

## On the Metro Domination Number of Cartesian Product of $P_m P_n$ and $C_m C_n$

G.C. Basavaraju, M. Vishukumar and P. Raghunath

Department of Mathematics, Reva University, Yalahanka, 64 Bangalore  
 Karnataka State, India

**Abstract:** Let  $G = (V, E)$  be a graph. A set  $S \subseteq V$  is called resolving set if for every  $u, v \in V$  there exist  $w \in S$  such that  $d(u, w) \neq d(v, w)$ . The resolving set with minimum cardinality is called metric basis and its cardinality is called metric dimension and it is denoted by  $\beta(G)$ . A set  $D \subseteq V$  is called dominating set if every vertex not in  $D$  is adjacent to at least one vertex in  $D$ . The dominating set with minimum cardinality is called domination number of  $G$  and it is denoted by  $\gamma(G)$ . A set which is both resolving set as well as dominating set is called metro dominating set. The minimum cardinality of a metro dominating set is called metro domination number of  $G$  and it is denoted by  $\gamma\beta(G)$ . In this study we determine on the metro domination number of cartesian product of  $P_m$  and  $C_m C_n$ .

**Key words:** Metric dimension, landmark, dominating set, metro dominating set, cardinality, product

### INTRODUCTION

All the graphs considered are simple, finite and connected. Given a graph  $G = (V, E)$  and  $u, v \in V$ ,  $d_G(u, v)$  (or simply  $d(u, v)$ ) denotes as distance between  $u$  and  $v$  in  $G$ , i.e., the length of a shortest  $u$ - $v$  path.

Harary and Melter (1976) introduce the metric dimension graph  $G$ . The vertex and edge sets of a graph  $G$  are represented as  $V(G)$  and  $E(G)$ . The distance between vertices  $u, w \in V(G)$  is represented as  $d_G(v, w)$  or  $d(v, w)$ . A vertex  $a \in V(G)$  resolves a pair of vertices  $v, w \in V(G)$  if  $d(v, a) \neq d(w, a)$ . A set of vertices  $S \subseteq V(G)$  resolves  $G$  and  $S$  is a resolving set of  $G$ , if every pair of distinct vertices of  $G$  are resolved by some vertex in  $S$ . A resolving set  $S$  of  $G$  with minimum cardinality is a metric basis of  $G$  and its cardinality is the metric dimension of  $G$ , denoted by  $\beta(G)$ .

Let  $D$  be a dominating set, if every vertex not in  $D$  is adjacent to at least one vertex in  $D$ . The minimum cardinality of a dominating set is called the domination number of the graph  $G$  and denoted by  $\gamma(G)$ . The cartesian product of two graphs  $G, H$  is a graph with vertex set  $V(G) \times V(H)$  and  $((g_1, h_1), (g_2, h_2)) \in E(GH)$  if and only if either  $g_1 = g_2$  and  $(h_1, h_2) \in E(H)$  or  $(g_1, g_2) \in E(G)$  and  $h_1 = h_2$  (Dirac, 1952).

Vizing (1963) was initiated by the domination number of the cartesian products of graphs and also were intensively investigated in the past (Vizing, 1963; El-Zahar and Pareek, 1991; Jacobson and Kinch, 1983). We define a metro dominating set and which can be served as a better alternating for the locating dominating set as A dominating set  $D$  of  $V(G)$  having the property

that for each pair of vertices  $u, v$  there exists a vertex  $a$  in  $D$  such that  $d(u, a) \neq d(v, a)$  is called the metro dominating set of  $G$ . The minimum cardinality of a metro dominating set of  $G$  is called metro domination number of  $G$ , denoted by  $\gamma\beta(G)$  (Buckley and Harary, 1990).

**Remark 1.1:** For any connected graph  $G$ ,  $\gamma\beta(G) \geq \max\{\gamma(G), \beta(G)\}$ .

### MATERIALS AND METHODS

**On the metro domination number of Cartesian product of  $P_m P_n$ :** In this study, we determine some of known result on the metro domination number of cartesian product of  $P_m P_n$ .

**Theorem 2.1:** For all  $m, n$ ,  $\gamma\beta(P_2, P_n) = \lceil n+1/2 \rceil$   $n \geq 4$ .

**Proof:** Consider  $P_2, P_n$  as two canonical copies of  $P_n$  with vertices labeled  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_n$  with for each  $i$ ,  $u_i, v_i$  the only edges between the two paths. Domination number  $\gamma(P_2, P_n) = \lceil n+1/2 \rceil$  (Jacobson and Kinch, 1983) and metric dimension  $\beta(P_m, P_n) = 2$  (Khuller *et al.*, 1996). The dominating set is also serves as metric set, thus, (Eq. 1):

$$\gamma\beta(P_2, P_n) \geq \left\lceil \frac{n+1}{2} \right\rceil \quad (1)$$

To prove the reverse inequality we find a metro dominating set of cardinality  $n+1/2$ . We define a set  $D$  as follows:

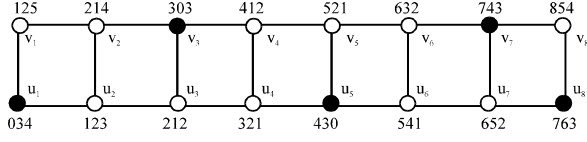


Fig. 1:  $\gamma\beta(P_2, P_8) = 5$

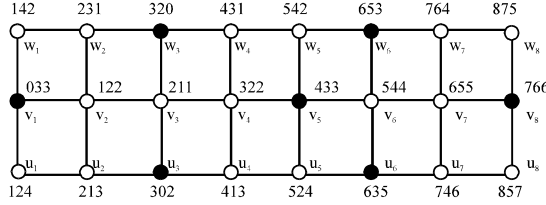


Fig. 2:  $\gamma\beta(P_3, P_8) = 7$

- $D_1 = \{u_{4k-3} : k \geq 1\}, n = 1 \pmod{4}$
- $D_2 = \{v_{4k-1} : k \geq 1\}, n = 3 \pmod{4}$

Choose D in above cases and then  $|D| = \lceil \frac{n+1}{2} \rceil$  proved and D is a dominating set, in fact for every  $v_j \in V-D$  by the choice of D, at least one of  $v_{j-1}$  or  $v_{j+1}$  must be in D and which dominates  $v_j$ , by using Khuller *et al.* (1996), D is resolving set. Hence, (Eq. 2):

$$\gamma\beta(P_2, P_n) \leq \left\lceil \frac{n+1}{2} \right\rceil \quad (2)$$

Therefore, from Eq. 1 and 2:

$$\gamma\beta(P_2, P_n) = \left\lceil \frac{n+1}{2} \right\rceil \quad (3)$$

**Ex:** The minimal metro dominating set of a graph G of Fig. 2 is 5:

**Theorem 2.2; For all m, n:**

$$(P_3, P_n) = \begin{cases} n - \left\lceil \frac{n-1}{4} \right\rceil & \text{if } n \geq 3, n \neq 4k+6, k \geq 1 \\ (n+1) - \lceil n-1 \rceil & \text{if } n = 4k+6, k \geq 1 \end{cases}$$

**Proof:** Consider  $P_3, P_n$  as three canonical copies of  $P_n$  with vertices labeled  $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$  and  $w_1, w_2, \dots, w_n$  with for each  $i, u_i, v_i, w_i$  the only edges between the three paths. Domination number  $\gamma(P_3, P_n) = n - \lceil n-1/4 \rceil$  (Jacobson and Kinch, 1983) and metric dimension  $\beta(P_3, P_n) = 2$  (Khuller *et al.*, 1996). The dominating set is also serves as metric set, thus, (Eq. 4):

$$\gamma\beta(P_3, P_n) \geq n - \left\lceil \frac{n-1}{4} \right\rceil \quad (4)$$

To prove the reverse inequality we find a metro dominating set of cardinality  $n - \lceil n-1/4 \rceil$ . We define a set D as follows:

- $D_1 = \{u_{4k-1} : k \geq 1\}, n = 3 \pmod{4}$
- $D_2 = \{v_{4k-3} : k \geq 1\}, n = 1 \pmod{4}$
- $D_3 = \{w_{4k-1} : k \geq 1\}, n = 3 \pmod{4}$

Choose D in above cases and then  $|D| = n - \lceil n-1/4 \rceil$  proved and D is a dominating set, in fact for every  $v_j \in V-D$ , by the choice of D, at least one of  $v_{j-1}$  or  $v_{j+1}$  must be in D and which dominates  $v_j$ , by using Khuller *et al.* (1996), D is resolving set. Hence, (Eq. 5):

$$\gamma\beta(P_3, P_n) \leq n - \left\lceil \frac{n-1}{4} \right\rceil \quad (5)$$

Therefore, from Eq. 4 and 5:

$$\gamma\beta(P_3, P_n) = n - \left\lceil \frac{n-1}{4} \right\rceil \quad (6)$$

**Ex:** The minimal metro dominating set of a graph G, of Fig. 3 is 7.

**Theorem 2.3:** For all m, n,  $\gamma\beta(P_4, P_n) = n, n = 4, 7, 8$  and  $n \geq 10$ .

**Proof:** Consider  $P_4, P_n$  as four canonical copies of  $P_n$  with vertices labeled  $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n$  and  $x_1, x_2, \dots, x_n$  with each for  $i, u_i, v_i, w_i, x_i$ , the only edges between the four paths. Domination number  $\gamma(P_4, P_n) = n$  (Jacobson and Kinch, 1983) metric dimension  $\beta(P_4, P_n) = 2$  (Khuller *et al.*, 1996). The dominating set is also serves as metric set, thus, (Eq. 7):

$$\gamma(P_4, P_n) \geq n \quad (7)$$

To prove the reverse inequality we find a metro dominating set of cardinality n. We define a set D as follows:

**Case; 1:** Suppose  $n = 4 + 3k$  for some integer  $k \geq 0$ . Let.  $D_1 = \{u_3, v_1, w_4, x_2\}$   $D_2 = \{x_{6t+1}, u_{6t} : t = 0, 1, \dots, \lfloor k/2 \rfloor\}$  and  $D_3 = \{u_{6t+3}, w_{6t+4}, x_{6t+2} : t = 0, 1, \dots, \lfloor k/2 \rfloor\}$ . Set  $D = D_1 \cup D_2 \cup D_3$ . It is easy to show that D is a dominating set of  $P_4, P_n$  of order n (Chartrand *et al.*, 2000; Elumalai and Karthikeyan, 2014).

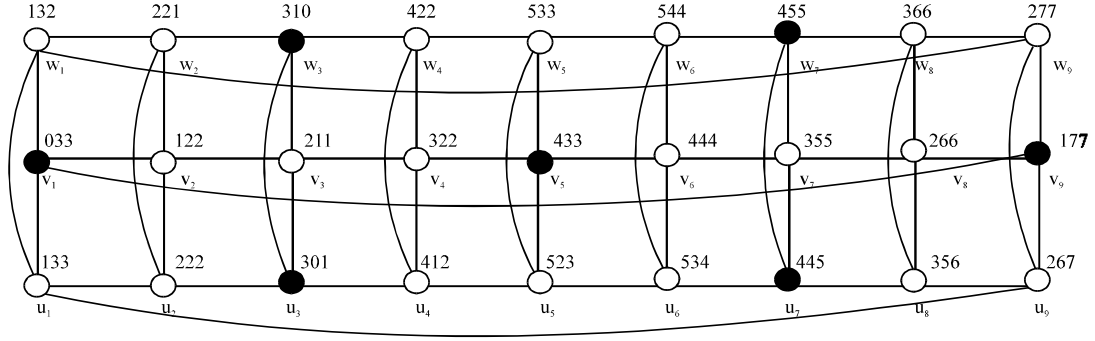


Fig. 3:  $\gamma\beta(C_2, C_8) = 7$

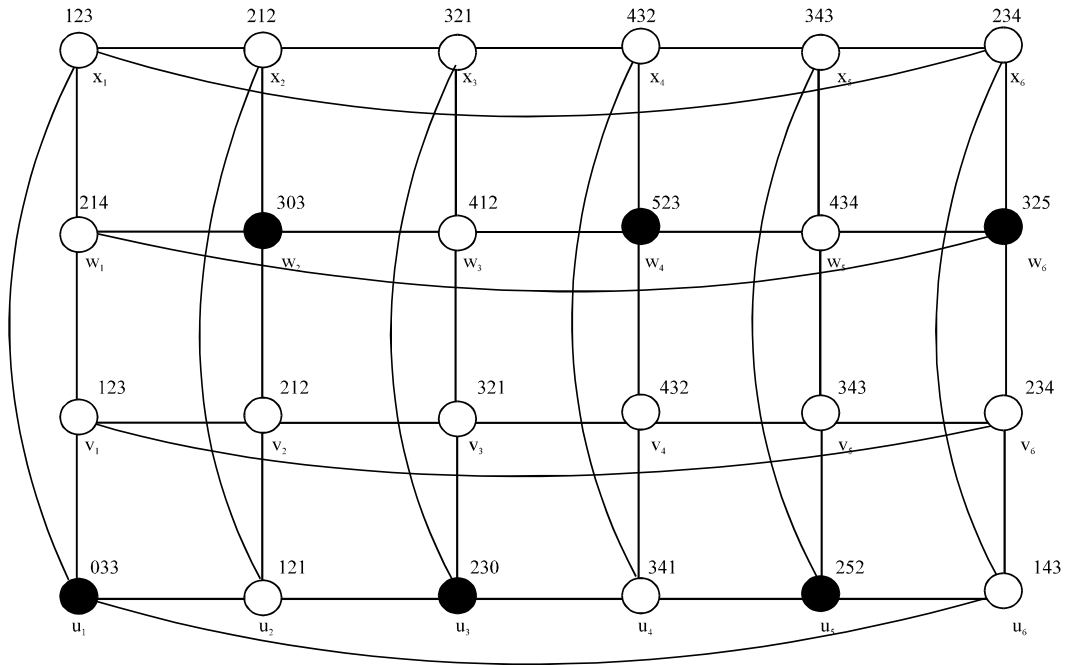


Fig. 4:  $\gamma\beta(C_2, C_6) = 6$

**Case 2:** Suppose  $n = 8+3k$  for some integer  $k \geq 0$ . Let.  $D_1 = \{u_3, v_1, w_4, x_2, x_7, u_5, w_8, x_6\}$ ,  $D_2 = \{x_{6t+4}, v_{6t+5}, u_{6t+3}; t = 0, 1, \dots, [k/2]\}$  and  $D_3 = \{u_{6t+11}, w_{6t+12}, x_{6t+10}; t = 0, 1, \dots, [k/2]\}$ . Set  $D = D_1 \cup D_2 \cup D_3$ . Again, it can shown that  $D$  is a dominating set of  $P_4, P_n$  of order  $n$ .

**Case 3:** Suppose  $n = 12+3k$  for some integer  $k \geq 0$ . Let.  $D_1 = \{u_3, v_7, u_{11}, v_1, u_5, v_9, w_4, w_8, w_{12}, x_2, x_6, x_{10}\}$ ,  $D_2 = \{x_{6t+8}, v_{6t+9}, u_{6t+7}; t = 0, 1, \dots, [k/2]\}$  and  $D_3 = \{u_{6t+11}, w_{6t+12}, x_{6t+10}; t = 0, 1, \dots, [k/2]\}$ . Set  $D = D_1 \cup D_2 \cup D_3$ . As above,  $D$  is dominating set of  $P_4, P_n$  of order  $n$ .

Choose  $D$  in above cases and  $|D| = n$  proved and  $D$  is a dominating set, in fact for every  $v_j \in V-D$ , by the choice of  $D$ , at least one of  $v_{j-1}$  or  $v_{j+1}$  must be in  $D$  and which dominates  $v_j$  by using Khuller *et al.* (1996),  $D$  is resolving set. Hence,  $\gamma\beta(P_4, P_n) \leq n$ . Therefore, from Eq. 1 and 2  $\gamma\beta(P_4, P_n) = n$ :

## RESULTS AND DISCUSSION

### Generalisation result

**Theorem 3.1:** For all  $m, n$ ,  $\gamma\beta(P_m, P_n) \leq mn/5$ .

**Proof:** Label the vertices of  $P_m \times P_n$  as  $y_{ij}$  for  $1 \leq i \leq m, 1 \leq j \leq n$ . Consider the vertices  $y_{ij}$ , where,  $j = 2 \pmod{5}$ . There are not more than  $mn/5$  of these vertices and they dominate of all  $P_m \times P_n$  except for approximately  $1/5$  th those vertices of the form  $y_{ij}, y_{mj}, y_{1i}, y_{ni}$ . It is sufficient to show that not more than  $2/5 (m+n)+2$  vertices of  $P_m \times P_n$  are not dominated by this set. Hence, by using Jacobson and Kinch (1983), for lower bound,  $1/5 \gamma\beta(P_m, P_n) \leq 1/mn (mn/5+2/5 (m+n)+2)$ . By using Khuller *et al.* (1996), we note that the dominating set which satisfies the above condition also serves as metric basis. Thus,  $\gamma\beta(P_m, P_n) \leq mn/5$ .

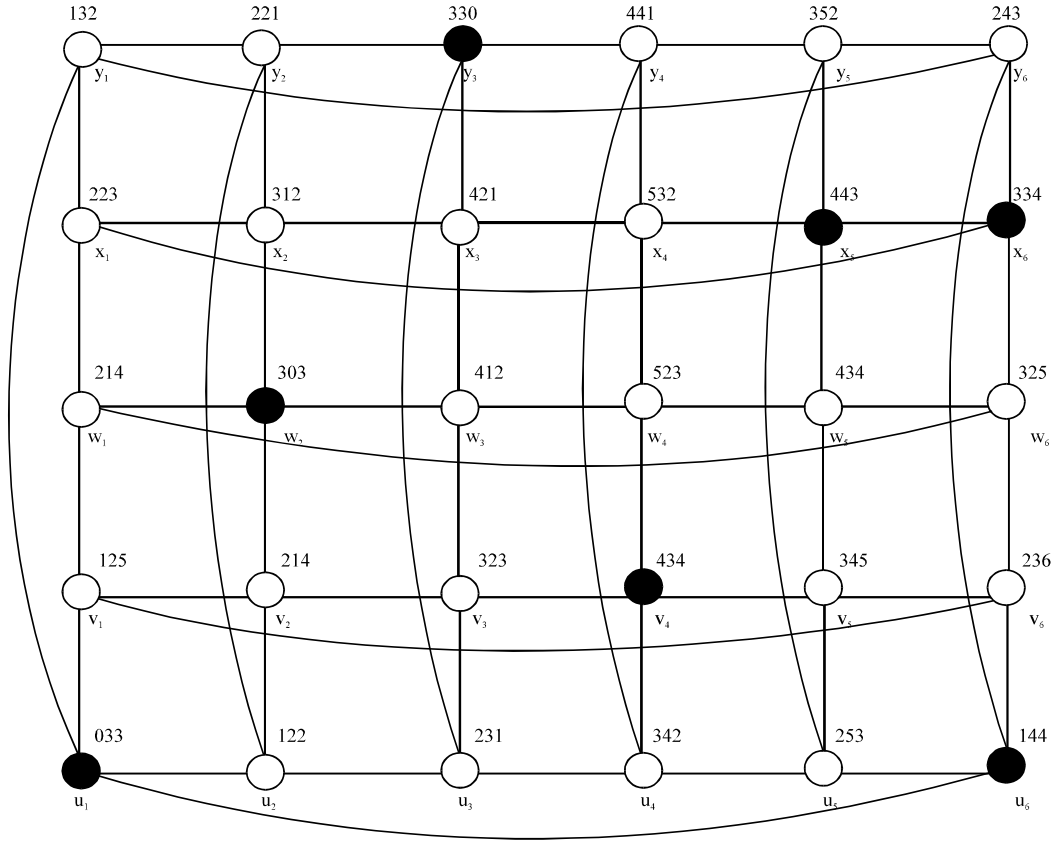


Fig. 5:  $\gamma\beta(C_5, C_6) = 7$

**On the metro domination number of cartesian product of  $C_m, C_n$ :** In this study we determine some of known result on the metro domination number of cartesian product of  $C_m, C_n$ , where  $m = 3, 4, 5$  and  $6$ .

**Theorem 4.1:** For all  $m, n$ ,  $\gamma\beta(C_3, C_n) = n - \lfloor n/4 \rfloor$ ,  $n \geq 4$ .

**Proof:** Let  $D$  be a dominating set of  $C_3, C_n$ . Let  $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$  and  $w_1, w_2, \dots, w_n$  are the vertices of  $C_1-C_3$ , respectively such that for each  $i, i = 1, 2, \dots, n-1$ . Domination number  $\gamma(C_3-C_n) = n - \lfloor n/4 \rfloor$  (Klavzar and Seifter, 1995) and metric dimension  $\beta(C_m, C_n)$  is 3 if  $m$  or  $n$  is odd and 4 otherwise (Caceres *et al.*, 2007). The dominating set is also serves as metric set, thus, (Eq. 8):

$$\gamma\beta(C_3, C_n) \geq n - \left\lceil \frac{n}{4} \right\rceil \quad (8)$$

To prove the reverse inequality we find a metro dominating set of cardinality  $n - \lfloor n/4 \rfloor$ . We define a set  $D$  as follows:

- $D_1 = u_{4i-1}, 1 \geq 1, n = 3(\text{mod } 4)$
- $D_2 = v_{4i-3}, 1 \geq 1, n = 1(\text{mod } 4)$
- $D_3 = w_{4i-1}, 1 \geq 1, n = 3(\text{mod } 4)$

Choose  $D$  in the above cases and then  $|D| = n - \lfloor n/4 \rfloor$  proved and  $D$  is a dominating set, in fact for every  $v_j \in V - D$ , by the choice of  $D$ , at least one of  $v_{j-1}$  or  $v_{j+1}$  must be in  $D$  and which dominates  $v_j$ , by using Caceres *et al.* (2007),  $D$  is resolving set. Hence, (Eq. 8):

$$\gamma\beta(C_3, C_n) \leq n - \left\lceil \frac{n-1}{4} \right\rceil \quad (9)$$

Therefore, from 11 and 12  $\gamma\beta(C_3, C_n) \geq n/4$ .

**Ex:** The minimal metro dominating set of a graph  $G$ , of Fig. 3 is 7.

**Theorem 4.2:** For all  $m, n$ ,  $\gamma\beta(C_4, C_n) = n$ ,  $n \geq 4$ .

**Proof:** Let  $D$  be a dominating set of  $C_4, C_n$ , let  $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n$  and  $x_1, x_2, \dots, x_n$  are the vertices of  $C_1-C_4$  respectively such that for each  $i, i = 1, 2, \dots, n-1$ . Domination  $\gamma(C_4, C_n) = n$  (Klavzar and Seifter, 1995) and metric dimension  $\beta(C_m, C_n)$  is 3 if  $m$  or  $n$  is odd and 4 otherwise (Caceres *et al.*, 2007). The dominating set is also serves as metric set, thus, (Eq. 9):

$$\gamma\beta(C_3, C_n) \geq n - \left\lceil \frac{n}{4} \right\rceil \quad (10)$$

To prove the reverse inequality we find a metro dominating set of cardinality  $n$ . We define a set  $D$  as follows:

- $D_1 = \{u_{2i-1} : 1 \leq i\}, n = 1(\text{mod } 2)$
- $D_2 = \{w_{2i} : 1 \leq i\} \text{ g, } n = 2(\text{mod } 2)$

Choose  $D$  in the above cases and then  $|D| = n$  proved and  $D$  is a dominating set, in fact for every  $v_j \in V - D$  by the choice of  $D$ , at least one of  $v_{j-1}$  or  $v_{j+1}$  must be in  $D$  and which dominates  $v_j$  by using Caceres *et al.* (2007),  $D$  is resolving set. Hence, (Eq. 10):

$$\gamma\beta(C_4, C_n) \geq n \quad (11)$$

Therefore, from Eq. 10 and 11:

$$\gamma\beta(C_4, C_n) = n \quad (12)$$

**Ex:** The minimal metro dominating set of a graph  $G$ , of Fig. 4 is 6.

**Theorem 4.3:** For all  $m, n$ :

$$(C_5, C_n) = \begin{cases} n+1 & \text{if } n \geq 3, n \neq 5k, k \geq 1 \\ n & \text{if } n = 5k, k \geq 1 \end{cases}$$

**Proof:** Let  $D$  be a dominating set of  $C_5, C_n$ . Let  $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n, x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  are the vertices of  $C_1 - C_5$  respectively such that for each  $i, i = 1, 2, \dots, n-1$ . Domination number  $\gamma(C_5 - C_n) = n+1$  (Klavzar and Seifter, 1995) and metric dimension  $\beta(C_m, C_n)$  is 3 if  $m$  or  $n$  is odd and 4 otherwise (Caceres *et al.*, 2007). The dominating set is also serves as metric set, thus,  $\gamma\beta(C_5, C_n) \geq n+1$ . To prove the reverse inequality we find a metro dominating set of cardinality  $n+1$ . We define a set  $D$  as follows:

- $D_1 = \{u_{5i-4} : 1 \leq i\}, n = 1(\text{mod } 5)$
- $D_2 = \{v_{5i-1} : 1 \leq i\}, n = 4(\text{mod } 5)$
- $D_3 = \{w_{5i-3} : 1 \leq i\}, n = 2(\text{mod } 5)$
- $D_4 = \{x_{5i} : 1 \leq i\}, n = 5(\text{mod } 5)$
- $D_5 = \{y_{5i-2} : 1 \leq i\}, n = 3(\text{mod } 5)$

Choose  $D$  in the above cases and then  $|D| = n+1$  proved and  $D$  is a dominating set in fact for every  $v_j \in V - D$  by the choice of  $D$ , at least one of  $v_{j-1}$  or  $v_{j+1}$  must be in  $D$ . and which dominates  $v_j$ , by using Caceres *et al.* (2007),  $D$

is resolving set. Hence, (Eq. 12):

$$\gamma\beta(C_5, C_n) \geq n+1 \quad (13)$$

Therefore, from Eq. 11 and 12:

$$\gamma\beta(C_5, C_n) \leq n+1 \quad (14)$$

**Ex:** The minimal metro dominating set of a graph  $G$ , of Fig. 5 is 7.

**Theorem 4.2:** For all  $m, n$ :  $\gamma\beta(C_6, C_n) = 2n, n \geq 4$ .

**Proof:** Let  $D$  be a dominating set of  $C_6, C_n$ . Let  $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n, x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$  and  $z_1, z_2, \dots, z_n$  are the vertices of  $C_1 - C_6$  and  $C_n$  respectively such that for each  $i, i = 1, 2, \dots, n-1$ . By using Klavzar and Seifter (1995) and metric dimension  $\beta(C_m, C_n)$  is 3 if  $m$  or  $n$  is odd and 4 otherwise (Caceres *et al.*, 2007). The dominating set is also serves as metric set, thus,  $\gamma\beta(C_6, C_n) \geq 2n$ . To prove the reverse inequality we find a metro dominating set of cardinality  $2n$ . We define a set  $D$  as follows:

- $D_1 = \{v_{2i-1} : 1 \leq i\}, n = 1(\text{mod } 2)$
- $D_2 = \{w_{2i} : 1 \leq i\}, n = 2(\text{mod } 2)$
- $D_3 = \{y_{2i-1} : 1 \leq i\}, n = 1(\text{mod } 2)$
- $D_4 = \{z_{2i} : 1 \leq i\}, n = 2(\text{mod } 2)$

Choose  $D$  in the above cases and then  $|D| = 2n$  proved and  $D$  is a dominating set, in fact for every  $v_j \in V - D$  by the choice of  $D$ , at least one of  $v_{j-1}$  or  $v_{j+1}$  must be in  $D$  and which dominates  $v_j$  by using Caceres *et al.* (2007),  $D$  is resolving set. Hence,  $\gamma\beta(C_6, C_n) \leq 2n$ . Therefore, from 1 and 2  $\gamma\beta(C_6, C_n) = 2n$ .

## CONCLUSION

Determination of the locating domination number of a graph is equivalent to finding the least number of monitors that can do the contain task in a given graph or network. Metro dominating set served as better alternating set for the locating dominating set. Using this concepts we are finding the results for Cartesian product of some graphs which is absolutely finer than the locating domination number.

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