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On the Diophantine Equation $x^2+y^2 = z^2+5$

[1]Shreemathi Adiga, [2]N. Anusheela and [3]M.A. Gopalan
[1]Department of Mathematics, Government First Grade College, Koteshwara,
Kundapura Taluk, 576222 Udupi, Karnataka, India
[2]Department of Mathematics, Government Arts College,
Udhagamandalam, 643002 The Nilgiris, India
[3]Department of Mathematics, Shrimati Indira Gandhi College,
620002 Trichy, Tamil Nadu, India

Abstract: This study concerns with the problem of obtaining non-zero distinct integer solutions to the given ternary quadratic Diophantine equation by introducing linear transformations. A few interesting relations among the solutions are given. From the resulting solutions of the given equation integer solutions to different choices of parabola and hyperbola are obtained. Construction of Diophantine 3-tuples and special dio 3-tuples is exhibited.

Key words: Ternary quadratic integer solutions, parabola, hyperbola, Diophantine 3-tuple, dio 3-tuple, relation

INTRODUCTION

The subject of Diophantine equations is one of the oldest and largest branches of number theory. The word Diophantine refers to the Hellenistic mathematician of the 3rd century Diophantus of Alexandria, Egypt who made a study of Diophantine equations and introduced symbolism into algebra. The study of Diophantine equations is the study of solutions of polynomial equations or systems of equations in integers, rational numbers or sometimes more general number rings. One of the fascinations of the subject is that the problems are usually easy to state and when they can be solved, sometimes involve sophisticated mathematical tools.

Diophantine equations play an important and significant role in number theory. It is worth mentioning that Diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations. In more technical language, they define an algebraic curve, algebraic surface or more general object and ask about the lattice points on it. In this context, one may refer Miller (1980), Dickson (1952); Gopalan *et al.* (2013a-c, 2014, 2015a, b, 2017), Gopalan and Geetha (2014); Meena *et al.* (2014); Mordell (1969), Vidhyalakshmi and Thenmozhi (2017), Vidhyalakshmi and Priya (2017).

Diophantine equations are numerously rich because of their variety. There is no universal method or algorithm for determining whether an arbitrary Diophantine equation has a solution or finding all the solutions, if it exists. Such an algorithm does exist for the solution of first-order Diophantine equations. However, the impossibility of obtaining a general solution was proven by Yuri Matiyasevich. There is a general theory for quadratic Diophantine equations in many variables. There are many quadratic Diophantine problems which kindled the interest among mathematicians.

Definition

Nasty number: A positive integer n is a nasty number if n = ab = cd and a+b = c-d or a-b = c+d where a, b, c, d are non-zero distinct positive integers (Miller, 1980).

Diophantine m-tuple: A set of m positive integers $\{a_1, a_2, a_3, ..., a_m\}$ is said to have the property D (n), $n\in Z-\{0\}$ if a_i, a_ja_j+n is a perfect square for all $1 \leq i\leq j \leq m$ and such a set is called a Diophantine m-tuple with property D(n).

Special dio m-tuples: A set of m positive integers $\{a_1, a_2, a_3, ...,\}$ is said to have the property D(n), $n\in Z-\{0\}$ if $a_ia_j+a_i+a_j+n$ is a perfect square for all $1\leq i\leq j\leq m$ and such a set is called a special dio m-tuples with property D(n).

MATERIALS AND METHODS

Section 1: The non-homogeneous ternary quadratic equation to be solved is:

Corresponding Author: Shreemathi Adiga, Department of Mathematics, Government First Grade College, Koteshwara, Kundapura Taluk, 576 222 Udupi, Karnataka, India

$$x^2+y^2 = z^2+5 \quad (1)$$

Different sets of solutions to Eq. 1 are illustrated.

Set 1: Introduction of the linear transformations:

$$z = 3\alpha,\ x = 2\alpha \quad (2)$$

Equation 1 leads to:

$$y^2 = 5\alpha^2+5 \quad (3)$$

which is the positive Pell equation. The smallest positive integer solution of Eq. 3 is:

$$\alpha_0 = 2,\ y_0 = 5$$

To obtain the other solutions of Eq. 3, consider its corresponding positive Pell equation represented by:

$$y^2 = 5\alpha^2+1$$

Whose general solution $(\tilde{\alpha}_n, \tilde{y}_n)$ is given by:

$$\tilde{y}_n = \frac{1}{2}f_n,\ \tilde{\alpha}_n = \frac{1}{2\sqrt{5}}g_n$$

Where:

$$f_n = \left(9+4\sqrt{5}\right)^{n+1}+\left(9-4\sqrt{5}\right)^{n+1},\ g_n = \left(9+4\sqrt{5}\right)^{n+1}-\left(9-4\sqrt{5}\right)^{n+1},\ n = -1,0,1,2,...,$$

Applying Brahmagupta lemma between (α_0, y_0) and $(\tilde{\alpha}_n, \tilde{y}_n)$, the other integer solutions of Eq. 3 are given by:

$$\alpha_{n+1} = f_n+\frac{\sqrt{5}}{2}g_n \quad (4)$$

$$y_{n+1} = \frac{5}{2}f_n+\sqrt{5}g_n,\ n = -1, 0, 1, 2, ..., \quad (5)$$

In view of Eq. 2, we have:

$$x_{n+1} = 2f_n+\sqrt{5}g_n,\ z_{n+1} = 3f_n+\frac{3\sqrt{5}}{2}g_n \quad (6)$$

Thus, Eq. 5 and 6 represent integer solutions to Eq. 1. A few numerical examples are given in Table 1. A few interesting relations among the solutions are given:

- $x_{n+3}-18x_{n+2}+x_{n+1} = 0$
- $8y_{n+1}-x_{n+2}+9x_{n+1} = 0$
- $8y_{n+2}-9x_{n+2}+x_{n+1} = 0$
- $8y_{n+3}-161x_{n+2}+9x_{n+1} = 0$
- $144y_{n+1}-x_{n+3}+161x_{n+1} = 0$
- $16y_{n+2}-x_{n+3}+x_{n+1} = 0$
- $144y_{n+3}-161x_{n+3}+x_{n+1} = 0$
- $y_{n+2}-9y_{n+1}-10x_{n+1} = 0$
- $y_{n+3}-161y_{n+1}-180x_{n+1} = 0$
- $y_{n+3}-18y_{n+2}+y_{n+1} = 0$

Each of the following expressions represents a nasty number:

- $12y_{2n+2}-12x_{2n+2}+12$
- $12y_{2n+2}-8z_{2n+2}+12$
- $6x_{2n+2}+12y_{2n+2}-12z_{2n+2}+12$

Each of the following expressions represents a cubical integer:

- $2y_{3n+3}-2x_{3n+3}+6y_{n+1}-6x_{n+1}$
- $1/3\ (6y_{3n+3}-4z_{3n+3}+18y_{n+1}-12z_{n+1}$
- $x_{3n+3}+2y_{3n+3}-2z_{3n+3}+3x_{n+1}+6y_{n+1}-6z_{n+1}$

Each of the following expressions represents a bi-quadratic integer:

- $2y_{4n+4}-2x_{4n+4}+8y_{2n+2}-8x_{2n+2}+6$
- $1/3\ (6y_{4n+4}-4z_{4n+4}+24y_{2n+2}-16z_{2n+2}+18$
- $x_{4n+4}+2y_{4n+4}-2z_{4n+4}+4x_{2n+2}+8y_{2n+2}-8z_{2n+2}+6$

Each of the following expressions in Table 2 represent hyperbolas. Each of the following expressions in Table 3 represent parabolas.

Set 2: Introduction of the linear transformations:

$$z = 12\alpha,\ x = 10\alpha \quad (7)$$

Table 1: Numerical examples according to n

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	4	5	6
0	76	85	114
1	1364	1525	2046
2	24476	27365	36714
3	439204	491045	658806

Table 2: Hyperbolas according to (P, Q)

Hyperbolas	(P, Q)
$5P^2-Q^2 = 20$	$(2y_{n+1}-2x_{n+1},\ 5x_{n+1}-4y_{n+1})$
$5P^2-Q^2 = 180$	$(6y_{n+1}-4z_{n+1},\ 10z_{n+1}-12y_{n+1})$
$5P^2-Q^2 = 20$	$(x_{n+1}+2y_{n+1}-2z_{n+1},\ 2x_{n+1}-4y_{n+1}+2z_{n+1})$

Equation 1 leads to:

$$y^2 = 44\alpha^2+5 \quad (8)$$

which is the positive Pell equation. The smallest positive integer solution of Eq. 8 is:

$$\alpha_0 = 1,\ y_0 = 7$$

To obtain the other solutions of Eq. 8, consider its corresponding positive pell equation represented by:

$$y^2 = 44\alpha^2+1$$

Whose general solution $(\tilde{\alpha}_n, \tilde{y}_n)$ is given by:

$$\tilde{y}_n = \frac{1}{2}f_n,\ \tilde{\alpha}_n = \frac{1}{2\sqrt{44}}g_n$$

Where:

$$f_n = \left(199+60\sqrt{11}\right)^{n+1} + \left(199-60\sqrt{11}\right)^{n+1},\ g_n = \left(199+60\sqrt{11}\right)^{n+1} - \left(199-60\sqrt{11}\right)^{n+1},\ n = -1, 0, 1, 2, ...,$$

Applying Brahmagupta lemma between (α_0, y_0) and $(\tilde{\alpha}_n, \tilde{y}_n)$ the other integer solutions of Eq. 8 are given by:

$$\alpha_{n+1} = \frac{1}{2}f_n + \frac{7}{4\sqrt{11}}g_n \quad (9)$$

$$y_{n+1} = \frac{7}{2}f_n + \sqrt{11}g_n,\ n = -1,\ 0, 1, 2, ..., \quad (10)$$

In view of Eq. 7, we have:

$$x_{n+1} = 5f_n + \frac{35}{2\sqrt{11}}g_n,\ z_{n+1} = 6f_n + \frac{21}{\sqrt{11}}g_n \quad (11)$$

Thus, Eq. 10 and 11 represent integer solutions to Eq. 1. A few numerical examples are given in Table 4. A few interesting relations among the solutions are given as:

Table 3: Parabolas according to (P, Q)

Parabolas	(P, Q)
$5P-Q^2 = 20$	$(2y_{2n+2}-2x_{2n+2}+2,\ 5x_{n+1}-4y_{n+1})$
$15P-Q^2 = 180$	$(6y_{2n+2}-4z_{2n+2}+6,\ 10z_{n+1}-12y_{n+1})$
$5P-Q^2 = 20$	$(x_{2n+2}+2y_{2n+2}-2z_{2n+2}+2,\ 2x_{n+1}-4y_{n+1}+2z_{n+1})$

- $x_{n+3}-398x_{n+2}+x_{n+1} = 0$
- $300y_{n+1}-x_{n+2}+199x_{n+1} = 0$
- $300y_{n+2}-199x_{n+2}+x_{n+1} = 0$
- $300y_{n+3}-79201x_{n+2}+199x_{n+1} = 0$
- $132x_{n+1}-199y_{n+3}+79201y_{n+2} = 0$
- $132x_{n+2}-y_{n+3}+199y_{n+2} = 0$
- $y_{n+1}-398y_{n+2}+y_{n+3} = 0$
- $x_{n+3}-119400y_{n+1}-79201x_{n+1} = 0$
- $y_{n+2}-199y_{n+1}-132x_{n+1} = 0$
- $y_{n+3}-79201y_{n+1}-52536x_{n+1} = 0$

Each of the following expressions represents a nasty number:

- $6/25\ (70y_{2n+2}-44x_{2n+2}+50)$
- $2/5\ (42y_{2n+2}-22z_{2n+2}+30)$
- $6/5\ (14y_{2n+2}+11z_{2n+2}-22x_{2n+2}+10)$

Each of the following expressions represents a cubical integer:

- $1/25\ (70y_{3n+3}-44x_{3n+3}+210y_{n+1}-132x_{n+1})$
- $1/15\ (42y_{3n+3}-22z_{3n+3}+126y_{n+1}-66z_{n+1})$
- $1/5\ (14y_{3n+3}+11z_{3n+3}-22x_{3n+3}+42y_{n+1}+33z_{n+1}-66x_{n+1})$

Each of the following expressions represents a bi-quadratic integer:

- $1/25\ (70y_{4n+4}-44x_{4n+4}+280y_{2n+2}-176x_{2n+2}+150)$
- $1/15\ (42y_{4n+4}-22z_{4n+4}+168y_{2n+2}-88z_{2n+2}+90)$
- $1/5\ (14y_{4n+4}+11z_{4n+4}-22x_{4n+4}+56y_{2n+2}+44z_{2n+2}-88x_{2n+2}+30)$

Each of the following expressions in Table 5 represent hyperbolas. Each of the following expressions in Table 6 represent parabolas.

Set 3: Introduction of the linear transformations

Table 4: Numerical examples according to n

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	10	7	12
0	4090	2713	4908
1	1627810	1079767	1953372
2	647864290	429744553	777437148

Table 5: Hyperbolas according to (P, Q)

Hyperbolas	(P, Q)
$11P^2-Q^2 = 27500$	$(70y_{n+1}-44x_{n+1},\ 154x_{n+1}-220y_{n+1})$
$11P^2-Q^2 = 9900$	$(42y_{n+1}-22z_{n+1},\ 77z_{n+1}-132y_{n+1})$
$11P^2-Q^2 = 1100$	$(14y_{n+1}+11z_{n+1}-22x_{n+1},\ 44x_{n+1}-44y_{n+1}-11z_{n+1})$

Table 6: Parabolas according to (P, Q)

Parabolas	(P, Q)
$275P-Q^2 = 27500$	$(70y_{2n+2}-44x_{2n+2}+50,\ 154x_{n+1}-220y_{n+1})$
$165P-Q^2 = 9900$	$(42y_{2n+2}-22z_{2n+2}+30,\ 77z_{n+1}-132y_{n+1})$
$55P-Q^2 = 1100$	$(14y_{2n+2}+11z_{2n+2}-22x_{2n+2}+10,\ 44x_{n+1}-44y_{n+1}-11z_{n+1})$

$$z = 6\alpha,\ x = 5\alpha \tag{12}$$

Equation 1 leads to:

$$y^2 = 11\alpha^2 + 5 \tag{13}$$

which is the positive Pell equation. The smallest positive integer solution of Eq. 13 is:

$$\alpha_0 = 1,\ \ y_0 = 4$$

To obtain the other solutions of Eq. 13, consider its corresponding positive Pell equation represented by:

$$y^2 = 11\alpha^2 + 1$$

Whose general solution $(\tilde{\alpha}_n, \tilde{y}_n)$ is given by:

$$\tilde{y}_n = \frac{1}{2}f_n,\ \tilde{\alpha}_n = \frac{1}{2\sqrt{11}}g_n$$

Where:

$$f_n = \left(10+3\sqrt{11}\right)^{n+1} + \left(10-3\sqrt{11}\right)^{n+1},\ g_n = \left(10+3\sqrt{11}\right)^{n+1} - \left(10-3\sqrt{11}\right)^{n+1},\ n = -1, 0, 1, 2, ..,$$

Applying Brahmagupta lemma between (α_0, y_0) and $(\tilde{\alpha}_n, \tilde{y}_n)$, the other integer solutions of Eq. 13 are given by:

$$\alpha_{n+1} = \frac{1}{2}f_n + \frac{2}{\sqrt{11}}g_n \tag{14}$$

$$y_{n+1} = 2f_n + \frac{\sqrt{11}}{2}g_n,\ n = -1, 0, 1, 2, ..., \tag{15}$$

In view of Eq. 12, we have:

$$x_{n+1} = \frac{5}{2}f_n + \frac{10}{\sqrt{11}}g_n,\ z_{n+1} = 3f_n + \frac{12}{\sqrt{11}}g_n \tag{16}$$

Thus, Eq. 15 and 16 represent integer solutions to Eq. 1. A few numerical examples are given in Table 7. A few interesting relations among the solutions are given as:

- $x_{n+3}-20x_{n+2}+x_{n+1} = 0$
- $15y_{n+1}-x_{n+2}+10x_{n+1} = 0$

Table 7: Numerical examples of Pell equation

n	x_{n+1}	y_{n+1}	z_{n+1}
-1	5	4	6
0	110	73	132
1	2195	1456	2634
2	43790	29047	52548
3	873605	579484	1048326

- $15y_{n+2}-10x_{n+2}+x_{n+1} = 0$
- $15y_{n+3}-199x_{n+2}+10x_{n+1} = 0$
- $33x_{n+1}-50y_{n+3}+995y_{n+2} = 0$
- $33x_{n+2}-5y_{n+3}+50y_{n+2} = 0$
- $33x_{n+3}-50y_{n+3}+5y_{n+2} = 0$
- $y_{n+1}-20y_{n+2}+y_{n+3} = 0$
- $x_{n+3}-300y_{n+1}-199x_{n+1} = 0$
- $5y_{n+2}-50y_{n+1}-33x_{n+1} = 0$

Each of the following expressions represents a nasty number:

- $6/25\ (40y_{2n+2}-22x_{2n+2}+50)$
- $2/5\ (24y_{2n+2}-11z_{2n+2}+30)$
- $3/5\ (16y_{2n+2}+11z_{2n+2}-22x_{2n+2}+20)$

Each of the following expressions represents a cubical integer:

- $1/25\ (40y_{3n+3}-22x_{3n+3}+120y_{n+1}-66x_{n+1})$
- $1/15\ (24y_{3n+3}-11z_{3n+3}+72y_{n+1}-33z_{n+1})$
- $1/10\ (16y_{3n+3}+11z_{3n+3}-22x_{3n+3}+48y_{n+1}+33z_{n+1}-66x_{n+1})$

Each of the following expressions represents a bi-quadratic integer:

- $1/25\ (40y_{4n+4}-22x_{4n+4}+160y_{2n+2}-88x_{2n+2}+150)$
- $1/15\ (24y_{4n+4}-11z_{4n+4}+96y_{2n+2}-44z_{2n+2}+90)$
- $1/10\ (16y_{4n+4}+11z_{4n+4}-22x_{4n+4}+64y_{2n+2}+44z_{2n+2}-88x_{2n+2}+60)$

Each of the following expressions in Table 8 represent hyperbolas. Each of the following expressions in Table 9 represent parabolas.

Set 4: Introducing the linear transformations:

$$z = u+v,\ y = u-v \tag{17}$$

Equation 1, it leads to:

$$x^2 = 4uv+5 \tag{18}$$

Table 8: Hyperbolas

Hyperbolas	(P, Q)
$11P^2-Q^2 = 27500$	$(40y_{n+1}-22x_{n+1},\ 88x_{n+1}-110y_{n+1})$
$11P^2-Q^2 = 9900$	$(24y_{n+1}-11z_{n+1},\ 44z_{n+1}-66y_{n+1})$
$11P^2-4Q^2 = 4400$	$(16y_{n+1}+11z_{n+1}-22x_{n+1},\ 44x_{n+1}-22y_{n+1}-22z_{n+1})$

Table 9: Parabolas

Parabolas	(P, Q)
$275P-Q^2 = 27500$	$(40y_{2n+2}-22x_{2n+2}+50,\ 88x_{n+1}-110y_{n+1})$
$165P-Q^2 = 9900$	$(24y_{2n+2}-11z_{2n+2}+30,\ 44z_{n+1}-66y_{n+1})$
$55P-2Q^2 = 2200$	$(16y_{2n+2}+11z_{2n+2}-22x_{2n+2}+20,\ 44x_{n+1}-22y_{n+1}-22z_{n+1})$

Table 10: Illustrations

v	u	x	y	z
1	k^2+k-1	$2k+1$	k^2+k-2	k^2+k
5	$5k^2-5k+1$	$10k-5$	$5k^2-5k-4$	$5k^2-5k+6$
11	$11k^2-15k+5$	$22k-15$	$11k^2-15k-6$	$11k^2-15k+16$
11	$11k^2-7k+1$	$22k-7$	$11k^2-7k-10$	$11k^2-7k+12$

Table 11: Solutions

x	y	z
$2k^2+4k-1$	$2k+2$	$2k^2-4k$
k^2+3k	$2k+3$	k^2+3k+2
$10k^2-3$	$10k$	$10k^2+2$
$5k^2+5k-4$	$10k+5$	$5k^2+5k-+6$
$22k^2-8k-5$	$22k-4$	$22k^2-8k+6$
$22k^2+8k-5$	$22k+4$	$22k^2+8k+6$

Choose u and v such that the R.H.S of Eq. 18 is a perfect square and from which the x-value is obtained. Substituting the corresponding values of u and v in Eq. 17, the y and z values are obtained. This process is illustrated in Table 10.

However, apart from the above sets of solutions to Eq. 1, we have some more choices of integer solutions to Eq. 1 which are presented in Table 11.

Section 2; Construction of Diophantine triples

Sequence 1: An attempt is made to form a sequence of Diophantine triples (a,b,c), (b,c,d), ..., (c,d,e) with the property D (1).

Case 1: Let $a = k^2+k-2$ and $b = k^2+k$. Consider $ab+1 = (k^2+k-1)^2$. Let c be any non-zero integer. Consider:

$$\left.\begin{aligned} ac+1 &= \left(k^2+k-2\right)c+1 = p_1^2 \\ bc+1 &= \left(k^2+k\right)c+1 = q_1^2 \end{aligned}\right\} \tag{19}$$

Performing some algebra, we have:

$$(k^2+k)p_1^2-(k^2+k-2)q_1^2 = 2 \tag{20}$$

Using the linear transformations:

$$p_1 = X+\left(k^2+k-2\right)T,\ q_1 = X+\left(k^2+k\right)T \tag{21}$$

Equation 20, we have:

$$X^2 = \left(k^2+k-2\right)\left(k^2+k\right)T^2+1 \tag{22}$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = k^2+k-1$. Hence, from Eq. 21, we have:

$$p_1 = 2k^2+2k-3$$

Now from Eq. 19, we get:

$$c = 4\left(k^2+k-1\right)$$

Hence, (a, b, c) is the Diophantine triple with the property D (1).

Case 2: Let $b = k^2+k$ and $c = 4\,(k^2+k-1)$. Consider $bc+1 = (2k^2+2k-1)^2$. Let d be any non-zero integer. Consider:

$$\left.\begin{aligned} bd+1 &= \left(k^2+k\right)d+1 = p_2^2 \\ cd+1 &= 4\left(k^2+k-1\right)d+1 = q_2^2 \end{aligned}\right\} \tag{23}$$

Performing some algebra, we have:

$$4\left(k^2+k-1\right)p_2^2-\left(k^2+k\right)q_2^2 = 3k^2+3k-4 \tag{24}$$

Using the linear transformations:

$$p_2 = X+\left(k^2+k\right)T,\ q_2 = X+4\left(k^2+k-1\right)T \tag{25}$$

Equation 24, we have:

$$X^2 = 4\left(k^2+k\right)\left(k^2+k-1\right)T^2+1$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 2k^2+2k-1$. Hence, from Eq. 25, we have:

$$p_2 = 3k^2+3k-1$$

Now from Eq. 23, we get:

$$d = 3\left(3k^2+3k-2\right)$$

Hence (b, c, d) is the Diophantine triple with the property D (1).

Case 3: Let $c = 4\,(k^2+k-1)$ and $d = 3\,(3k^2+3k-2)$ Consider $cd+1 = (6k^2+6k-5)^2$. Let e be any non-zero integer. Consider:

$$\left.\begin{aligned} ce+1 &= 4\left(k^2+k-1\right)e+1 = p_3^2 \\ de+1 &= 3\left(3k^2+3k-2\right)e+1 = q_3^2 \end{aligned}\right\} \tag{26}$$

Performing some algebra, we have:

$$3\left(3k^2+3k-2\right)p_3^2-4\left(k^2+k-1\right)q_3^2 = 5k^2+5k-2 \tag{27}$$

Table 12: Numerical examples according to k

k	(a, b, c)	(b, c, d)	(c, d, e)
2	(4, 6, 20)	(6, 20, 48)	(20, 48, 130)
3	(10, 12, 44)	(12, 44, 102)	(44, 102, 280)
5	(28, 30, 116)	(30, 116, 264)	(116, 264, 730)
4	(18, 20, 76)	(20, 76, 174)	(76, 174, 480)

Using the linear transformations:

$$p_3 = X+4\left(k^2+k-1\right)T,\ q_3 = X+3\left(3k^2+3k-2\right)T \quad (28)$$

Equation 27, we have:

$$X^2 = 12\left(k^2+k-1\right)\left(3k^2+3k-2\right)T^2+1$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 6k^2+6k-5$. Hence, from Eq. 28, we have:

$$p_3 = 10k^2+10k-9$$

Now from Eq. 26, we get:

$$e = 5\left(5k^2+5k-4\right)$$

Hence, (c, d, e) is the Diophantine triple with the property D (1). In all the above cases (a, b, c), (b, c, d), (c, d, e), ... will form a sequence of Diophantine triples. For simplicity and clear understanding, sequence of Diophantine triples are exhibited in Table 12.

Sequence 2: An attempt is made to form a sequence of Diophantine triples, (a, b, c), (b, c, d), (c, d, e), ..., with the property D (25).

Case 1: Let $a = 5k^2+5k-4$ and $b = 5k^2+5k+6$. Consider $ab+25 = (5k^2+5k+1)^2$. Let c be any non-zero integer. Consider:

$$\left.\begin{aligned} ac+25 &= \left(5k^2+5k-4\right)c+25 = p_1^2 \\ bc+25 &= \left(5k^2+5k+6\right)c+25 = q_1^2 \end{aligned}\right\} \quad (29)$$

Performing some algebra, we have:

$$\left(5k^2+5k+6\right)p_1^2-\left(5k^2+5k-4\right)q_1^2 = 250 \quad (30)$$

Using the linear transformations:

$$p_1 = X+\left(5k^2+5k-4\right)T,\ q_1 = X+\left(5k^2+5k+6\right)T \quad (31)$$

Equation 30, we have:

$$X^2 = \left(5k^2+5k-4\right)\left(5k^2+5k+6\right)T^2+25$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 5k^2+5k+1$. Hence, from Eq. 31, we have:

$$p_1 = 10k^2+10k-3$$

Now from Eq. 29, we get:

$$c = 4\left(5k^2+5k+1\right)$$

Hence, (a, b, c) is the Diophantine triple with the property D (25).

Case 2: Let $b = 5k^2+5k+6$ and $c = 4\ (5k^2+5k+1)$. Consider $bc+25 = (10k^2+10k+7)^2$. Let d be any non-zero integer. Consider:

$$\left.\begin{aligned} bd+25 &= \left(5k^2+5k+6\right)d+25 = p_2^2 \\ cd+25 &= 4\left(5k^2+5k+1\right)d+25 = q_2^2 \end{aligned}\right\} \quad (32)$$

Performing some algebra, we have:

$$4\left(5k^2+5k+1\right)p_2^2-\left(5k^2+5k+6\right)q_2^2 = 25\left(15k^2+15k-2\right) \quad (33)$$

Using the linear transformations:

$$p_2 = X+\left(5k^2+5k+6\right)T,\ q_2 = X+4\left(5k^2+5k+1\right)T \quad (34)$$

Equation 33, we have:

$$X^2 = 4\left(5k^2+5k+6\right)\left(5k^2+5k+1\right)T^2+25$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 10k^2+10k+7$. Hence, from (Eq. 32), we have:

$$p_2 = 15k^2+15k+13$$

Now from Eq. 32, we get:

$$d = 3\left(15k^2+15k+8\right)$$

Hence, (b, c, d) is the Diophantine triple with the property D (25).

Case 3: Let $c = 4\ (5k^2+5k+1)$ and $d = 3(15k^2+15k+8)$. Consider $cd+25 = (30k^2+30k+11)^2$. Let e be any non-zero integer. Consider:

$$\left.\begin{array}{l} ce+25 = 4\left(5k^2+5k+1\right)e+25 = p_3^2 \\ de+25 = 3\left(15k^2+15k+8\right)e+25 = q_3^2 \end{array}\right\} \quad (35)$$

Performing some algebra, we have:

$$3\left(15k^2+15k+8\right)p_3^2-4\left(5k^2+5k+1\right)q_3^2 = 25\left(25k^2+25k+20\right) \quad (36)$$

Using the linear transformations:

$$p_3 = X+4\left(5k^2+5k+1\right)T,\ q_3 = X+3\left(15k^2+15k+8\right)T \quad (37)$$

Equation 36, we have:

$$X^2 = 12\left(5k^2+5k+1\right)\left(15k^2+15k+8\right)T^2+25$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 30k^2+30k+11$. Hence, from Eq. 37, we have:

$$p_3 = 50k^2+50k+15$$

Now from Eq. 35, we get:

$$e = 25\left(5k^2+5k+2\right)$$

Hence, (c, d, e) is the Diophantine triple with the property D (25). In all the above cases (a, b, c), (b, c, d), (c, d, e), ... will form a sequence of Diophantine triples. For simplicity and clear understanding, sequence of Diophantine triples are exhibited in Table 13.

Sequence 3: An attempt is made to form a sequence of Diophantine triples (a, b, c), (b, c, d), (c, d, e), ... with the property D (121).

Case 1: Let $a = 11k^2-7k-10$ and $b = 11k^2-7k+12$. Consider $ab+121 = (11k^2-7k+1)^2$. Let c be any non-zero integer. Consider:

$$\left.\begin{array}{l} ac+121 = \left(11k^2-7k-10\right)c+121 = p_1^2 \\ bc+121 = \left(11k^2-7k+12\right)c+121 = q_1^2 \end{array}\right\} \quad (38)$$

Table 13: Numerical examples according to (a, b, c)

k	(a, b, c)	(b, c, d)	(c, d, e)
2	(26, 36, 124)	(36, 124, 294)	(124, 294, 800)
3	(56, 66, 244)	(66, 244, 564)	(244, 564, 1550)
5	(146, 156, 604)	(156, 604, 1374)	(604, 1374, 3800)
4	(96, 106, 404)	(106, 404, 924)	(404, 924, 2550)

Performing some algebra, we have:

$$\left(11k^2-7k+12\right)p_1^2-\left(11k^2-7k-10\right)q_1^2 = 2662 \quad (39)$$

Using the linear transformations:

$$p_1 = X+\left(11k^2-7k-10\right)T,\ q_1 = X+\left(11k^2-7k+12\right)T \quad (40)$$

Equation 39, we have:

$$X^2 = \left(11k^2-7k-10\right)\left(11k^2-7k+12\right)T^2+121$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 11k^2-7k+1$. Hence, from Eq. 40, we have:

$$p_1 = 22k^2-14k-9$$

Now from Eq. 38, we get:

$$c = 4\left(11k^2-7k+1\right)$$

Hence, (a, b, c) is the Diophantine triple with the property D (121)

Case 2: Let $b = 11k^2-7k+12$ and $c = 4(11k^2-7k+1)$. Consider $bc+121 = (22k^2-14k+13)^2$. Let d be any non-zero integer. Consider:

$$\left.\begin{array}{l} bd+121 = \left(11k^2-7k+12\right)d+121 = p_2^2 \\ cd+121 = 4\left(11k^2-7k+1\right)d+121 = q_2^2 \end{array}\right\} \quad (41)$$

Performing some algebra, we have:

$$4\left(11k^2-7k+1\right)p_2^2-\left(11k^2-7k+12\right)q_2^2 = 121\left(33k^2-21k-8\right) \quad (42)$$

Using the linear transformations:

$$p_2 = X+\left(11k^2-7k+12\right)T,\ q_2 = X+4\left(11k^2-7k+1\right)T \quad (43)$$

Equation 42, we have:

$$X^2 = 4\left(11k^2-7k+12\right)\left(11k^2-7k+1\right)T^2+121$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 22k^2-14k+13$. Hence, from Eq. 43, we have:

$$p_2 = 33k^2-21k+25$$

Now from Eq. 41, we get:

$$d = 3\left(33k^2-21k+14\right)$$

Hence, (b, c, d) is the Diophantine triple with the property D (121).

Case 3: Let $c = 4\ (11k^2-7k+1)$ and $d = 3\ (33k^2-21k+14)$. Consider $cd+121 = (66k^2-42k+17)^2$. Let e be any non-zero integer. Consider:

$$\left.\begin{aligned} ce+121 = 4\left(11k^2-7k+1\right)e+121 = p_3^2 \\ de+121 = 3\left(33k^2-21k+14\right)e+121 = q_3^2 \end{aligned}\right\} \quad (44)$$

Performing some algebra, we have:

$$3\left(33k^2-21k+14\right)p_3^2-4\left(11k^2-7k+1\right)q_3^2 = 121\left(55k^2-35k+38\right) \quad (45)$$

Using the linear transformations:

$$p_3 = X+4\left(11k^2-7k+1\right)T,\ q_3 = X+3\left(33k^2-21k+14\right)T \quad (46)$$

Equation 45, we have:

$$X^2 = 12\left(11k^2-7k+1\right)\left(33k^2-21k+14\right)T^2+121$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 66k^2-42k+17$. Hence, from Eq. 46, we have:

$$p_3 = 110k^2-70k+21$$

Now from Eq. 44, we get:

$$e = 5\left(55k^2-35k+16\right)$$

Hence (c, d, e) is the Diophantine triple with the property D (121). In all the above cases (a, b, c), (b, c, d), (c, d, e), ... will form a sequence of Diophantine triples. For simplicity and clear understanding, sequence of Diophantine triples are exhibited in Table 14.

Table 14: Numerical examples

k	(a, b, c)	(b, c, d)	(c, d, e)
2	(20, 42, 124)	(42, 124, 312)	(124, 312, 830)
3	(68, 90, 316)	(90, 316, 744)	(316, 744, 2030)
5	(230, 252, 964)	(252, 964, 2202)	(964, 2202, 6080)
4	(138, 160, 596)	(160, 596, 1374)	(596, 1374, 3780)

RESULTS AND DISCUSSION

Section 3; Construction of special dio-triples

Sequence 1: An attempt is made to form a sequence of special dio-triples (a, b, c), (b, c, d), (c, d, e), ... with the property D (26).

Case 1: Let $a = 5k^2+5k-4$ and $b = 5k^2+5k+6$. Consider $ab+a+b+26 = (5k^2+5k+2)^2$. Let c be any non-zero integer. Consider:

$$\left.\begin{aligned} ac+a+c+26 = \left(5k^2+5k-3\right)c+a+26 = p_1^2 \\ bc+b+c+26 = \left(5k^2+5k+7\right)c+b+26 = q_1^2 \end{aligned}\right\} \quad (47)$$

Performing some algebra, we have:

$$\left(5k^2+5k+7\right)p_1^2-\left(5k^2+5k-3\right)q_1^2 = 250 \quad (48)$$

Using the linear transformations:

$$p_1 = X+\left(5k^2+5k-3\right)T,\ q_1 = X+\left(5k^2+5k+7\right)T \quad (49)$$

Equation 48, we have:

$$X^2 = \left(\left(5k^2+5k-4\right)\left(5k^2+5k+6\right)+10k^2+10k+3\right)T^2+25$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 5k^2+5k+2$. Hence, from Eq. 48, we have:

$$p_1 = 10k^2+10k-1$$

Now from Eq. 47, we get:

$$c = 20k^2+20k+7$$

Hence, (a, b, c) is the special dio-triple with the property D (26).

Case 2: Let $b = 5k^2+5k+6$ and $c = 20k^2+20k+7$. Consider $bc+b+c+26 = (10k^2+10k+9)^2$. Let d be any non-zero integer. Consider:

$$\left.\begin{aligned} bd+b+d+26 = \left(5k^2+5k+7\right)d+b+26 = p_2^2 \\ cd+c+d+26 = \left(20k^2+20k+8\right)d+c+26 = q_2^2 \end{aligned}\right\} \quad (50)$$

Performing some algebra, we have:

$$\left(20k^2+20k+8\right)p_2^2-\left(5k^2+5k+7\right)q_2^2 = 25\left(15k^2+15k+1\right) \quad (51)$$

Using the linear transformations:

$$p_2 = X+\left(5k^2+5k+7\right)T,\ q_2 = X+\left(20k^2+20k+8\right)T \quad (52)$$

Equation 51, we have:

$$X^2 = \left(\left(5k^2+5k+6\right)\left(20k^2+20k+7\right)+25k^2+25k+14\right)T^2+25$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 10k^2+10k+9$. Hence, from Eq. 52, we have:

$$p_2 = 15k^2+15k+16$$

Now from Eq. 50, we get:

$$d = 45k^2+45k+32$$

Hence, (b, c, d) is the special dio-triple with the property D (26).

Case 3: Let $c = 20k^2+20k+7$ and $d = 45k^2+45k+32$. Consider $cd+c+d+26 = (30k^2+30k+17)^2$. Let e be any non-zero integer. Consider:

$$\left.\begin{array}{l} ce+c+e+26 = \left(20k^2+20k+8\right)e+c+26 = p_3^2 \\ de+d+e+26 = \left(45k^2+45k+33\right)e+d+26 = q_3^2 \end{array}\right\} \quad (53)$$

Performing some algebra, we have:

$$\left(45k^2+45k+33\right)p_3^2-\left(20k^2+20k+8\right)q_3^2 = 625\left(k^2+k+1\right) \quad (54)$$

Using the linear transformations:

$$p_3 = X+\left(20k^2+20k+8\right)T,\ q_3 = X+\left(45k^2+45k+33\right)T \quad (55)$$

Equation 54, we have:

$$X^2 = \left(\left(20k^2+20k+7\right)\left(45k^2+45k+32\right)+65k^2+65k+40\right)T^2+25$$

Table 15: Numerical examples

k	(a, b, c)	(b, c, d)	(c, d, e)
2	(26, 36, 127)	(36, 127, 302)	(127, 302, 824)
3	(56, 66, 247)	(66, 247, 572)	(247, 572, 1574)
5	(146, 156, 607)	(156, 607, 1382)	(607, 1382, 3824)
4	(96, 106, 407)	(106, 407, 932)	(407, 932, 2574)

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 30k^2+30k+17$. Hence, from Eq. 55, we have:

$$p_3 = 50k^2+50k+25$$

Now, from Eq. 53, we get:

$$e = 125k^2+125k+74$$

Hence, (c, d, e) is the special dio-triple with the property D (26). In all the above cases (a, b, c), (b, c, d), (c, d, e), ... will form a sequence of special dio-triples. For simplicity and clear understanding, sequence of special dio-triples are exhibited in Table 15.

Sequence 2: An attempt is made to form a sequence of special dio-triples (a, b, c), (b, c, d), (c, d, e), ... with the property D (122).

Case 1: Let $a = 11k^2-7k-10$ and $b = 11k^2-7k+12$. Consider $ab+a+b+122 = (11k^2-7k+2)^2$. Let c be any non-zero integer. Consider:

$$\left.\begin{array}{l} ac+a+c+122 = \left(11k^2-7k-9\right)c+a+122 = p_1^2 \\ bc+b+c+122 = \left(11k^2-7k+13\right)c+b+122 = q_1^2 \end{array}\right\} \quad (56)$$

Performing some algebra, we have:

$$\left(11k^2-7k+13\right)p_1^2-\left(11k^2-7k-9\right)q_1^2 = 2662 \quad (57)$$

Using the linear transformations:

$$p_1 = X+\left(11k^2-7k-9\right)T,\ q_1 = X+\left(11k^2-7k+13\right)T \quad (58)$$

Equation 57, we have:

$$X^2 = \left(\left(11k^2-7k-10\right)\left(11k^2-7k+12\right)+22k^2-14k+3\right)T^2+121$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 11k^2-7k+2$. Hence, from Eq. 58, we have:

$$p_1 = 22k^2-14k-7$$

Now from Eq. 56, we get:

$$c = 44k^2-28k+7$$

Hence, (a, b, c) is the special dio-triple with the property D (122).

Case 2: Let $b = 11k^2-7k+12$ and $c = 44k^2-28k+7$. Consider $bc+b+c+122 = (22k^2-14k+15)^2$. Let d be any non-zero integer. Consider:

$$\left.\begin{aligned} bd+b+d+122 &= (11k^2-7k+13)d+b+122 = p_2^2 \\ cd+c+d+122 &= (44k^2-28k+8)d+c+122 = q_2^2 \end{aligned}\right\} \quad (59)$$

Performing some algebra, we have:

$$(44k^2-28k+8)p_2^2-(11k^2-7k+13)q_2^2 = 121(33k^2-21k-5) \quad (60)$$

Using the linear transformations:

$$p_2 = X+(11k^2-7k+13)T,\ q_2 = X+(44k^2-28k+8)T \quad (61)$$

Equation 60, we have:

$$X^2 = \left(\begin{aligned}&(11k^2-7k+12)(44k^2-28k+7)+\\&55k^2-35k+20\end{aligned}\right)T^2+121$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 22k^2-14k+15$. Hence, from Eq. 61, we have:

$$p_2 = 33k^2-21k+28$$

Now from Eq. 59, we get:

$$d = 99k^2-63k+50$$

Hence, (b, c, d) is the special dio-triple with the property D (122).

Case 3: Let $c = 44k^2-28k+7$ and $d = 99k^2-63k+50$. Consider $cd+c+d+122 = (66k^2-42k+23)^2$. Let e be any non-zero integer. Consider:

$$\left.\begin{aligned} ce+c+e+122 &= (44k^2-28k+8)e+c+122 = p_3^2 \\ de+d+e+122 &= (99k^2-63k+51)e+d+122 = q_3^2 \end{aligned}\right\} \quad (62)$$

Performing some algebra, we have:

$$(99k^2-63k+51)p_3^2-(44k^2-28k+8)q_3^2 = 121(55k^2-35k+43) \quad (63)$$

Using the linear transformations:

$$p_3 = X+(44k^2-28k+8)T,\ q_3 = X+(99k^2-63k+51)T \quad (64)$$

Equation 63, we have:

$$X^2 = \left(\begin{aligned}&(44k^2-28k+7)(99k^2-63k+50)+\\&143k^2-91k+58\end{aligned}\right)T^2+121$$

which is the Pellian equation with the initial solution $T_0 = 1$, $X_0 = 66k^2-42k+23$. Hence, from Eq. 64, we have:

$$p_3 = 110k^2-70k+31$$

Now from Eq. 62, we get:

$$e = 275k^2-175k+104$$

Hence, (c, d, e) is the special dio-triple with the property D (122). In all the above cases (a, b, c), (b, c, d), (c, d, e), ... will form a sequence of special dio-triples. For simplicity and clear understanding, sequence of special dio-triples are exhibited in Table 16.

Sequence 3: An attempt is made to form a sequence of special dio-triples (a, b, c), (b, c, d), (c, d, e), ... with the property D (2).

Case 1: Let $a = k^2+3k$ and $b = k^2+3k+2$. Consider $ab+a+b+2 = (k^2+3k+2)^2$. Let c be any non-zero integer. Consider:

$$\left.\begin{aligned} ac+a+c+2 &= (k^2+3k+1)c+a+2 = p_1^2 \\ bc+b+c+2 &= (k^2+3k+3)c+b+2 = q_1^2 \end{aligned}\right\} \quad (65)$$

Table 16: Numerical examples

k	(a, b, c)	(b, c, d)	(c, d, e)
2	(20, 42, 127)	(42, 127, 320)	(127, 320, 854)
3	(68, 90, 319)	(90, 319, 752)	(319, 752, 2054)
5	(230, 252, 967)	(252, 967, 2210)	(967, 2210, 6104)
4	(138, 160, 599)	(160, 599, 1382)	(599, 1382, 3804)

Performing some algebra, we have:

$$\left(k^2+3k+3\right)p_1^2-\left(k^2+3k+1\right)q_1^2=2 \quad (66)$$

Using the linear transformations:

$$p_1=X+\left(k^2+3k+1\right)T,\ q_1=X+\left(k^2+3k+3\right)T \quad (67)$$

Equation 66, we have:

$$X^2=\left(\left(k^2+3k\right)\left(k^2+3k+2\right)+2k^2+6k+3\right)T^2+1$$

which is the Pellian equation with the initial solution $T_0=1$, $X_0=k^2+3k+2$. Hence, from Eq. 67, we have:

$$p_1=2k^2+6k+3$$

Now from Eq. 65, we get:

$$c=4k^2+12k+7$$

Hence, (a, b, c) is the special dio-triple with the property D (2).

Case 2: Let $b=k^2+3k+2$ and $c=4k^2+12k+7$. Consider $bc+b+c+2=(2k^2+6k+5)^2$. Let d be any non-zero integer. Consider:

$$\left.\begin{aligned} bd+b+d+2&=\left(k^2+3k+3\right)d+b+2=p_2^2\\ cd+c+d+2&=\left(4k^2+12k+8\right)d+c+2=q_2^2\end{aligned}\right\} \quad (68)$$

Performing some algebra, we have:

$$\left(4k^2+12k+8\right)p_2^2-\left(k^2+3k+3\right)q_2^2=3k^2+9k+5 \quad (69)$$

Using the linear transformations:

$$p_2=X+\left(k^2+3k+3\right)T,\ q_2=X+\left(4k^2+12k+8\right)T \quad (70)$$

Equation 69, we have:

$$X^2=\left(\left(k^2+3k+2\right)\left(4k^2+12k+7\right)+5k^2+15k+10\right)T^2+1$$

which is the Pellian equation with the initial solution $T_0=1$, $X_0=2k^2+6k+5$. Hence, from Eq. 70, we have:

$$p_2=3k^2+9k+8$$

Now from Eq. 68, we get:

$$d=9k^2+27k+20$$

Hence, (b, c, d) is the special dio-triple with the property D (2).

Case 3: Let $c=4k^2+12k+7$ and $d=9k^2+27k+20$. Consider $cd+c+d+2=(6k^2+18k+13)^2$. Let e be any non-zero integer. Consider:

$$\left.\begin{aligned} ce+c+e+2&=\left(4k^2+12k+8\right)e+c+2=p_3^2\\ de+d+e+2&=\left(9k^2+27k+21\right)e+d+2=q_3^2\end{aligned}\right\} \quad (71)$$

Performing some algebra, we have:

$$\left(9k^2+27k+21\right)p_3^2-\left(4k^2+12k+8\right)q_3^2=5k^2+15k+13 \quad (72)$$

Using the linear transformations:

$$p_3=X+\left(4k^2+12k+8\right)T,\ q_3=X+\left(9k^2+27k+21\right)T \quad (73)$$

Equation 72, we have:

$$X^2=\left(\left(4k^2+12k+7\right)\left(9k^2+27k+20\right)+13k^2+39k+28\right)T^2+1$$

which is the Pellian equation with the initial solution $T_0=1$, $X_0=6k^2+18k+13$. Hence, from Eq. 73, we have:

$$p_3=10k^2+30k+21$$

Now from Eq. 71, we get:

$$e=25k^2+75k+54$$

Hence, (c, d, e) is the special dio-triple with the property D (2). In all the above cases (a, b, c), (b, c, d), (c, d, e), ... will form a sequence of special dio-triples. For simplicity and clear understanding, sequence of special dio-triples are exhibited in Table 17.

Table 17: Numerical examples

k	(a, b, c)	(b, c, d)	(c, d, e)
2	(10, 12, 47)	(12, 47, 110)	(47, 110, 304)
3	(18, 20, 79)	(20, 79, 182)	(79, 182, 504)
5	(40, 42, 167)	(42, 167, 380)	(167, 380, 1054)
4	(28, 30, 119)	(30, 119, 272)	(119, 272, 754)

Section 4; Generation of solutions: Let (x_0, y_0, z_0) be any solution of Eq. 1. The solution may be in real integers or in gausssian integers or in irrational numbers. Let (x_1, y_1, z_1) be the second solution of Eq. 1 where:

$$x_1 = h\text{-}x_0,\ y_1 = h\text{-}y_0,\ z_1 = h+z_0 \quad (74)$$

In which h is an unknown to be determined. Substitution of Eq. 74 in Eq. 1 gives:

$$h = 2x_0+2y_0+2z_0 \quad (75)$$

Using Eq. 75 in 74, the second solution (x_1, y_1, z_1) of Eq. 1 is expressed in the matrix form as:

$$(x_1, y_1, z_1)^t = M(x_0, y_0, z_0)^t$$

where, t is the transpose and:

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

The repetition of the above process leads to the general solution $(x_{n+1}, y_{n+1}, z_{n+1})$ of Eq. 1 written in the matrix form as:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{Y_n-(-1)^n}{2} & \frac{Y_n+(-1)^n}{2} & X_n \\ \frac{Y_n+(-1)^n}{2} & \frac{Y_n-(-1)^n}{2} & X_n \\ X_n & X_n & Y_n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix},\ n = 0, 1, 2, ...,$$

where (x_n, y_n) is the general solution of the Pellian equation $Y^2 = 2X^2+1$. That is:

$$Y_n = \frac{1}{2}\left(\left(3+2\sqrt{2}\right)^{n+1} + \left(3-2\sqrt{2}\right)^{n+1}\right)$$

$$X_n = \frac{1}{2\sqrt{2}}\left(\left(3+2\sqrt{2}\right)^{n+1} - \left(3-2\sqrt{2}\right)^{n+1}\right)$$

CONCLUSION

This research study concerns with the study of the ternary quadratic Diophantine equation given by $x^2+y^2 = z^2+5$. Employing the solutions of the given equation integer solutions to special hyperbolas and parabolas are obtained. Further, using the solutions, Diophantine 3-tuples and special dio 3-tuples are constructed.

REFERENCES

Dickson, L.E., 1952. History of the Theory of Numbers. Vol. 2, Carnegie Institution for Science, Washington DC., USA.,.

Gopalan, M.A. and V. Geetha, 2014. Lattice points on the homogeneous cone. Intl. J. Sci. Technol., 2: 291-295.

Gopalan, M.A., D. Maheswari and J. Maheswari, 2015a. On ternary quadratic Diophantine equation $2(x^2+Y^2)$-3xy= $43z^2$. J. P. J. Math. Sci., 13: 9-23.

Gopalan, M.A., S. Mallika and S. Vidhyalakshmi, 2014. On ternary Diophantine equation $8(x^2+y^2)$-15xy = $80z^2$. Bull. Math. Stat. Res., 2: 429-433.

Gopalan, M.A., S. Vidhyalakshmi and E. Premalatha, 2013b. On the homogeneous quadratic equation with three unknowns x^2-xy+y^2 = $(k^2+3)z^2$. Bull. Math. Stat. Res., 1: 38-41.

Gopalan, M.A., S. Vidhyalakshmi and S. Nivethitha, 2013a. On ternary quadratic equation x^2+y^2–xy = $7z^2$. Intl. J. Eng. Res. Online, 1: 16-22.

Gopalan, M.A., S. Vidhyalakshmi and U.K. Rajalakshmi, 2017. On ternary quadratic Diophantine equation $5(x^2+y^2)$-6xy = $196z^2$. I. J. R.D.O. J. Math, 3: 1-10.

Gopalan, M.A., S. Vidhyalakshmi, S. Devibala and J. Umavathy, 2015b. On the ternary quadratic Diophantine equation $3(x^2+y^2)$-5xy = $60Z^2$. Intl. J. Appl. Res., 1: 234-238.

Gopalan, M.A., V. Sangeetha and M. Somanath, 2013c. On the ternary quadratic equation $5(x^2+y^2)$-9xy = $19z^2$. Intl. J. Innovative Res. Sci. Eng. Technol., 2: 2008-2010.

Meena, K., S. Vidhyalakshmi, M.A. Gopalan and S.A. Thangam, 2014. On homogeneous ternary quadratic Diophantine equation $2(x^2+y^2)$-3xy = $16z^2$. Intl. J. Eng. Sci. Math., 3: 63-69.

Miller, B., 1980. Nasty numbers. Math. Teach., 73: 649-649.

Mordell, L.J., 1969. Diophantine Equations. Academic Press, Cambridge, Massachusetts, USA.,.

Vidhyalakshmi, S. and A. Priya, 2017. On the non-homogeneous ternary quadratic Diophantine equation $2(x^2+y^2)$-3xy+(x+y)+1 = z^2. J. Math. Inf., 10: 49-55.

Vidhyalakshmi, S. and S. Thenmozhi, 2017. On the ternary quadratic Diophantine equation $3(x^2+y^2)$-5xy = $75z^2$. J. Math. Inf., 10: 11-19.