{n+1}, x_{m+2}))\prec υ, \cdots, (2.16a)$$

Now, let (2.16a) hold for some m≥n:

$$H(D_*(hx_{n+1}, hx_{m+2}))\prec H(D_*(hx_n, hx_{n+1})\oplus H(D_*(hx_{n+1}, hx_{m+2}))\prec$$
$$υ\ominus\xi(υ)\oplus\xi(H(D_*(x_{n+1}, x_{m+2})))\prec υ\ominus\xi(υ) = υ$$

Therefore, $\{x_n\}$ be a Cauchy sequence and since, X is a complete fuzzy cone metric space, therefore, $\{x_n\}$ is a converge to x. Since:

$$\int_0^{H(D_*(x_{n+1}, x))}\psi d_p = \int_0^{H(D_*(hx_n, hx))}\psi d_p \preceq \xi\int_0^{D_*(x_n, x)}\psi d_p$$

Thus:

$$\lim_{n\to\infty} H(D_*(x_{n+1}, x)) = 0 \Rightarrow hx = x$$

Let w∈X such that hx = w:

$$\int_0^{H(D_*(x, w))}\psi d_p = \int_0^{H(D_*(hx, hw))}\psi d_p \preceq \xi\int_0^{H(D_*(x, w))}\psi d_p$$

This is contradiction ⇒x = w. Therefore, h has a unique fixed point.

Theorem 3.14: Let (X, D_*) be a complete fuzzy cone metric space and suppose that h: X→X be a ξ-f integrable such that:

$$\int_0^{H(D_*(hx, hy))}\psi d_p \preceq \xi\int_0^{H(D_*(x,hx)\oplus D_*(y, hy))}\psi d_p$$

For all x, y∈X where, ξ, H and ψ as previously defined, then h has a unique fixed point.

Proof: Let x_0∈X and let $x_1 = hx_0$. Choosing $\{x_n\}$, since, h is ξ-f integrable, we have:

$$\int_0^{H(D_*(x_{n+1}, x_n))}\psi d_p = \xi\int_0^{H(D_*(hx_n, hx_{n-1}))}\psi d_p \preceq \xi$$
$$\int_0^{H(D_*(x_n, hx_n)\oplus D_*(x_{n-1}, hx_{n-1}))}\psi d_p \preceq \xi^2\int_0^{H(D_*(x_n, x_{n-1}))}\psi d_p \preceq, \ldots, \preceq$$
$$(\xi^{n-1}+\xi^{n-2}+, \ldots, +\xi^n)\int_0^{H(D_*(x_1, x_0))}\psi d_p \preceq \Im^m\int_0^{H(D_*(x_1, x_0))}\psi d_p,$$
$$\Im^m = \frac{\xi^m}{1-\xi}$$

Next, we will show that $\{x_n\}$ is a Cauchy seq., so for, any n, p∈N:

$$\int_0^{H(D_*(x_n, x_{n+p}))}\psi d_p \preceq \int_0^{H(D_*(x_n, x_{n+p-1})\oplus D_*(x_{n+p-1}, x_{n+p}))}\psi d_p$$

Then $\lim_{n, p\to\infty} H(D_*(x_n, x_{n+p})) = 0$ by (H_2), $\{x_n\}$ is a Cauchy seq. Since, X is a complete, then there exists u such that $D_*(x_n, u)\to 0$ as n→∞:

$$\int_0^{H(D_*(x_{n+1}, u))}\psi d_p = \int_0^{H(D_*(x_n, hx_n)\oplus D_*(u, hu))}\psi d_p \preceq \xi\int_0^{D_*(x_n, u)}\psi d_p$$

Thus, $\lim_{n\to\infty} H(D_*(x_{n+1}, hu)) = 0 \Rightarrow hu = u$. Let w, another fixed point of h such that w≠u for a fixed ε>0 and every m∈N there exists n∈N such that $H(D^*(x_{n+1}, x_n))\preceq \frac{\varepsilon(1-k)}{2k}$ and $H(D_*(x_{n+1}, w))\preceq\frac{\varepsilon(1-k)}{2}$ for all n ≥ m.

$$H(D_*(w, hw))\preceq H(D_*(w, hx_n))\oplus H(D_*(hx_n, hw))$$
$$\preceq H(D_*(w, hx_n))\oplus\xi(H(D_*(hx_n, x_n)\oplus D_*(w, hw)))$$
$$\preceq H(D_*(w, hx_n))\oplus k(H(D_*(hx_n, x_n)\oplus D_*(w, hw)))$$

This implies that:

$$H\left(D_*\left(w, hw\right)\right) \preceq \frac{1}{1-k} H\left(D_*\left(w, hx_n\right)\right) \oplus$$

$$\frac{k}{1-k} H\left(D_*\left(hx_n, x_n\right)\right) \prec \frac{\varepsilon}{2} \oplus \frac{\varepsilon}{2} = \varepsilon$$

This implies that H (D_*(w, hw))→0. Now:

$$\int_0^{H\left(D_*(u, w)\right)} \psi d_P = \int_0^{H\left(D_*(hu, hw)\right)} \psi d_P \preceq$$

$$\xi \int_0^{H\left(D^*(u, hu) \oplus D_*(w, hw)\right)} \psi d_P \preceq k \int^{H\left(D_*(u, hu) \oplus D_*(w, hw)\right)} \psi d_P$$

which gives H (D_*(u, w)) = 0. Hence, w = u, then h has a unique fixed point on X.

CONCLUSION

In this study, we defined the concept of the partition of the closed interval in the fuzzy cone metric space which has an important role in the applications of conical shapes as well as other concepts such as functions that can be integrated into spaces. This study helps researchers to find other scientific applications through their application to identify integrations.

REFERENCES

AL-Mayahi, N.F. and S.H. Hadi, 2015. On-contraction in fuzzy metric space and its application. Gen., 26: 104-118.

Abbas, M. and G. Jungck, 2008. Common fixed point results for noncommuting mappings without continuity in cone metric spaces. J. Math. Anal. Appl., 34: 416-420.

Bag, T., 2013. Fuzzy cone metric spaces and fixed point theorems of contractive mappings. Ann. Fuzzy Math. Inf., 6: 657-668.

Bag, T., 2015. Fuzzy cone metric spaces and fixed point theorems on fuzzy T-kannan & fuzzy T-chatterjea type contractive mappings. Fuzzy Inf. Eng., 7: 305-315.

Di Bari, C. and P. Vetro, 2008. ϕ-pairs and common fixed points in cone metric spaces. Rep. Math. Cir. Palermo, 57: 279-285.

Huang, L.G. and X. Zhang, 2007. Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl., 232: 1468-1476.

Mizumoto, M. and J. Tanaka, 1979. Some Properties of Fuzzy Numbers. In: Advances in Fuzzy Set Theory and Applications, Gupta, M.M., R.K. Ragade and R.R. Yager (Eds.). North-Holland Publishing Company, Amsterdam, ISBN:9780444853721, pp: 153-164.

Oner, T., M.B. Kandemir and B. Tanay, 2015. Fuzzy cone metric spaces. J. Nonlinear Sci. Appl., 8: 610-616.

Vetro, P., 2007. Common fixed points in cone metric spaces. Rep. Math. Cir. Palermo, 56: 464-468.