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Existence and Uniqueness of Weak Solution for Quasilinear Problems with a p(x)-Biharmonic Operator

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Abstract: In this study, we show the existence and uniqueness of weak solution of a problem which involves the p(x)-biharmonic operator with some different bound-ary conditions. The proof of the result is made by Browder theorem and the theory of variable exponent Sobolev spaces.

Key words: Weak solution, p(x)-biharmonic operator, variable exponent spaces, variational methods, browder theorem, uniqueness

INTRODUCTION

In recent years, increasing attention has been paid to the study of diffrential equations and variational problems involving variable exponent. The interest in studying such problems was stimulated by their various physical applications. Indeed, there are many applications concerning nonlinear elasticity theory and in modelling electrorheological uids (Acerbi and Mingione, 2005; (Diening, 2002; Halsey and Martin, 1993; Ruzicka, 2000) and from the study of elastic mechanics (Zhikov, 1987) and raise many difficult mathematical problems. After this pioneering models, many other applications of differential operators with variable exponents have appeared in a large range of fields such as image restoration (Chen *et al*., 2006) and mathematical Biology (Fragnelli, 2010).

Fourth order elliptic equations arise in many applications such as micro electro mechanical systems, thin film theory, thin plate theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells and phase field models of multiphase systems (Danet, 2014; Ferrero and Warnault, 2009; Myers, 1998) and the references therein. There is also another important class of physical problems leading to higher order partial differential equations. An example of this is Kuramoto-Sivashinsky equation which models 1 pattern formation in different physical contexts such as chemical reaction-diffusion systems and a cellular gas flame in the presence of external stabilizing factors (Wang and Canessa, 1993). Numerous researchers investigated the existence and multiplicity of solutions for the problems involving biharmonic, p-biharmonic and p(x)-biharmonic operators.

We refer the readers to Afrouzi and Shokooh (2015), Heidarkhani *et al*. (2017), Heidarkhani (2012), Bisci *et al*. (2014), Yin and Liu (2013), Yucedag (2015) and the references there in. In this research, we consider the following problems involving p(x)-biharmonic.

Navier problem:

$$\begin{aligned} &\Delta^2_{p(x)} u + e_p(x)|u|^{p(x)-2} u = f(x,u(x)) \\ &\text{for } x \in \Omega u = \Delta u = 0 \text{ for } x \in \partial\Omega \end{aligned} \tag{1}$$

Neumann problem:

$$\begin{aligned} &\Delta^2_{p(x)} u + e_p(x)|u|^{p(x)-2} u = f(x,u(x)) \text{ for } x \in \Omega \\ &\frac{\partial \upsilon}{\partial \upsilon} = \frac{\partial}{\partial \upsilon}(|\Delta u|)^{p(x)-2}\Delta u) = 0 \text{ for } x \in \partial\Omega \end{aligned} \tag{2}$$

No flux problem:

$$\begin{aligned} &\Delta^2_{p(x)} u + e_p(x)|u|^{p(x)-2} u = f(x,u(x)) \text{ for } x \in \Omega \\ &u = \text{constant } \Delta u = 0 \text{ for } x \in \partial\Omega \int_{\partial\Omega} \frac{\partial u}{\partial \upsilon}(|\Delta u|)^{p(x)-2}\Delta u) \\ &ds = 0 \text{ for } x \in \partial\Omega \end{aligned} \tag{3}$$

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Steklov problem:

$$\begin{aligned} &-\Delta^2_{p(x)} u = e_p(x)|u|^{p(x)-2}u \ \text{ for } x \in \partial\Omega \\ &(|\Delta u|)^{p(x)-2}\frac{\partial u}{\partial \upsilon} = g(x, u(x)) \text{ for } x \in \partial\Omega \end{aligned} \quad (4)$$

Robin problem:

$$\begin{aligned} &\Delta^2_{p(x)} u = f(x, u, (x)) \text{ for } x \in \partial\Omega \ (|\Delta u|)^{p(x)-2}\frac{\partial u}{\partial \upsilon} + \\ &m(x)|u|^{p(x)-2}u = 0 \text{ for } x \in \partial\Omega \end{aligned} \quad (5)$$

where, $\Omega \subset \mathbb{R}^N$ is a nonempty bounded domain with a sufficient smooth boundary $\partial\Omega$ and υ is the outward unit normal to $\partial\Omega$. $\Delta^2_{p(x)} u: = \Delta(|\Delta u|)^{p(x)-2}\Delta u)$ is the, so called p(x)-biharmonic operator of fourth order, $p \in C(\bar{\Omega})$ with $1 < p^- := \inf_{x\in\bar{\Omega}} p(x) \leq p(x) \leq p^+ := \sup_{x\in\bar{\Omega}} p(x)$, $f \in C(\Omega\times\mathbb{R})$ $g \in C(\partial\Omega\times\mathbb{R})$, $e_p\ \Omega\to\mathbb{R} \in L^\infty(\Omega)$ is a real function with $e^-_p = \inf_{x\in\Omega} ep(x) > 0$: and m: $\partial\Omega\to\mathbb{R}\in L^\infty$ is a real function with $m- = \inf_{x\in\partial\Omega} m(x) > 0$.

Precise that elliptic equations involving the p(x)-biharmonic equations are not trivial generalizations of similar problems studied in the constant case, since, the p(x)-biharmonic operator is not homogeneous and thus, some techniques which can be applied in the case of the p-biharmonic operators will fail in that new situation such as the Lagrange Multiplier Theorem.

To our best of knowledge, there seems few results about uniqueness of solutions to p(x)-biharmonic equations. Although, a natural extension of the theory, the problem addressed, here is a natural continuation of recent papers. By Allaoui *et al.* (2015) for the p(x)-Laplacian problems, researchers have obtained existence and uniqueness of weak solution which generalizes the corresponding result by Abdelkadery and Ourraouiz (2013), Afrouzi *et al.* (2009) and Khafagy (2011) for the case when p is constant.

Motivated by the above papers and the ideas introduced by Afrouzi *et al.* (2009), the purpose of this research is to extend the results by Allaoui *et al.* (2015) to the case of p(x)-biharmonic operator with some different boundary conditions. Our technical approach is based on Browder Theorem and the theory of variable exponent Sobolev spaces. More precisely, we assume f (x, u) and g (x, u) satisfies the following hypothesis:

- (H_1) f and g are carathodory functions which are decreasing with respect to the second variable
- (H_2) there exist positive constants b_i, c_i (i = 1, 2) and $q \in C(\Omega)$, $r \in C(\bar{\Omega})$ such that:

$$|f(x,t)| \leq b_1 + b_2|t|^{q(x)}, \text{ a.e } x \in \Omega, t \in \mathbb{R}$$

And:

$$|g(x,t)| \leq c_1 + c_2|t|^{r(x)}, \text{ a.e } x \in \partial\Omega, t \in \mathbb{R}$$

Where:

$$1 \leq q(x) \leq \sup_{x\in\Omega} q(x) = q^+ < p^-$$

And:

$$1 \leq r(x) \leq \sup_{x\in\Omega} r(x) = r^+ < p^-$$

$$(H_3) f(x, 0) \neq 0, g(x, 0) \neq 0$$

The goal of this study is to prove the following result

Theorem 1.1: Suppose that f and g satisfies the Hypothesis $(H_1$-$H_3)$. Then the problems (1.1)-(1.5) have a unique weak solution.

MATERIALS AND METHODS

Preliminaries: In this study, we introduce some notation that will clarify what follows. Thus, when we refer to a Banach space X, we denote by X^* its dual and by (., .) the duality pairing between X^* and X. By |.|, we denote the absolute value of a number or the Euclidean norm when it is defined on $\mathbb{R}^N (N \geq 2)$.

For the reader's convenience, we recall some background facts concerning the Lebesgue-Sobolev spaces with variable exponent and introduce some notation. For more details, we refer the reader to Radulescu and Repovs (2015), Radulescu (2015) set:

$$C+(\Omega) := \{h : h \in (\bar{\Omega}) \text{ and } h(x) > 1, \forall x \in \bar{\Omega}\}$$

For $p(x) \in C+(\Omega)$, define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by:

$$L^{p(x)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \text{ measureable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\right\}$$

We define a norm, the so-called Luxemburg norm on this space by equation:

$$|u|_{p(x)} = \inf\{\beta > 0 : \int_\Omega |\frac{u(x)}{\beta}|p(x) dx \leq 1\}$$

And $(L^{p(x)}(\Omega), |u|_{p(x)})$ becomes a Banach space and we call it variable exponent Lebesgue space. Define the variable exponent Sobolev space $W^{m,p(x)}(\Omega)$ by:

$$W^{m,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid D^{\alpha}u \in L^{p(x)}(\Omega), |\alpha| \le m\}$$

Where:

$$D^{\alpha}u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} u$$

With $\alpha = (\alpha_{1, \ldots}, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^{N} \alpha_i$. The space $W^{m, p(x)}(\Omega)$, equipped with the norm:

$$\|u\|_{m, p(x)} := \sum_{|\alpha| \le m} |D^{\alpha}u|_{p(x)}$$

Becomes a separable, reflexive and uniformly convex Banach space. When $e_p(x)$ satisfies $e_p \in L^{\infty}(\Omega)$ such that e_p: = ess inf $x \in R^N$ $e_p(x) > 0$, we defined the weighted varaible exponent Lebesgue space $L^{p(x)}_{e_{p(x)}}(\Omega)$ by:

$$L^{p(x)}_{e_{p(x)}}(\Omega) = \left\{ \begin{array}{l} \text{u: is a measurable real-valued funcion,} \\ \int_{\Omega} e_{p(x)} |u(x)|^{p(x)} dx < \infty \end{array} \right\}$$

With the norm:

$$|u|_{p(x), e_{p(x)}} := |u|_{L^{p(x)}_{e_{p(x)}}(\Omega)} = \inf \left\{ \beta > 0 : \int_{\Omega} e_{p(x)} \left| \frac{u(x)}{\beta} \right|^{p(x)} dx \le 1 \right\}$$

Then obviously $L^{p(x)}_{e_{p(x)}}(\Omega)$ is a Banach space (Cruz-Uribe *et al*., 2011). Now, we denote:

$$X := W^{2, p(x)}(\Omega) \cap W_0^{1, p(x)}(\Omega)$$

where, $W_0^{m,p(x)}(\Omega)$ denote the closure of $C_0^{\infty}(\Omega)$ in $W^{m, p(x)}(\Omega)$: For $u \in X$, we define:

$$|u\|_{ep} = \inf\{\beta > 0 : \int_{\Omega} \left(\left| \frac{\Delta u(x)^{p(x)}}{\beta} \right|^{p(x)} + e_p(x) \left| \frac{u(x)}{\beta} \right|^{p(x)} \right) dx \le 1\}$$

Clearly, we observe that X endowed with the above norm is a separable and reflexive Banch space.

Remark 2.1: From Zang and Fu (2008) the norm $\|u\|_{2, p(x)}$ is equivalent to the norm $|\Delta u|_{p(x)}$ in the space X. Consequently, the norms $\|u\|_{2, p(x)}$, $\|u|e_p|$ and $|\Delta u|_{p(x)}$ are equivalent. For the rest of this study, we use $\|u\|e_p$ instead of $\|u\|_{2,p(x)}$ on X.

In order to discuss problem (1.3), we need to choose a variable exponenet space that is more appropriate for our study than the ones presented in the previous part. Therefore, we introduce the following subspace of $W^{2, p(x)}(\Omega)$:

$$X = \{u \in W^{2, p(x)}(\Omega) : u/\partial\Omega = \text{constant}\}$$

Notice that X can be viewed also as:

$$X = \{u + c : u \in W^{2, p(x)}(\Omega) \cap : W_0^{1, p(x)}(\Omega), c \in \mathbb{R}\}$$

and the (X, $\|.\|w^{2, p(x)}(\Omega)$ is separable and reflexive Banch space ([6, Theorem 4]).

Proposition 2.2; Repovs (2015): The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$ where ,q(x) is the conjugate function of p (x), i.e:

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1$$

$$\left| \int_{\Omega} u(x)\upsilon(x)dx \right| \le (\frac{1}{p^-} + \frac{1}{q^-}) |u|_{p(x)} |\upsilon|_{q(x)} \le 2 |u|_{p(x)} |\upsilon|_{q(x)}$$

Proposition 2.3; Repovs (2015): Let $\rho(u) = \int_{\Omega} |u|^{p(x)} dx$. For $u, u_n \in L^{p(x)}(\Omega)$, we have:

$$|u|_{p(x)} < (=;>);1 \Leftrightarrow \rho(u) < (=;>)1$$

$$|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}$$

$$|u_n|_{p(x)} \to 0 \Leftrightarrow \rho(u_n) \to 0$$

$$|u_n|_{p(x)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$$

From proposition 2.3 for $u \in W^{2, p(x)}(\Omega)$ the following inequalities hold:

$$\|u\|^{p^-}_{e_p} \le \int_{\Omega} \left(|\Delta u|^{p(x)} + e_p(x) |u|^{p(x)} \right) dx \le \|u\|^{p^+}_{e_p}, \text{if} \|u\|_{e_p} \ge 1$$

$$\|u\|^{p^+}_{e_p} \le \int_{\Omega} \left(|\Delta u|^{p(x)} + e_p(x) |u|^{p(x)} \right) dx \le \|u\|^{p^-}_{e_p}, \text{if} \|u\|_{e_p} \ge 1$$

For all $x \in \Omega$ and $k \ge 1$ denote by:

$$p_k^*(x) = \begin{cases} \dfrac{Np(x)}{N - kp(x)} & \text{for } kp(x) < N \\ +\infty & \text{for } kp(x) \ge N \end{cases}$$

Proposition 2.4; Repovs (2015): For $p \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$ there is a continuous and compact embedding:

$$W^{k,\, p(x)}\left(\overline{\Omega}\right) \rightarrow L^{r(x)}\left(\overline{\Omega}\right)$$

Lemma 2.5; Fan and Zhao (2001): If $f: \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a caratheodory function and:

$$|f(x, s)| \leq a(x) + b|s|^{\frac{p1(x)}{p2(x)}},\ \forall (x, s) \in \overline{\Omega} \times \square$$

where, $p1(x) \in C(\overline{\Omega})$, $a(x) \in L^{p2(x)}(\Omega)$, $p2(x) > 1$, $a(x) \geq 0$ and $b \geq 0$ is a constant then the Nemytskii operator from $L^{p1(x)}(\Omega)$ defined nby $Nf(u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Definition 2.6 : For simplicity, let $X = W^{2,\, p(x)}(\Omega) \cap W_0^{1,\, p(x)}(\Omega)$, $W^{2,\, p(x)}(\Omega)$ or $(W^{2,\, p(x)}(\Omega) \cap W_0^{1,\, p(x)}(\Omega)) \oplus \mathbb{R}$.

We say that a function $u \in X$ is a weak solution of (1.1-1.3) if:

$$\int_\Omega |\Delta u(x)|^{p(x)-2} \Delta u(x) \Delta \upsilon(x) dx + \int_\Omega e_p(x) |u(x)|^{p(x)-2} u(x) \upsilon(x) dx = \int_\Omega f(x, u(x)) \upsilon(x) dx$$

We say that a function $\upsilon \in X$ is a weak solution of Steklov problem (1.4) if:

$$\int_\Omega |\Delta u(x)|^{p(x)-2} \Delta u(x) \Delta \upsilon(x) dx + \int_\Omega e_p(x) |u(x)|^{p(x)-2} u(x) \upsilon(x) dx = \int_{\partial\Omega} g(x, u(x)) \upsilon(x) dx$$

We say that a function $u \in X$ is a weak solution of Robin problem (1.5) if:

$$\int_\Omega |\Delta u(x)|^{p(x)-2} \Delta u(x) \Delta \upsilon(x) dx + \int_{\partial\Omega} m(x) |u(x)|^{p(x)-2} u(x) \upsilon(x) dx = \int_\Omega f(x, u(x)) \upsilon(x) dx$$

hold for all $\upsilon \in X$.
Define the operators I, J, K and L: $X \rightarrow X^*$ by:

$$\langle I(u), \upsilon \rangle = \int_\Omega |\Delta u(x)|^{p(x)-2} \Delta u(x) \Delta \upsilon(x) dx$$
$$\langle J(u), \upsilon \rangle = \int_\Omega e_p(x) |u(x)|^{p(x)-2} u(x) \upsilon(x) dx$$
$$\langle K(u), \upsilon \rangle = \int_\Omega f(x, u(x)) \upsilon(x) dx$$
$$\langle L(u), \upsilon \rangle = \int_{\partial\Omega} g(x, u(x)) \upsilon(x) dx$$

for all $u, \upsilon \in X$ and we define the operator $T: X \rightarrow X^*$ by:

$$\langle T(u), \upsilon \rangle = \langle I(u), \upsilon \rangle + a\langle J(u), \upsilon \rangle - b\langle K(u), \upsilon \rangle - c\langle L(u), \upsilon \rangle - d\int_{\partial\Omega} m(x) |u(x)|^{p(x)-2} u(x) \upsilon(x) dx$$

Theorem 2.7; Leray and Lions (1965): Let X be re exive real Banach space. Moreover, let $T : X \rightarrow X^*$ be an operator which is: bounded, demicontinuous, coercive and monotone on the space X. Then the equation $T(u) = f$ has at least one solution $u \in X$ for each $f \in X^*$. If moreover, T is strictly monotone operator then for every $f \in X^*$ the equation $T(u) = f$ has precisely one solution $u \in X$.

Definition 2.8: Let X be re exive real Banach space. An operator $A : X \rightarrow X^*$ verifies:

$$\langle A(u) - A(\upsilon),\ u - \upsilon \rangle \geq 0$$

for any $u, \upsilon \in X$ is called a monotone operator. An operator A is called strictly monotone if for $u \neq \upsilon$ the strict inequality holds in (2.3). An operator A is called strongly monotone if there exists $C>0$ such that:

$$\langle A(u) - A(\upsilon),\ u - \upsilon \rangle \geq C\|u - \upsilon\|_X^2$$

for any $u, \upsilon \in X$.

RESULTS AND DISCUSSION

In this study, we prove our main result by using Browder theorem. We see that, $u \in X$ is a weak solution of (Eq. 1-5) if and only if $T(u) = 0$ in X^* with a-d checking some conditions in each equation. to prove the result we show that T satisfies the assertions of the theorem (2.7). Next, we split the proof in several steps.

Step 1: We prove that T is bounded in fact, let $\|u\|_{ep}$ M: Since and J are the Frechet derivative of the functional $\int_\Omega \frac{1}{p(x)} |\Delta u(x)|^{p(x)}$ and $\int_\Omega \frac{1}{p(x)} e_p(x) |u(x)|^{p(x)} dx$, respectively and then I with J are bounded. We have some deduction for $\int_{\partial\Omega} m(x) |u(x)|^{p(x)-2} u dx$. Moreover, form proposition (2.2) and lemma (2.5) there exists $C_1>0$ such that:

$$\|K(u)\| x^* = \sup_{\|\upsilon\|=1} L(u), \upsilon \rangle |$$

$$\leq | \sup_{\|\upsilon\|=1} 2|f|_{p'(x)} |\upsilon|_{p(x)}$$

$$\leq C_1 |f|_{p'(x)}$$

Similarly, in view of lemma (2.5) there exists $C_2>0$ such that:

$$\|L(u)\|x^* \leq C_2 |g| L_{p'(x)}(\partial\Omega)$$

So, L is a bounded operate.

Step 2: We prove that T is continuous.We have I and J are continuous operators because that are the Frechet derivative of the functional $\int_\Omega \frac{1}{p(x)}|\Delta u(x)|^{p(x)}dx$ and $\int_\Omega \frac{1}{p(x)} e_p(x)|u(x)|^{p(x)} dx$ respectively therefore I and J are continous. On the other hand, let $(u_n)_n \subset X$ be a sequence such that $u_n \rightharpoonup u$. Using the compact embedding of X into $L^{q(x)}(\Omega)$ there exists a subsequence, noted also $(u_n)_n$, such that $u_n \to u$ in $L^{q(x)}(\Omega)$. According to the Krasnoselki's theorem, the Nemytskii operator:

$$N_f = L^{q(x)}(\Omega) \to L\frac{q(x)}{q(x)\text{-}1}(\Omega)u \mapsto f(.,u)$$

Is continuous. Hence, $N_f(u_n) \to N_f(u)$ in $L\frac{q(x)}{q(x)-1}(\Omega)$. Also, in view of the Holder's in equality and the continuous embedding of X into $L^{q(x)}(\Omega)$ we obtain:

$$|\langle L(un)\rangle| \text{-}L(u),\upsilon| = |\int_\Omega (f(x,u_n)\text{-}f(x,u))\upsilon(x)dx|$$

$$\leq 2\|N_f(u_n)\text{-}N_f(u)\| \frac{q(x)}{q(x)\text{-}1} |\upsilon(x)| q(x) \leq$$

$$C\| N_f(u_n)\text{-}N_f(u)\| \frac{q(x)}{q(x)\text{-}1} \|\upsilon\| e_p$$

Thus, $K(u_n) \to K(u)$ in X^*.

Further, it is known that the Nemytskii operator N_g: $u \mapsto g(x, u)$ is a continuous bounded operator from $L^{r(x)}(\partial\Omega)$ into $L\frac{r(x)}{r(x)-1}(\partial\Omega)$ and analogously, L is completely continuous. Step 3: we prove that T is strongly monotone. We recall the elementary in equality for $\alpha.\beta \in \mathbb{R}^N$:

$$\begin{cases} |\alpha\text{-}\beta|\gamma \leq 2\gamma(|\alpha|)^{\gamma\text{-}2}\alpha\text{-}|\beta|\gamma\text{-}2\beta).(\alpha\text{-}\beta) & \text{if } \gamma \geq 2 \\ |\alpha\text{-}\beta|^2 \leq \frac{1}{\gamma\text{-}1}(|\alpha|+|\beta|)^{2\text{-}\gamma}(|\alpha|^{\gamma\text{-}2}\alpha\text{-}|\beta|^{\gamma\text{-}2}\beta).(\alpha\text{-}\beta) & \text{if } 1<\gamma<2 \end{cases}$$

where. Denotes the standard inner product in $\mathbb{R}^N$. Let us define the sets of Ω dependent on p:

$$U_p := \{x \in \Omega : p(x) \geq 2\}$$

$$V_p := \{x \in \Omega : 1 < p(x) < 2\}$$

Now, we show that I+J is strongly monotone. Indeed:

$$\langle (I+J)(u)\text{-}(I+J)(\upsilon)\rangle, u\text{-}u = \int_\Omega (|\Delta u|^{p(x)-2}\Delta u\text{-})|\Delta\upsilon|^{p(x)-2}\Delta\upsilon$$
$$(\Delta u\text{-}\Delta\upsilon)dx + \int_\Omega (|u|^{p(x)} u\text{-}|\upsilon|^{p(x)}\upsilon)(u\text{-}\upsilon)dx$$

By help of the elementary inequality (3.1), we get:

$$\langle (I+J)(u)\text{-}(I+J)(\upsilon), u\text{-}u\rangle \geq \int_{Up} \frac{1}{2^p(x)}(|\Delta(u\text{-}\upsilon)|^{p(x)} + |u\text{-}\upsilon|^{p(x)})dx +$$
$$(p(x)\text{-}1)\int_{Vp} |\Delta(u\text{-}\upsilon)|^{p(x)} \left(\frac{|\Delta u\text{-}\Delta\upsilon|}{|\Delta u|+|\Delta\upsilon|}\right)^{2\text{-}p(x)} dx$$

Since, the fact that:

$$\leq \left(\frac{|\Delta u\text{-}\Delta\upsilon|}{|\Delta u|+|\Delta\upsilon|}\right)^{2\text{-}p(x)} \leq 1$$

and:

$$\leq \left(\frac{|u\text{-}\upsilon|}{|u|+|\upsilon|}\right)^{2\text{-}p(x)} \leq 1$$

It then comes:

$$\langle (I+J)(u)\text{-}(I+J)(\upsilon),\ u\text{-}u\rangle \geq \frac{1}{2^{p+}}\int_{Up}\left(|\Delta(u\text{-}\upsilon)|^{p(x)} +)|u\text{-}\upsilon|^{p(x)}\right) +$$
$$(P^{-}\text{-}1)\int_{Vp}\left(|\Delta(u\text{-}\upsilon)|^{p(x)} + |u\text{-}\upsilon|^{p(x)}\right)dx$$

From proposition 2.3, taking $c_o = \min\{1/2^{p+}, p^-\text{-}1\}$. Hence, I+J is strongly monotone (Zeidler, 2013). Since, f is decreasing with respect to the second variable, then:

$$\langle K(u)\text{-}K(\upsilon), u\text{-}\upsilon\rangle = \int_\Omega (f(x,u)\text{-}f(x,\upsilon))(u\text{-}\upsilon)dx \leq 0$$

Also:

$$\langle L(u)\rangle\text{-}l(\upsilon), u\text{-}\upsilon = \int_{\partial\Omega}(g(x,u)\text{-}g(x,\upsilon))(u\text{-}\upsilon)dx \leq 0$$

Consequently, T is strongly monotone. Step 4 we prove that T is is coercive for all $u \in X$, we have wether a = b = 1= , c = d = 0:

$$\frac{1}{\|u\|}\langle Tu,u\rangle = \frac{1}{\|u\|}\int_\Omega\left(|\Delta u|^{p(x)} + m(x)|u|^{p(x)}\right)dx - \int_\Omega f(x,u)udx \geq$$
$$\frac{1}{\|u\|}\min\left\{\|u\|e_p(x)^{p^-}\|u\|p^+_{e_p}\right\}\text{-}2|f|_{p'(x)}|u|_{p(x)} \geq$$
$$\frac{1}{\|u\|}\left(\min\left\{\|u\|p^-_{e_p},\|u\|p^+_{e_p}\right\}\text{-}C_1\|u\|_{e_p}\right)$$

If a = c = 1, b = d = 0, we have:

$$\frac{1}{\|u\|}\langle Tu, u\rangle = \frac{1}{\|u\|}\int_\Omega\left(|\Delta u|^{p(x)} + e_p(x)|u|^{p(x)}\right)dx - \int_\Omega g(x, u)udx \geq$$
$$\frac{1}{\|u\|}\min\left\{\|u\|e_p(x)p^-, \|u\|p^+_{e_p}\right\} - 2|g|L^{r^1(x)}(\partial\Omega)|u|_{p(x)}$$
$$\frac{1}{\|u\|}\left(\min\left\{\|u\|p^-_{e_p}, \|u\|p^+_{e_p}\right\} - C_2\|u\|e_p\right)$$

For a = c = 0, b = d = 1:

$$\frac{1}{\|u\|_m}\langle Tu, u\rangle = \frac{1}{\|u\|_m}\int_\Omega\left(|\Delta u|^{p(x)} + m(x)|u|^{p(x)}\right)dx - \int_\Omega f(x,u)udx \geq$$
$$\frac{1}{\|u\|}\min\left\{\|u\|p^-_m, \|u\|p^+_m\right\} - 2|f|_{q'(x)}|u|_{q(x)} \geq$$
$$\frac{1}{\|u\|_m}\left(\min\left\{\|u\|p^-_m, \|u\|p^+_m\right\} - C_1\|u\|_m\right)$$

With $\|u\|_m = |\Delta u|_{p(x)} + m(x)|u|_{Lp(x)(\partial\Omega)}$ is equivalent to $\|u\|e_p$. It means that the coercivity of T holds. The previous steps guarantee the existence of solution of the problems. For the uniqueness of weak solution for problems studied, suppose that u and υthat u ≠ v. By the strong monotonicity of T, it follows that:

$$0 = \langle Tu - T\upsilon, u - \upsilon\rangle \geq C_p \| u - \upsilon \|^p \geq 0$$

Then u = υ and the proof now is completed. This solution cannot be trivial provided that we suppose f(x, 0)≠ = 0 and g(x, 0) ≠ 0 because in this case T(0) ≠ 0.

CONCLUSION

The proof of the results is made by Browder theorem and the theory of variable exponent Sobolev space.

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