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# Industrial Application of Optimal Filtering for States Polynomials Incompletely Measurable with Cross Noise

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**Abstract:** Our study discusses the optimal filtration problem for the states of the linear system of polynomials with the polynomial cross noise over the comments with an arbitrary, not necessarily invertible, the observation matrix is treated proceeding from the general term for stochastic variation. For this case, we use, the Ito differentials of the best estimate of the variance and the error corresponding to the filtering problem indicated are drift first. Derived from this is a transformation of the observation equation to reduce the original problem of an invertible observable matrix. The procedure for obtaining a closed system of filter equations for a linear polynomial any state with the cross-noise polynomial over observations is then established, yields that closed the explicit form of equations in particular filtering boxes of linear equations and bilinear status. As an example, the performance of the optimum filter of the optimal filter for a quadratic state with an independent state noise and a conventional extended Kalman-Bucy filter is presented as an analysis of the results obtained in Matlab.

**Key words:** Kalman-Bucy filter, optimal filter, simulation, independent, cross-noise, performance

## INTRODUCTION

In the last decade, the developments in the digital signal processing part has taken a great impulse (Montiel *et al*., 2017; Serrezvela *et al*., 2017; Kallianpur, 2013) due to the advances of information technologies that allow computational developments much faster and efficient. We have seen significant contributions in the field of robotics, control engineering and digital signal processing based on new theoretical proposals (Serrezuela *et al*., 2016a,b; Grewal, 2011; Oksendal, 2013). We then express the mathematical development of the filtering optimal for states incompletely measurable polynomials with cross-noise.

Be $(\Omega, F, P)$ a complete probability space with a family of σ-alebra $F_t$, $t\geq 0$ Growing and continuous on the right, and be $(N1(t), F_t, t\geq 0)$y $(N_2(t), F_t, t\geq 0)$ two independent centralized Poisson processes. The random process Ft-measurable (x(t), y(t)) is described by a nonlinear stochastic differential equation with a term (Kushner, 2012; Hazewinkel and Williams, 2012).

**Polynomial drift for system state:**

$$dx(t) = f(x, t)\, dt+b(t)\, dN_1(t),\ x(0) = x0 \tag{1}$$

And a linear differential equation for the observational process:

$$dy(t) = (A_0(t)+A(t)x(t))dt+B(t)dN_2(t) \tag{2}$$

Here, $x(t) \in R^n$ It is the state vector and $y(t) \in R^m$ is the linear observation vector of which follows that $m\leq n$. The initial condition is a Poisson vector such that x0, $N_1(t)\in R^p$, y $N_2(t)\in R^q$ are independent of each other. The observation matrix $A(t)\in R^{m\times n}$ it is not necessarily invertible, it is not even required to be a square matrix. It is assumed that $B(t)B^T(t)$ is a positive definite matrix and therefore, $m\leq q$. All coefficients in Eq. 1 and 2 are deterministic functions of appropriate dimensions (Kushner, 2012; Rojas *et al*., 2016).

The nonlinear function f (x, t) will be considered a polynomial function of n variables, Where the components of the state vector $x(t) \in R^n$, are coefficients that depend on time since, $x(t) \in R^n$ is a vector, a special definition is required for the polynomial in the case where $n>1$. In accordance with (Montiel *et al*., 2017), a polynomial of degree p of a vector $x(t) \in R^n$ is considered of the p-linear form of n components of x (t) and can be expressed as follows:

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$$f(x,t) = a_0(t)+a_1(t)+a_2(t)xx^T+ ,...,+ap(t)x,...,p\text{ times},...,x \quad (3)$$

where $a_0(t)$ is a vector of dimension n, $a_1$ is a matrix of dimension n×n, $a_2$ is a 3D dimension tensor n×n×n, $a_p$ is a (p+1) D dimension tensor n×, ..., (p+1) times, ..., × n, yx×, ..., p times, ..., ×x is a pD dimension tensor n×, ..., p times, ..., ×n which is obtained by multiplying the vector x (t), p times by itself. Such a polynomial can also be expressed in the form of the following summation:

$$fk(x,t) = a_0\, k(t)+\sum a_1 k_i(t)x_i(t)\sum a_2\, k_{ij}(t)x_i(t)x_j(t)+ ,...,+\sum_{i_1,...,i_p} a_p\, ki1,...,ip(t)xi1(t),...,x_{ip}(t),k, i, j, i_1,...,i_p, = 1,..., n$$

The problem of estimation is to find the optimal estimate x (t) of the state of the system x (t), based on the observation process Y (t) = {y (s), t0≤s≤t}, minimize the second Euclidean standard:

$$J = E\left[(x(t)-x(t))^T(x(t-x(t))F_t^Y\right]$$

For each t. Here, E [z (t)|$F^Y_t$represents the expected conditional value of a stochastic process z (t) = (x (t)-x (t))$^T$ (x (t)-x (t))with respect to σ-alebra $F^Y_t$ generated by the observation process Y(t) in the time interval [$t_0$, t]. How do you know (Krishnan, 2013; Rodriguez and Carvajal, 2015). This optimal estimate is given by the conditional expected value:

$$\hat{x}(t)=m(t)=E(x(t)|F_t^Y)$$

Of the system state x(t) with respect to the σ-alebra $F^Y_t$ Generated by the observation process Y(t) in the time interval [$t_0$, t]. As usual, the matrix function:

$$P(t) = E\left[(x(t)-m(t))(x(t)-m(t)^T|F_t^Y\right]$$

Is the estimation of the variance of the estimation error. The proposed solution to this optimal filtering problem is based on the formulas of the Ito differentials of the conditional expected value E(x(t)|$F^Y_t$) and its variance P(t) (Havlicek *et al*., 2011; Serrezuela and Zarta, 2017) and will be developed in the following section.

## MATERIALS AND AMETHODS

**Optimal filter design:** The optimum filtering equations will be obtained using the differential formula Ito of the conditional expected value mt = E(x(t)|$F^Y_t$). In the case of the term linear drift $A_0(t)$+A(t)x(t) in the observation equation (Hu *et al*., 2012; Kristensen, 2010):

$$dm(t) = E(f(x,t)|F_t^Y)dt+E(x(t)[A(t(x(t)-m(t))]^T|F_t^Y)\times(B(t)B^T(t))^{-1}(dy(t)-(A_0(t)+A(t)m(t)) \quad (4)$$

where, f (x, t) is the term polynomial drift in the state equation. The Eq. 4 should be complemented by the initial condition $m(t_0)=E(x(t_0|F_{t0}^Y)$ trying to form a closed system of filtering equations, the Eq. 4 could be complemented with the equation for the variance of the error P (t). For this, the formula for the differential Ito of the variance can be used $E((x(t)-m(t))(x(t)-m(t))^T|F_t^Y$ in the case of the term linear drift $A_0(t)$+A (t) x (t) in the observation Eq. 2:

$$dP(t) = E((x(t)-m(t))(x(t)-m(t))^T|F_t^Y\, dP(t) = (E((x(t)-m(t))(f(x,t))^T|F_t^Y)+E(f(x,t)(x(t)-m(t))^T)|F_t^Y)+b(t)b^T(t)-(E(x(t)(x(t)-m(t))^T F_t^Y)A^T(t)\times(B(t)B^T(t))^{-1}A(t)E((x(t)-m(t)x^T(t))|F_t^Y))dt$$

Using the formula of variance P(t) = E ((x)(t)-m(t)xT(t))|$F^Y_t$. The latter equation can be represented as:

$$dP(t) = E((x(t)-m(t))(x(t)-m(t))^T|F_t^Y\, dP(t) = (E((x(t)-m(t))(f(x,t))^T|F_t^Y)+E(f(x,t)(x(t)-m(t))^T)|F_t^Y)+b(t)b^T(t)-P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt \quad (5)$$

Equaion 5 must be complemented with the initial condition:

$$P(t_0) = E[(X(t_0))-m(t_0)))(X(t_0)-m(t_0))^T|F_t^Y$$

Equation 4 and 5 for the optimal estimates m (t) y P(t) respectively, they don't form a closed system of filtering equations for the nonlinear state (Eq. 1) On linear observations (Eq. 2). This means that the system (Eq. 4 and 5): E(f(x, t)|$F^Y_t$)yE((x(t)-m(t))$f^T$(x, t))|$F^Y_t$) which aren't yet expressed as functions of the system variables m(t) and P(t).

As it is shown by Hu *et al*. (2012) and Stengel (2012) in the case of Gaussian white noise in the state y observational equations, it is possible to obtain a closed system of the equations of filtered for the state of the system Eq. 1 with the term polynomial drift over observations linear. In the case that is being considered, of white cross-noise, the following transformation.

First, note that it can always be assumed that matrix A is of full rank and equal to m, representing the dimension of linearly independent observations $y(t) \in R^m$ if not the linearly dependent rows of matrix A corresponding to linearly dependent excess observations should be eliminated. Once this is done, the number of Poisson processes in the observation equations can be reduced to m. The dimension of linearly independent observations. Adding and re-enumerating Poisson processes in each observation Eq. 2.

Therefore, it can always be assumed that the matrix B is a square matrix of dimension m×m such that $B(t)\,B^T(t)$ is a definite positive matrix. Then, the new matrices $\bar{A}(t)\, y\, \bar{B}(t)$ are defined as follows: the matrix $\bar{A}(t) \in R^{m\times n}$ is obtained from the matrix A ($t \in R^{m\times n}$) adding n-m linearly independent rows such that the resulting matrix $\bar{A}(t)$ be invertible. The matrix $\bar{B}(t) \in R^{m\times n}$ is obtained from the matrix $\bar{B}(t) \in R^{m\times n}$ placing B (t) in the upper left corner of $\bar{B}(t)$ defining the others n-m entries of the main diagonal of B(t) equal to infinity and zeroing all other inputs of B(t) outside the main diagonal and submatrix B(t). In other words, B(t = diag[B(t), $\beta I_{(n-m)\times(n-m)}$] where $\beta = \infty$ and $I_{(n-m)\times(n-m)}$ is the identity matrix of dimension (n-m)×(n-m). Then, the new observation equation is given by:

$$y(t) = (\bar{A}_0(t)+(\bar{A}(t)x(t))dt+\bar{B}(t)dN_2(t) \quad (6)$$

where, $\bar{y}(t) \in R^n, \bar{A}_0(t) = [A_0^T(t), 0n\text{-}m]T \in Rn, y0n\text{-}m$ is a vector of n-m zeros the key point of the transformation was that the new process of observation y(t) is physically equivalent to the old process $\bar{y}(t)$ the last ones n-m the dummy components of y (t), the corresponding the $\bar{y}(t)$ coincides with y(t). In addition, the observation matrix $\bar{A}(t)$ is invertible and the matrix $(\bar{B}(t)\bar{B}^T(t))^{-1} \in R^{m\times n}$ exists and is equal to the square matrix of dimension n×m, which is formed by occupying the upper left corner with the submatrix $(B\,(t)\,B)^T(t))^{-1} \in R^{m\times n}$ and all other entries are zeros. In terms of the new observation Eq. 6, the filtering Eq. 4 and 5 take the form:

$$dm(t) = E(f(x,t)|\,F_t^Y dt+P(t)\bar{A}^T(t)(\bar{B}(t)\bar{B}^T(t))^{-1}\times(d\bar{y}(t)-(\bar{A}_0(t)+\bar{A}(t)m(t))dt) \quad (7)$$

$$dP(t) = (E((x(t)-m(t))(f(x,t))^T\,|\,F_t^Y)+E(f(x,t)(x(t)-m(t))^T)\,|\,F_t^Y+b(t)bT(t)-P(t)\bar{A}^T(t)(\bar{B}(t)\bar{B}^T(t))^{-1}\bar{A}(t)P(t))dt \quad (8)$$

With the initial conditions:

$$m(t0) = E(x(t0)\,|\,F_t^Y)\; yP(t0) = E[(x(t0)-m(t0)(x(t0)-m(t0)^T\,|\,F_t^Y$$

Since, the new matrix $\bar{A}(t)$ is invertible for any t≥t0, the random variable x (t)-m (t) is conditionally Poisson with respect to the new process of observation y (t) and therefore, with respect to the original observation process y (t) for any t≥t0 (Serrezud *et al*.,; Crisan *et al*., 2013). Therefore, the following considerations apply to the equations of filtered Eq. 4 and 5. If the function f (x, t) is a polynomial function of the state x with coefficients that depend on the time t, then the expressions for the terms $E(f(x,t)|F^Y_t$ in Eq. 7 and $(E((x(t)\text{-}m(t))(fT(x,t))|F^Y_t)$ in Eq. 8 would include only the polynomial terms of x. Then, these polynomial terms can be represented as functions of m(t) and P(t) using the following property of a Poisson random variable x (t)-m(t): all moments of a Poisson random variable can be represented as functions of the variance P(t).

**For example:** m1 = E [(x (t)-m (t))| Y (t)] = 0, $m_2$ = E [(x (t)-m (t))$^2$| Y (t)] P, $m_3$ = E [(x (t)-m (t))$^3$ | Y (t)] = P, $m_4$ = E [(x (t)-m (t))$^4$| Y (t)] = $3P^2$+P etc. After representing all the polynomial terms in Eq. 7 and 8, it is possible to obtain a closed form of the filtering equations, which is generated by expressing $E(f(x,t)|F^Y_t) yE((x(t)\text{-}m(t))fT(x,t))|F^Y_t)$ as functions of m(t) and P(t).

Finaly, in view of the definition of the matrices $\bar{A}(t)\, y\, \bar{B}(t)$ and the new observational process $\bar{y}(t)$ the filtering Eq. 7and s8 can be rewritten in terms of the original observation Eq. 2 usin y(t), A (t), yB (t). In Fig.1 we can see the block diaram of the Kalman-Bucy filter implemented in MATLAB.

$$dm(t) = E(f(x,t)\,|\,F_t^Y dt+P(t)A^T(t)(B(t)B^T(t))^{-1}\times(dy(t)-(A_0(t)+A(t)m(t))dt) \quad (9)$$

$$dP(t) = (E((x(t)-m(t))(f(x,t))^T\,|\,F_t^Y)+E(f(x,t)(x(t)-m(t))^T\,|\,F_t^Y)+b(t)b^T(t)-P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt \quad (10)$$

With the initial conditions:

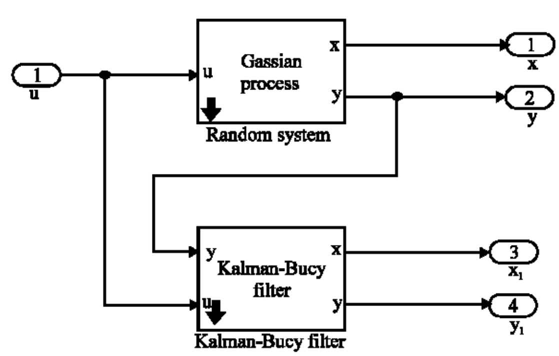


Fig. 1: Kalman-Bucy filter implemented in MATLAB

$$m(t_0) = E(x(t_0) \mid F_t^Y)\ y P(t_0) = E[(x(t_0)-m(t_0))(x(t_0)-m(t_0))T \mid F_t^Y]$$

In addition, the closed form of the filtering Eq. 9 and 10 will be obtained for a third order function $f(x, t)$ in Eq. 1 as follows: Note, however, that the application of the same procedure will result in the design of a closed system of the filtering equations for any polynomial function $f(x, t)$ in (Eq. 1).

**Optimal filter for a third order polynomial state:** Be:

$$f(x, t) = a_0(t)+a_1(t)x+a_2(t)xx^T+a_3(t)xxx^T \quad (11)$$

A third order polynomial function where x is a vector of dimension n, $a_0(t)$ is a vector of dimension n, $a_1(t)$ is an array of dimension n×n, $a_2(t)$ is a 3D tensor of dimension n×n×n, $a_3(t)$ is a 4D dimension tensor n×n×n×n.

In this case, the representation for $E(f(x, t)|F^Y_t) y E((x(t)-m(t))(f(x, t))^T|F^Y_t)$ as functions of m(t) y P(t) is derived as follows:

$$\begin{aligned}E(f(x,t) \mid F_t^Y) &= a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t)+a_2(t)P(t)+3a_3(t)m(t)P(t)+\\ &a_3(t)m(t)m(t)m^T(t)+a_3(t)P(t)*1\end{aligned} \quad (12)$$

$$\begin{aligned}&E(f(x,t)(x(t)-m(t))^T) \mid F_t^Y)+E((x(t)-m(t))(f(x,t))^T \mid F_t^Y) = a_1(t)P(t)+P(t)a_1^T(t)+2a_2(t)\\ &m(t)P(t)+a_2(t)P(t)*1+(a_2(t)(2m(t)P(t)+P(t)*1))^T+a_3(t)[(p(t)*(1*1^T))+3P(t)P(t)+3m(t)m^T\\ &(t)P(t)+3(m(t)P(t))*1^T]+(a_3(t)[(P(t)*(1*1^T))+3P(t)P(t)+3m(t)m^T(t)P(t)+3(m(t)P(t))*1^T])^T\end{aligned} \quad (13)$$

Here, vector 1 represents a vector of dimension n with all its components equal to 1 and vector $a_3\ P(t)*1 \in R^n$ and the matrices $a_3\ P(t)*1\ *1^T \in R^{n\times n}$ and $a_3(t)\ m(t)\ P(t)\ *1^T \in R^{n\times n}$ are defined as:

$$a_3(t)P(t)*1)_i = \sum_{j,k,1} a_{3ijkl}(t)P_{jk}(t)1_i, i=1,\ldots,n$$

$$a_3(t)P(t)*1*1^T)_{ij} = \sum_{j,k,1} a_{3ihkl}(t)P_{hk}(t)1_i 1_j, i,j=1,\ldots,n,$$

$$a_3(t)m(t)P(t)*1*1^T)_{ij} = \sum_{j,k,1} a_{3ihkl}\, m_h(t)P_{kl}(t)1_j,\ i,j=1,\ldots,n$$

Substituting the expression Eq. 12 into Eq. 9 and the expression Eq. 13 into Eq. 10, we obtain the following filtering equations for the optimal estimate m(t) and the variance of error P(t):

$$\begin{aligned}dm(t) &= (a_0(t)+a_1(t)m(t)+a_2(t)m(t)m^T(t)+a_2(t)P(t)+3a_3(t)m(t)P(t)+a_3(t)m(t)m(t)\\ &m^T(t)+a_3(t)p(t)*1+P(t)A^T(t)(B(t)B^T(t))^{-1}[dy(t)-(A_0(t)+A(t)m(t))dt]\end{aligned} \quad (14)$$

$$\begin{aligned}&m(t_0) = E(x(t_0) \mid F_t^Y)),\ dP(t) = a_1(t)P(t)+P(t)a_1^t(t)+2a_2(t)m(t)P(t)+a_2(t)P(t)*1+\\ &a_2(t)(2m(t)P(t)+P(t)*1))^T+(a_3(t)[(P(t)*1(1*1^T))+3P(t)P(t)+3m(t)m^T(t)P(t)+\\ &3m(t)P(t))*1^T]+(a_3(t)[(P(t)*1(1*1^T))+3P(t)P(t)+3m(t)m^T(t)P(t)+3m(t)P(t))\\ &*1^T])+b(t)b^T-P(t)A^T(t)B(t)B^T(t))^{-1}A(t)P(t)dt.\ P(t0) = E((x(t_0)-m(t_0))(x(t_0)-m(t_0))^T \mid F_t^Y))\end{aligned} \quad (15)$$

By means of the previous derivation, the following result will be tested.

**Theorem:** The optimum filter of finite dimension for the third order state (Eq. 1), where the polynomial function f (x, t) of the third order defined by (Eq. 11), on incomplete linear observations (Eq. 2) is given by Eq. 14 for the estimated optimum $m(t) = E(x(t)|F^Y_t)$ and Eq. (3.15)for the estimated error variance $P(t) = E[(x(t)-m(t)(x(t)-m(t))^T|F^Y_t]$.

Thus, based on the unclosed general system of filtering Eq. 7 and 8, it is proved that the closed system of the filtering equations can be obtained for any polynomial state (Eq. 1) over incomplete linear observations (Eq. 2). Therefore, the specific form (Eq. 14 and 15) of the closed system of the filtering equations corresponding to a third order state is derived. In the next section, we will verify the performance of the optimal filter designed for a third order state over incomplete linear observations against a conventional quadratic average filter for stochastic polynomial systems with Gaussian noises, obtained.

## RESULTS AND DISCUSSION

**Analysis and results:** In this section we present an example of optimal filter design for a third-order two-dimensional state and for linear scalar observations and compare it with a conventional mean-square filter for stochastic polynomial systems with Gaussian noises. Be x(t) a two-dimensional real state satisfying the next third-order system:

$$\dot{x}1(t) = x_2(t), x_1(0) = x_{10}, \dot{x}_2(t)= 0.1x_2^3(t)+\psi_1(t), x_2(0) = x_{20} \quad (16)$$

And y(t) the scalar observation process given by the following linear equation:

$$y(t) = x_1(t)+\psi_2(t), \quad (17)$$

where $\Psi_1$ are white cross-noises which are the derivatives of two independent Poisson standard processes (Kushner, 2012; Hazewinkel and Williams, 2012; Krishnan, 2013)). Equation. 16 and 17 present the conventional form of Eq. 1, 2 which are in fact used in practice (Montiel *et al*., 2017; Serrezuela and Chavarro, 2016).

The filtering system 16 and 17 includes two components of the state $x(t) = [x_1(t), x_2(t)]^T \in R^2$ and only one channel of observation $y(t) \in R$, measuring the State component $x_1$ (t) Therefore, the observation matrix $A = [10] \in R^{(1\times2)}$ and is non-invertible. Moreover, the nonlinear component of the state $x_2$ (t).

It is not measurable. The problem is to find the optimum estimate for the third-order state Eq. 16, using incomplete linear observations (Eq. 17) disturbed with random, independent disturbances with isolated pulses modeled as white cross-noises. We will show how to calculate the coefficients of the polynomial vector Eq. 3 for the system Eq. 16.

Even the coefficients of the matrix $a_1$ is a matrix of dimension 2×2, equal to $a_1 = [01|0\ 0]$, sthe coefficient of 3D tensor $a_2$ consists only of zeros, since, there are no quadratic or bilinear terms in (Eq. 16) and the coefficient of the 4D tensor $a_3$. Has only one entry other than zero, $a_{32222} = 0.1$ and all other entries are zeros. Hence, according to (Eq. 14 and 15), this single term other than zero should enter the Equation for $m_2$, multiplied by $3m_2p_{22}+m_2^3+p_{22}$, the Equation for $p_{21} = p_{12}$ multiplied by $3m_2^2p_{12}+3p_{22}p_{12}+p_{22}\ 3m_2p_{22}$ in view of the symmetry of the variance matrix P and the equation for $p_{22}$ multiplied by 2 $p_{22}+6P^2_{22}+6m_2p_{22}+6m_2^2p_{22}$. Figure 2 we can observe the behavior of both current and expected states developed by the Kalman-Bucy filter, performed for about one hundred samples.

In Fig. 3 we can observe the behavior of the results of the position estimation and the speed estimation results of the system.

As a result, the filtering Eq. 14 and 15 take the following particular form for the system (Eq. 16 and17):

$$\dot{m}_1(t) = m_2(t)+P_{11}(t)[y(t)-m_1(t)] \quad (18)$$

$$\dot{P}_{12}(t)=1.1P_{22}(t)+0.3m_2^2(t)P_{12}(t)+ 0.3m_2(t)P_{22}(t)+0.3P_{22}(t)P_{12}(t)- P_{11}(t)_{P12}(t)P_{22}(t) = 1+0.2P_{22}(t)+ 0.6m_2^2(t)P_{22}(t)+0.6m_2(t)P_{22}(t)+ 0.6P_{22}^2(t)$$

With the initial condition:

$$P(0)=E\left((x(0)-m(0))(x(0)-m(0))t \mid Y(0)\right)=P_0$$

The estimates obtained by solving Eq. 18 and 19 are compared with the estimates that satisfy the conventional quadratic average filter equations for the third order state (Eq. 16) on incomplete linear observations (Eq. 17):

$$m'_{k1}(t) = m_{k2}(t)+P_{k11}(t)[y(t)-m_{k1}(t)]$$

$$m'_{k2}(t) = 0.1m_{k2}^3(t)+0.3m_{22}(t)m_{k2}(t)+ P_{k2}(t)[y(t)-m_{k1}(t)]$$

With the initial condition m (0) = E (x (0)| y (0)) = $m_0$:

$$P'_{k11} = 2P_{k12}(t)-P_{K11}^2(t) \quad (19)$$

$$P'_{k12}(t) = P_{k22}(t)+0.3m_{k2}^2(t)P_{K12}^2(t)+ 0.3P_{k22}(t)P_{k12}(t)-P_{k11}(t)P_{k12}(t)$$

$$P'_{k22}(t) = 1+0.6m_{k2}^2(t)P_{k22}(t)+ 0.6P_{k22}^2(t)-P_{k12}^2(t)$$

With the initial condition The estimates obtained by solving Eq. 18 and 19 are compared with the estimates that satisfy the conventional quadratic average filter equations for the third order state (Eq. 16) on incomplete linear observations (Eq. 17).

$$m'_{k1}(t) = m_{k2}(t)+P_{k11}(t)[y(t)-m_{K1}(t)] \quad (20)$$

$$m'_{k2}(t) = 0.1m_{k2}^3(t)+0.3P_{k22}(t)m_{k2}(t)+ P_{k12}(t)[y(t)-m_{k1}(t)]\ P'_{k22}(t) = 1+0.6m_{k2}^2(t)P_{k22}(t)+0.6P_{k22}^2(t)-P_{k22}^2(t)$$

With the initial condition m (0) = E (x (0) | y (0)) = $m_0$:

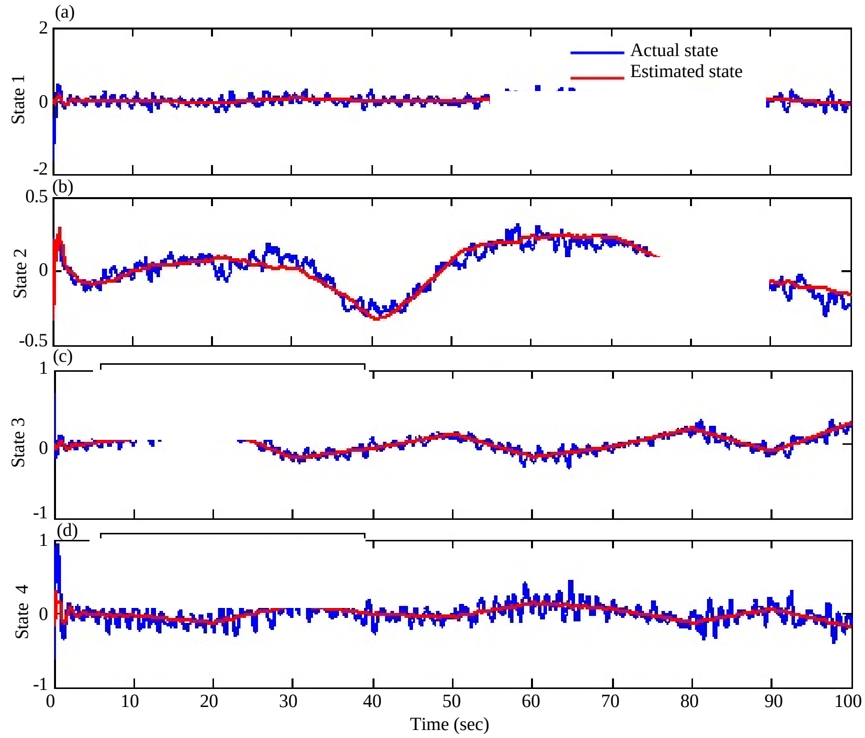


Fig. 2: Behavior of both current and expected states developed by the Kalman-Bucy filter

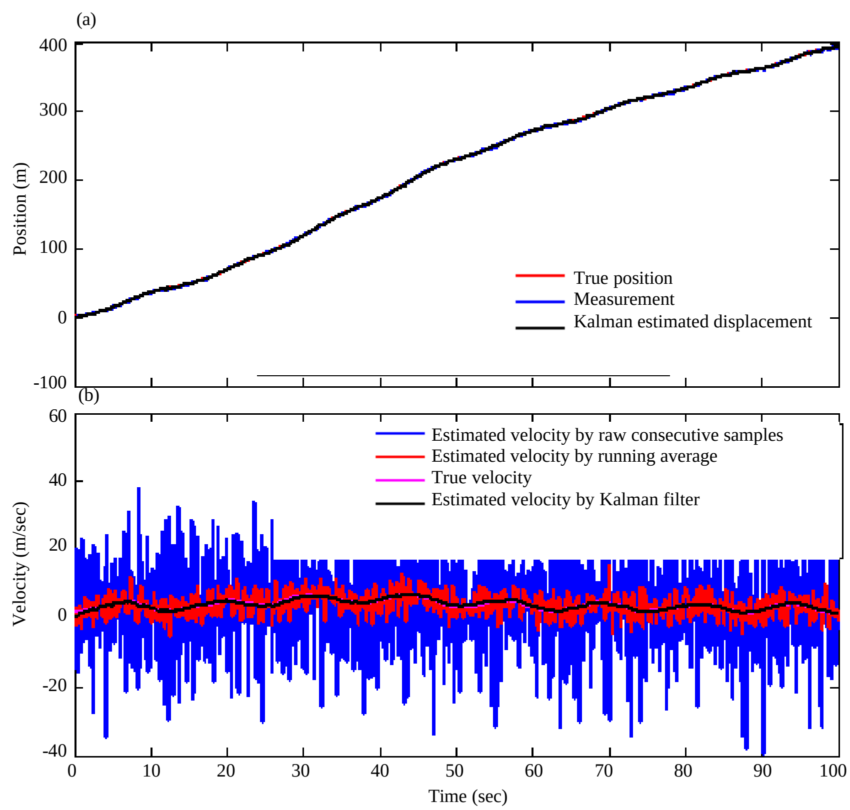


Fig. 3: Behavior of the results of the position estimation and the speed estimation results by the Kalman-Bucy filter

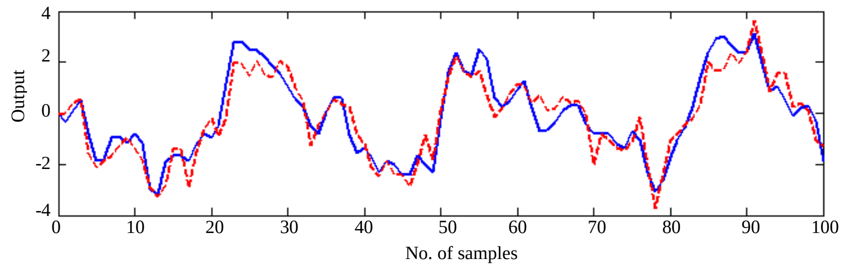


Fig. 4: The errors between the components of the reference state x_1 (t) and x_2 (t)

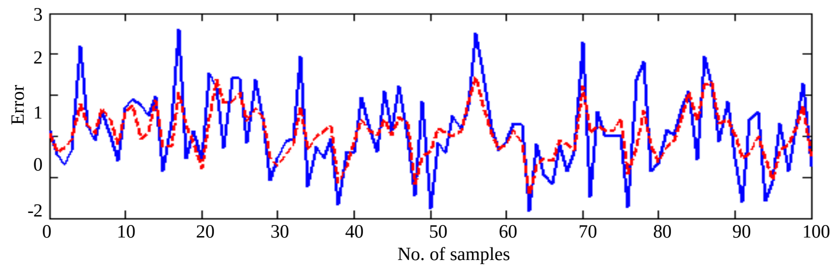


Fig. 5: The error between the real state $x_2$ (t)

$$P'_{k11} = 2P_{k12}(t) - P^2_{k11}(t) \tag{21}$$

$$P'_{k12} = P_{k22}(t)+0.3m^2_{k2}(t)P_{k12}(t)+0.3P_{k22}(t)P_{k12}(t)$$
$$-P_{k11}(t)P_{k12}(t)\ P'_{k12}(t) = 1+0.6m^2_{k2}(t)P_{k22}(t)+$$
$$0.6P^2_{k22}(t)-P^2_{k12}(t)$$

The results of the numerical simulation are obtained by solving the systems of the filtering Eq. 18-21. The values obtained from the estimates $m_1$ (t), $m_2$ (t), $m_{k1}$ (t) and $m_{k2}$ (t)s which satisfy Eq. 18 and 20, respectively, are compared with the actual values of the state variables $x_1$ (t) and $x_2$ (t) in (Eq. 16). The following initial values were assigned for each of the two filters Eq. 18-21 and the reference system Eq. 16 and 17, wrapped in a simulation:

$$x_{10} = -2.5,\ x_{20} = -0.35,\ m_{10} = -14.6,\ x_{20} = -1.38,\ P_{110} = 20,\ _{P120} = 0.9,\ _{P22}0 = 0.06$$

The realizations of the white cross-noises $\psi_1$ (t) and $\psi_2$ (t) in Eq. 20 were generated using the suggested Simulink table. The graphs of the errors between the components of the reference state $x_1$ (t) and $x_2$ (t) which satisfy Eq. 16 and the components of the optimal filter estimate $m_1$ (t) and $m_2$ (t) satisfying (Eq. 18) are shown in Fig. 1. The graphs of the errors between the components of the reference state $x_1$ (t) and $x_2$ (t) satisfying equations Eq. 16 and the components of the polynomial filter estimate in conventional quadratic average $m_{k1}$ (t) and $m_{k2}$ (t) satisfying Eq. 20 are shown in Fig. 2. It can be seen that the estimation error given by the optimum filter reaches quickly and then keeps its values closed to zero. This presents a clear advantage of the optimally designed filter. In contrast, the estimation error given by the polynomial filter in the conventional quadratic average diverges to infinity at time, 100 samples have been taken for the simulation. This leads to the well-justified conclusion that the conventional quadratic polynomial filter designed for Gaussian systems is inapplicable for polynomial systems damaged with white cross-noises in which case the filter designed in this chapter should be used (Fig. 4 and 5). Graph of the error between the real state $x_1$ (t), which satisfies (Eq. 16) and the estimated of the optimal filter $m_1$ (t), satisfying (Eq. 18) in the simulation interval [0, 2].

Figure 4 shows the graph of the error between the real state $x_2$ (t), satisfying (Eq. 16) and the optimal filter estimate $m_2$ (t), satisfying (Eq. 18) in the simulation interval [0, 2]. Note that the variance of the optimal filter error P (t) does not converge to zero as time tends to the asymptotic time point, since, the third order polynomial dynamics is stronger than the Ricatti quadratic terms on the right side of Eq. 19. Thus, it can be concluded that the obtained optimum filter (Eq. 18 and 19) for a third order two-dimensional state over incomplete linear observations provides better estimates than the conventional filter for Gaussian noise polynomial systems.

## CONCLUSION

We can conclude that the optimal filter obtained in this study for a quadratic state with a quadratic noise crossed over incomplete linear observations produces better estimates than the optimal filter for a quadratic state with a state-independent noise or a conventional extended Kalman-Bucy filter. Likewise, the simulations carried out in Matlab software support our claims and give weight to our theoretical discoveries.

In this way, the various applications that can be derived in the field of digital signal processing for the optimum filter for a quadratic state with a state-independent or cross-noise noise using the Kalman-Bucy filter are proposed.

## RECOMMENDATIONS

A future reserach could be the implementation of such algorithms in digital signal processing devices such as: DSP, FPGA or microcontrollers that allow the verification of what has been expressed mathematically and what has been the object of the simulation in Matlab.

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