Journal of Engineering and Applied Sciences 13 (24): 10349-10355, 2018
ISSN: 1816-949X


# Visible Sub-Modules of a Module X Over a Ring R is Introduced

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**Abstract:** The concept of visible submodule of a module X over a ring R is introduced (R is commutative ring with identity and X is unitary R-module) where is a new concept not previously presented. As well as the description of the visible radical submoule and many of the results own this concept has mad. Also, we have presented a concept of V closure operation. Through this study we have been able to obtain many of the results and characteristics that belong to those concepts above.

**Key words:** Visible submodule, visible radical of submodule, strongly cancellation module, pure submodule, vclosure operation, concepts

## INTRODUCTION

In this study the concept of visible submodule has been presented as this concept is new and has not been addressed by anyone before us. A proper submodule K of a module X over a ring R is said to be visible, if K = UK for every a nonzero ideal U of R. Section 2 has been introduced a visible submodule and several properties with important characterization of such a submodules. Section 3 has been defined a visible radical of a submodule K and which is defined as the intersection of all visible submodule of X containing K and we dented by $Vrad_x(K)$. The definition of $Vrad_x(K)$ is gotten from the generalization of visible radical of an ideal G of R is denoted by $\sqrt{G}$.

The concept of V closure operation (for pithiness, $V_{CL}$ operation) has also been provided in this study, where $q{:}S\to S$, S is the set of all visible submodules of a module X over R is called $V_{CL}$ operation if $U\sqsubseteq q()U$, $q(q(U)\sqsubseteq q(U)$ $U\sqsubseteq K$ implies $q(U)\sqsubseteq q(K)$.

(V)Aq(U) = q(AU for all nonzero ideals A of R and submodules U, K of X. This concept is stranger than the concept of closure operation in Lu (1990), where we can make the 4th condition in the concept of $V_{CL}$ operation to achieve equality rather than containment, thanks to the use of the concept of visible submodule. Resulted in this emergence of the concept of $V_{CL}$ operations which a more general of the concept is located in Ali (2005).

In this study we have demonstrated a lot of important properties and characteristics, we have also provided several important and useful results in this search.

In our study, we need to the following fundamental concept. A module X is called faithful if ann(X) = {r∈R; rx = 0, x∈X is the zero ideal of . We call that a module X over is a multiplication module, if for every submodule U of X, then U is written as U = LX for some ideal L of R (Azizi and Jayaram, 2017).

According to Lu (1990) a proper submodule Uof X is said to be irreducible when $X_1\cap X_2 = 2$, then $X_1 = U$ or $X_2$ for every submodules $X_1$ and $X_2$ of X. If S is a multiplicative set of R and U is a submodule of X, then U(S) = {m∈X:j∈S such that jm∈U} is a submodule of X contain U.

A cancellation ideal of R is an ideal J of R such that XJ = YJ for all ideals X, Y, then X = Y (Ali, 2005) and a module X over R is called strongly cancellation module, if for each ideals X, Y of R such that XU = YU then X = Y for every submodule U of X (Elewi, 2016).

**Visible submodules:** In this study a new type of submoule was defined and named as visible submodule. Many essential properties and some characterizations a round this concept have been built (Anderson *et al.*, 2017).

**Definition (2.1):** A proper submodule K of an R-module X is said to be visible whenever K = AK for every a nonzero ideal A of. A proper ideal of a ring R is named visible ideal if A = JA for every a nonzero ideal J of R.

**Remarks and examples (2.2):**

- A zero submodule of any R is always visible
- Consider $Z_4$ as a Z-module. A submodule of $(\bar{2})$ is not visible. Since, for every a nonzero ideal A of Z, implies $(\bar{2}) \neq A(\bar{2})$.
- Two submodules $(\bar{2})$ and $(\bar{3})$ of the Z-module $Z_6$ are not visible for the same reason of No. 2

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- All a nonzero proper cyclic submodule of the module Q as a Z-module is not visible
- Let, L be a submodules of an R-module X such that K ≅ L. Then K is visible submodule ⇔L is visible submodule
- Let $X_1$ and $X_2$ be two R-module and $\psi$:$X_1$→$X_2$ be an R-homo. Then
- if K is a visible submodule of $X_2$, then $\psi^{-1}$(K) is also visible submodule of $X_1$
- If K is a visible submodule of $X_1$, then $\psi$(K) is visible submodule of $X_2$

**Proof (4):** Let L be a cyclic submodule of Q, generated by an element e/g, where e.g., are two nonzero element in Z. Let (s) be an ideal of Z, where s is a positive integer and x>1. Then (s)g = (g), that is (s)(e/g)≠(e/g). Therefore, L is not visible submodule (Atani, 2005).

**Proof (5):** Let $\psi$:K→L be an epimorphism. Then $\psi$(K) = L. Assume that K is a visible submodule which implies K = AK for every a nonzero ideal A of R. Therefore, L = $\psi$(K) = $\psi$(AK) = A$\psi$(K) = AL. Thus, L is visible submodule. Suppose that L is visible submodule. Let A be a nonzero ideal of $\psi$(K) = L = AL A$\psi$(L) = $\psi$(AL) but $\psi$ is (1-1) then L = AL produce L is visible submodule.

**Proof (6):** For every ideal I of R and ≠0 we have IK = K where K is proper submodule of $X_2$. Then:

- I$\psi^{-1}$(K) = $\psi^{-1}$(IK) = $\psi^{-1}$(K)
- I$\psi$(K) = $\psi$(IK) = $\psi$(K). Therefore, $\psi$(K) is visible submodule of $X_2$

**Proposition (2.3):** Let D be a proper submodule of an R-module X. Then the coming are equivalent:

- D is visible submodule
- D = ID for each a nonzero finitely generated (briefly FG) ideal I of R.
- D = (a)D for each 0≠a∈I and 0≠I is any ideal of R

**Proof:**

- ⇒(2):Let D be a visible submodule of X. Consequently, ∀0 ≠I, I is an ideal of R, we have D = ID, we can take I is finitely generated ideal
- ⇒(3):Let D be a proper submodule of X and 0≠I be a FG ideal of R. Therefore, directly from (Eq. 2) we get D = (a)D where 0≠a∈I
- ⇒(3):Let 0 ≠a∈I and 0≠I be an ideal of R. Then a⊆I which implies that (a)D⊆ID. Therefore, by (Eq. 3) we get D⊆ID and so on ID⊆D. Thus, ID = D and hence, D is visible submodule

**Proposition (2.4):** Let X be an R-module and E be a visible submodule of X. If L is a submodule of E, then E/L is a visible submodule of X/L.

**Proof:** Let 0≠A be an ideal of R. Now, A(E/A) = AE/L. But AE = E (since, E is visible submule of X). Then (AE+L)/L = (E+L)/L. Therefore, E/L is visible submodule of M/L.

**Proposition (2.5):** Let x be an R-module and L be two submodules of X. If D, L are visible submodule, then D+L is visible (Dauns, 1980; Kasch, 1982).

**Proof:** Let A be a nonzero ideal of R and L be two submodules of X. Then A(D+L) (since, D and L are visible submodule). Therefore, D+L is visible submodule of X.

**Remarke (2.6):** As a generalization of proposition (2.5), we get: if $\{N_k\}^n_{k=1}$ is a finite collection of a submodule of an R-module X and $N_k$ is visible submodule for all K, then the sum of all these submodules is visible submodule ofX.

**Proposition (2.7):** Every submodule of a visible submodule is also visible.

**Proof:** Let N be a visible submodule of an R-module X and let K be a proper submodule of that is K⊏N. Therefore, N = IN for every a nonzero ideal I of. Then K⊏IN which implies that:

$$IN+K = IN \tag{1}$$

Also, from the above inclusion, we get IK⊏IN. And hence:

$$IK+IN = IN \tag{2}$$

Form Eq. 1 and 2, we get IN+K = IN+K and hence, K = IK. Therefore, K is visible submodule.

**Corollary (2.8):** If either $N_1$ or $N_2$ is visible submodule of an-module, then $N_1 \cap N_2$ is also visible.

**Proof:** It is clearly that $N_1 \cap N_2 \sqsubset N_1$ and $N_1 \cap N_2 \sqsubset N_2$ but $N_1$ is visible, then by proposition (2.7), $N_1 \cap N_2$ is also visible. Similarly with $N_2$ is visible, we get $N_1 \cap N_2$ is visible submodule. As a directly result of corollary (2.8) , we give the following generalization.

**Corollary (2.9):** Let $\{N\}^n_{i=1}$ be a family of submodules of an R-module X such that at least one of them is visible, then $\cap^n_{i=1} N_i$ is visible submodule. The converse of proposition (2.7) need not to be true, for example:

The module $Z_{12}$ as a $Z_{36}$-module. Since, $(\bar{0})$ is contains in any submodule of any R-module X and $(\bar{0})$ is visible submodule by remarks and examples (1). But a submodule $(\bar{6})$ of module $Z_{12}$ is not visible, since, there exists $(\bar{2})$ is a nonzero ideal of $Z_{36}$ such that $(\bar{6}) \neq (\bar{2})(\bar{6}) = (\bar{0})$.

Therefore, $(\bar{6})$ is not visible submodule in $Z_{12}$. However, under a certain condition the converse of proposition (2.7) holds: The module X over is named fully cancellation if for each submodules W, K and for each ideal C of we have CW = CK implies W = K (Ali, 2005). Next, we can use above concept to present the coming result.

**Proposition (2.10):** Let be a ring which all nonzero ideals are idempotent. Let D be a visible submodule of a fully cancellation R-modules. If K is a proper submodule of X containing, then K is a visible submodule of X.

**Proof:** Suppose that, I be anonzero ideal of R. To prove that K = K, we have D⊆K, then ID⊆IK which implies that:

$$IK = ID+IK \quad (3)$$

Also D⊆IK (since, D is visible submodule), then IK = ID+IK. Therefore,

$$IK = ID+I^2K \quad (4)$$

Now, form Eq. 1 and 2, we get ID+IK (since, D is visible submodule) and hence, $IK = D+I^2$. But X is fully cancellation module, then IK = K hence, K is visible submodule.

**Proposition (2.11):** Let D be a visible submodule of a strongly cancellation R-module. Then nn(ID) = ann(I), for every a nonzero ideal I of R.

**Proof:** Let x∈$ann_x$(I). Then xI = 0 and hence, xID = 0 which implies that x∈ann(ID). Therefore, ann(I)⊆ann(ID). Now, let y∈ann(ID). Then ID = 0 but D is visible submodule, then yD = 0 and hence yD = 0D, we have X is strongly cancellation module. Then y = 0 thus, yI = 0 and hence, ∈ann(I), we obtain ann(ID)⊆ann(I). Therefore, ann(ID) = ann(I).

**Proposition (2.12):** Let D be a visible submodule of strongly cancellation R-module. Then every a nonzero ideal I of R is cancellation.

**Proof:** Let 0 ≠ I be an ideal of R s.t AI = BI where A, B are two ideals of let D be a submodule of X. Then AID = BID, but D is visible submodule which implies that AD = BD and hence A = B (since, D is strongly cancellation submodule).

**Proposition (2.13):** For each a nonzero ideal A of R and for each nonempty collection $\{W_\propto\}$ of visible submodule of an R-module X. We have $A(\cap_\propto W_\propto) = \cap_\propto AW_\propto$.

**Proof:** It is known that for each $\cap_\propto W_\propto \subseteq W_\propto$ but $W_\propto$ is visible submodule for each $\propto$ and hence, $AW_\propto$ for each $\propto$ also by proposition (2.7), we get $\cap_\propto W_\propto$ is visible submodule of $\cap_\propto W_\propto$ of X.

Implies $\cap_\propto AW_\propto = \cap_\propto W_\propto = A(\cap_\propto W_\propto)$ (since, $W_\propto$ is visible submodule for each $\propto$).

**Proposition (2.14):** Let N be a visible submodule of an R-module X. Then, N is pure submodule of X.

**Proof:** Let N be a proper submodule of a module X. Then N = IN for every a nonzero ideal I of R. Since, ⊆X, then IN⊆IX. Therefore, N∩IX = N∩IX and hence, N∩IX = (N∩X) = IN by proposition (2.13). Which completes the proof.

**Proposition (2.15):** Let X be a multiplication cancellation R-module. Then every proper submodule N of X is visible submodule if and only if (N:X) is visible ideal of.

**Proof:** Suppose that (N:X) is visible ideal of X. Let x∈N. Then (x)⊆N and hence, ((x:X)⊆(N:X). Therefore, $((x):_R X)\subseteq(N:_R X)$ = I(N:X) and hence, $((x:_R X)X\subseteq 1(N:_R X)X$ which implies that (x)⊆IN (since, X is multiplication module).

Therefore, ∈IN and hence, N⊆IN also, it is known that IN⊆N. Thus, from two above inclusion, we have N = IN, that is N is visible submodule. Let N be a visible submodule to prove that (N:X) is visible ideal. Let $x\in(N:_R X)$. Then (x)X⊆N, implies (x)X⊆IN (since, N is visible submodu). Then (x)X⊆I(N:X). But X is cancellation module. Therefore, (x)X⊆I(N:X) and hence, (x)X∈I(N:X). Then (N:X)⊆I(N:X). Conversely, I(N:X)⊆(N:X). Therefore, (N:X)⊆I(N:X). This end the proof.

**Corollary (2.16):** Let N be a proper submodule of a (F.G) faithful multiplication R-module X. Then N is visible if and only if (N:X) is visible ideal of R.

**Proof:** From Ali (2005), we get X is cancellation and by proposition (2.15) we obtain the result.

**Proposition (2.17):** Let X be a FG faithful multiplication R-module and I be a proper ideal of R. Then the following hold:

- If I is visible ideal of R then IX is visible submodule of X
- If N is visible submodule of then ann(N)

**Proof:** Let I be a visible ideal of R. Then JI = I for each ideal J of R0 ≠J and hence, JIX = IX. Therefore, IX is visible submodule. Suppose that IX is visible submodule of X then JIX = IX (since, X is cancellation module because X is FG faithful multiplication module). Therefore, JI = I and hence, I is visible ideal of R let x∈ann(N:X). Then x(N:X) = 0 which implies xN = x(N:X)N = 0, therefore, x∈ann(N).

Now, let N be a visible submodule of X. Then N = IN for every ideal 0 ≠ I of and by proposition (2.14) , we have N is pure, from this fact, we write N = N∩IM for every ideal I of R. But N is visible, therefore, IN = N∩IM. Taking I = $ann_R$(N) and hence, nn(N)N = N∩ann(N). 0 = N∩ann(N)X. This lead us (0:X) = ((N∩ann(N)X:X) = (N:X)∩ann(NX:X) = (N:X)∩(IX:X) = (N:X)∩I (since, X is faithful FG and multipli. module) = (N:X)∩ann(N) = (N:X)ann(N) by proposition (2.15) and proposition (2.14) Then ann(X) = (N:X)ann(N). But X is faithful which implies that 0 = (N:X)ann(N). Therefore, ann(N)⊑ann(N:X). Which completes the proof.

**Proposition (2.18):** A visible submodule of an R-module X is an idempotent submodule.

**Proof:** N is visible submodule of X, then N = IN for every 0 ≠ I, I is an ideal of R thus, N is an idempotent (choose I = (N:$_R$ X)).

**Proposition (2.19):** Assume X is (F.G) faithful multiplication R-module and K is visible submodule of then $\cap_{k\in I}J_kK = (\cap_{k\in I}J_k)K$ for every a nonempty collection $J_k(k\in I)$ of visible ideal of R.

**Proof:** K is visible submodule of X, then by corollary (2.16), we have (K:X) is visible ideal of R. Suppose that $J_k(k\in I)$ is any collection of visible ideals of R. Now, $(\cap_{k\in I}J_k)K$ = K = (K:X) by proposition (2.18) which is equal (K:X) $(\cap_{k\in I}J_k)K = (\cap_{k\in I}J_k)$ (K:X)K = $(\cap_{k\in I}J_k)$ (K:X)AX for some ideal A of. (since, X is multiplication module), we want to show that $(\cap_{k\in I}J_kK:_R X) = \cap_{k\in I}J_k (K:_R X)$ obviously, $\cap_{k\in I}J_k(K:X)\sqsubseteq(\cap_{k\in I}J_kK:X)$. Conversely, let, $y\in(\cap_{k\in I}J_kK:X)$. Then $X\sqsubseteq\cap_{k\in I}J_kK = \cap_{k\in I}J_k(K:X)$ but we have X is cancellation module Therefore $y\in\cap_{k\in I}J_k(K:X)$.

$$\text{Now, } (\cap_{k\in I}J_k)(K:X)AX = (\cap_{k\in I}J_kK:XAX)$$

$$= A\left(\underset{k=I}{\cap}J_kK:X\right)M$$

$$= A\ k\ IJkK$$

But $J_k$ is visible ideal for all k∈I, then by corollary (2.9), we get $\cap_{k\in I}J_k$ is visible ideal also by proposition (2.17) we obtain that $\cap_{k\in I}J_kK$ is visible, that is $(\cap_{k\in I}J_kK) = \cap_{k\in I}J_kK$ and hence, $(\cap_{k\in I}J_k)K = \cap_{k\in I}J_k$.

**The visible radical of a submodule:** During this study, the concept of visible radical of a submodule has been described. Also, we proved that the equality of the fourth condition of the concept of $V_{CL}$ module is achieved with this type of module and without condition. Many properties and results of these concepts are given.

**Definition (3.1):** A visible radical of a submodule K of an R-module X, denoted by $Vrad_x(K)$ is defined as the intersection of all visible submodule of X which contain K. If there exists no visible submodule of X containing, we write $Vrad_x(K)$ = X. If X = and D is an ideal of R then $Vrad_x(D)$ is the intersection of all visible ideals of R containing D.

**Definition (3.2):** If D is an ideal of, then $\sqrt{D}$ is represent the intersection of all visible ideal containing D. The following results give some fundamental properties of visible radical.

**Proposition (3.3):** If θ:X→X be an epimorphism from an R-module X into R-module X, and H be a submodule of X with ker θ⊑K, then:

- $\theta(Vrad_xH) = Vrad_x\ \theta(H)$
- $\theta^{-1}(Vrad_xH) = Vrad_x\theta^{-1}(H)$, where H is a submodule of X

**Proof:** We have $(Vrad_xH)$ = ∩W where W is visible X with ⊑W, therefore, $\theta(Vrad_xH)$ = θ(∩W). Since, skerθ⊑H⊑W, and by Kasch (1982) we get $\theta(Vrad_xH)$ = ∩θ(W) where intersection over all visible submodule θW of X (the harmomorphic image of visible submodule is also visible. With θ(H)⊑θ(W) and hence, (i) is verified.

Let H be a submodule of X. Then $Vrad_x(H)$ = ∩W where ∩ is over all visible submodule W of X with H⊑W, then by proposition (2.14), $\theta^{-1}(Vrad_xH) = \theta^{-1}(\cap W) = \cap\theta^{-1}(W)$ where ∩ is over all visible submodule $\theta^{-1}(W)$ of X with $\theta^{-1}(H)\sqsubseteq\theta^{-1}(W)$. Hence, $\theta^{-1}(Vrad_xH) = Vrad_x(\theta^{-1}(H))$.

**Proposition (3.4 ):** Let, W be two submodule of R-module X Then:

- $k\sqsubseteq Vrad_xK$
- If ⊑W, then $Vrad_xK\sqsubseteq Vrad_xW$
- $Vrad_x(Vrad_xK) = Vrad_xK$
- $Vrad_xK\cap W\sqsubseteq Vrad_xK\cap Vrad_xW$
- $Vrad_xK+W = Vrad_x(Vrad_xK+Vrad_xW)$
- $Vrad_x(W) = Vrad_x(AW)$ for every visible submodule W of X and for every a nonzero ideal A of R
- $Vrad_x(W)$ for every a nonzero ideal A of R
- $Vrad_x(AW) = Vrad_x(A\ Vrad_xW)$

**Proof:** Since, $Vrad_x K = \cap P$, where the intersection is taken all visible submodule P of X with $K \sqsubseteq P$, also $K \sqsubseteq vradK$. Let P be a visible submodule of Xwith $\sqsubseteq P$ but we have $K \sqsubseteq W \sqsubseteq P$, therefore, $K \sqsubseteq P$ that is $Vrad_x K \sqsubseteq Vrad_x W$. Since, $Vrad_x(Vrad_x K) = \cap P$ where the intersection is taken on all visible submodule Pof X with $Vrad_x K \sqsubseteq P$ and from (Eq. 1), $K \sqsubseteq Vrad_x K$, then directly $Vrad_x(Vrad_x K) \sqsubseteq Vrad_x K$. Also by (Eq. 1) we obtain $Vrad_x K \sqsubseteq Vrad_x(Vrad_x K)$. Thus, the equality holds.

It is clear that $K \cap W \sqsubseteq W$ and $\cap W \sqsubseteq K$, then by (Eq. 2), we obtain $Vrad_x(K \cap W) \sqsubseteq Vrad\, K$ and $Vrad\,(K \cap W)$. Therefore, $Vrad_x(K \cap W) \sqsubseteq Vrad_x K \cap Vrad_x W$. We have $K \sqsubseteq vrad_x K$ and $W \sqsubseteq vrad_x W$. Then $K+W \sqsubseteq Vrad_x K + Vrad_x W$. Also by (Eq. 2), we get $Vrad_x(K+W) \sqsubseteq Vrad_x(Vrad_x + Vrad_x W)$.

Now, to prove another inclusion, let P be a visible submodule of X such that $K+W \sqsubseteq P$ from this step with $K \sqsubseteq P$ we get $W \sqsubseteq P$. Therefore, $Vrad_x K \sqsubseteq P$ and $Vrad_x W \sqsubseteq P$. Thus, $Vrad_x K + Vrad_x W \sqsubseteq P$ and consequently, $Vrad_x(Vrad_x K + Vrad_x W) \sqsubseteq P$. Thus, $Vrad_x(Vrad_x K + Vrad_x W) \sqsubseteq Vrad_x(K+W)$. Therefore, $Vrad_x(Vrad_x K + Vrad_x W) = Vrad_x(K+W)$.

It is clear that $W \sqsubseteq W$, then by using No. (Eq. 2), we get $Vrad_x AW \sqsubseteq Vrad_x W$. Another inclusion: let $Vrad_x W = \cap_{W \sqsubseteq P}$ where P is a visible submodule of X. Therefore, by proposition (2.7), we have also $\cap_{W \sqsubseteq P}$ is visible submodule of X implies W = AW for every a nonzero ideal A of R, therefore, $Aw \sqsubseteq P$ hence, the intersection over visible submodule of X containing AW which gives the visible radical of AW that is $Vrad_x(AW) = \cap_{W \sqsubseteq P} P$ and hence, $Vrad_x(W) \sqsubseteq Vrad_x(AW)$. Thus, $Vrad_x(W) = Vrad_x(AW)$. $Vrad_x W = \cap_{W \sqsubseteq P} P$ where P is visible submodule but W is also visible by proposition (2.7) and hence, W is pure submodule by proposition (4), we get $AW = W \cap AX$ for every ideal A of R. And hence, $Vrad_x(AW) = Vrad_x(W \cap AX)$. And form No. (6), we get $Vrad_x(AW) = Vrad_x(W \cap AX)$. By depending on (Eq. 1), we get $W \sqsubseteq Vrad_x W$, implies $AW \sqsubseteq A\, Vrad_x W$ and hence, $Vrad_x(AW) \sqsubseteq Vrad_x(A\, VradW)$.

**Conversely:** We have $AVrad_x \sqsubseteq Vrad_x(AW)$ (since, W is visible submodule this leads to use ( ). Therefore, $Vrad_x(AVrad_x W) \sqsubseteq Vrad_x(Vrad_x(AW))$. Thus, the equality holds. Immediate form proposition (3.4), we get the coming corollary.

**Corollary (3.5):** Let K be a submoduleof an R-module X. Then we have:

- $Vrad_x K \sqsubseteq Vrad_x K(S)$
- $Vrad_x K \sqsubseteq Vrad_x[K:_R I]$ for every ideal I of R

**Proof:** Since, K (S) is a submodule of X and $\sqsubseteq K(S)$ also for every ideal I of R we have $K \sqsubseteq [K:_R I]$. Then the result follows directly by proposition ((3.4), No. (Eq. 2)). In the following proposition we give a condition under it the equality f proposition ((3.4) (Eq. 4)) holds.

**Proposition (3.6 ):** Let, W be two submoduleof an R-module X if every visible submodule P of P which contain $K \cap W$ is completely irreducible. Then $Vrad_x(K \cap W) = Vrad_x K \cap Vrad_x W$.

**Proof:** From proposition ((3.4) (Eq. 4)) we obtain $Vrad_x(K \cap W) \sqsubseteq Vrad_x K \cap Vrad_x W$. Now, to prove another side, if $Vrad_x(K \cap W) = X$, then $Vrad_x K = VradW = X$. If $Vrad_x(K \cap W) \neq X$, then $\exists$ a visible submodule P of X s.t $K \cap W$ but P is completely irreducible submodule, then either $K \sqsubseteq P$ or $W \sqsubseteq P$ and hence $Vrad_x K \sqsubseteq P$ or $Vrad_x W \sqsubseteq P$. Since every visible submodule containing $K \cap W$ is completely irreducible then $Vrad_x K \sqsubseteq Vrad_x(K \cap W)$ or $Vrad_x W \sqsubseteq Vrad_x(K \cap W)$ and hence, $Vrad_x K \cap Vrad_x W \sqsubseteq Vrad_x(K \cap W)$. Therefore, $Vrad_x(K \cap W) = Vrad_x K \cap Vrad_x W$.

**Proposition (3.7):** If X is a (F.G) faithful multiplication R-module and T is visible submodule of X, then $T = \sqrt{(T:X)}\,T$.

**Proof:** Let F be the set of all visible ideals P of R that contain (T:M). Therefore, (T:M). And hence by proposition (2.19), we get $\sqrt{(T:X)}\,T = (\cap_{P \in F} P)T = \cap_{P \in F} PT$. Now, for each visible ideal P of R we can write T = PT (since, P is visible) also for each $P \in F$, $T = (T:X)T \sqsubseteq PT \sqsubseteq T$. Therefore, $K = \cap_{P \in F} PT$ (since, $\cap_{P \in F} P$ is visible ideal of ), then it is equal to $\sqrt{(T:X)}\,T$. Hence, $T = \sqrt{(T:X)}\,T$.

**Proposition (3.8):** If S is a visible ideal of a ring R, then $S = S\sqrt{(S)}$.

**Proof:** We have $S \sqsubseteq \sqrt{(S)}$, then $S.S = S\sqrt{(S)}$ but S is visible, then S is an idempotent. Therefore, $\sqsubseteq S\sqrt{(S)}$.

**Conversely:** $S\sqrt{(S)} \sqsubseteq S \cap \sqrt{(S)} = S$ (since, $\sqsubseteq \sqrt{(S)}$) that is $S\sqrt{(S)} \sqsubseteq S$ and hence, $S = S\sqrt{(S)}$.

**Proposition (3.9):** Let T be a submodule of FG faithful multiplication R-module. Then $T = \sqrt{(T:X)}\,X = Vrad_x T$.

**Proof:** When $Vrad_x T = X$, the results is end. Otherwise, if P is any visible submodule of X which contains T, then $(T:X) \sqsubseteq (P:X)$ but P is visible submodule, then proposition (2.15), (P:M) is visible ideal of R and hence by proposition (2.7), we get (T:M) is visible ideal of R. Therefore, $(T:X) = \sqrt{(T:X)}(T:X)$ form proposition (3.8). Which implies that $(T:X)\sqrt{(T:X)}$ is equal to (T:M) which contains in (P:M). And hence, $(T:X)^2\sqrt{(T:X)}$ which inclsion in (P:X) (T:X), (since, every visible ideal is idempotent). Therefore, $\sqrt{(T:X)}(T:X)$ which contains in (P:X) (T:X)X. Since, (T:X)X is a submodule of X and by (Elewi, 2016) we get $\sqrt{(T:X)}\sqsubseteq$(P:X) and hence, $\sqrt{(T:X)}X$ which contains in (P:X)X = P. Since, P is any arbitrary visible submodule containing T, then we obtain $\sqrt{(T:X)}X\sqsubseteq Vrad_xT$.

**Conversely:** We have X is multiplication module, $Vrad_xT$ = ($Vrad_xT$:X)X. Since, T is visible submodule hence, by proposition (2.15) we have (T:X) is visible ideal of R. To show that ($Vrad_xT$:X)$\sqsubseteq\sqrt{(T:X)}$. Let P be any visible ideal such that (T:X)$\sqsubseteq$P. Look, P is visible ideal , then from proposition (2.17) PX is visible submodule of X containing T = (T:X)X. To prove this let x$\in$T. Then x$\in$(T:X)X. Therefor, (T:X)x$\sqsubseteq$(T:X)$^2$X = (TX)X. And hence, P(T:X)x$\sqsubseteq$(T:X)X = P$^2$(T:X)PX which implies that x$\in$PX (since, P(T:X) is an ideal of and X is fully cancellation module). That is T$\sqsubseteq$X. Thus, ($Vrad_xT$:X)X = $Vrad_xT\sqsubseteq$PX. Hence, ($Vrad_xTX$)$\sqsubseteq$(PX:X) = P (since, X is cancellation module). Consequently, ($Vrad_xTX$)$\sqsubseteq\sqrt{(T:X)}$. The result end.

**Proposition (3.10):** Let X be a (F. G) faithful multiplication-module. T be a visible submodule of X Then:

- $T=\sqrt{(T:X)}T$
- (T:X)$Vrad_xT$ = T = ($Vrad_x$ T:X)T
- If (T:X) is (F.G) (principle ideal generated by idempotent element ), then $Vrad_xT$ is a visible submodule of X and moreover, T = VradT

**Proof:** K is submodule of then by proposition (2.18), we get that, T is an idempotent ideal of R, therefore, T = (T:X)T, hence, $\sqrt{(T:X)}T=\sqrt{(T:X)}(T:X)T$. And by proposition (3.7) we obtain $\sqrt{(T:X)}T=(T:X)T=T$. It follows from No. (Eq. 1) and proposition (3.7) we get $T=\sqrt{(T:X)}:T$ is equal to $\sqrt{(T:X)}(T:X)X=(T:X)\sqrt{(T:X)}X$ is equal to (T:X)VradT. Suppose that (T:X) is (F.G) ideal of R.

Therefore, $(T:X)\sqrt{(T:X)}$ by" [on radicals of submodules of F.G modules]", hence, $(T:X)X=\sqrt{(T:X)}X=VradT$. Now, we will introduce the concept of Vclosure operation (for short $V_{CL}$ operation). Let X be an R-module and S be the set of all visible submodules of q:S→S we call H a $V_{CL}$ operation if:

- q$\sqsubseteq$q(G)
- q(q(G$\sqsubseteq$q(G)
- G$\sqsubseteq$K, implies q(G)$\sqsubseteq$q(K)
- Aq(G) = q(AG)

For all nonzero ideals A of R and submodules G, K of X. Next, we give a characterization for $V_{CL}$ operation.

**Proposition (3.11):** A mapping q:S→S is a $V_{CL}$ operation if and only if q(X):q(B) for all X, B$\in$S.

**Proof:** Suppose that q is $V_{CL}$ operation. Since, $\sqsubseteq$q(B), then q(X):q(B)$\sqsubseteq$q(X):B for all X, B$\in$S. Another inclusion q(X)$\sqsupseteq$(q(X):B):B)$\sqsupseteq$((q(X):B):q(B)). Thus, (q(X):q(B))$\sqsupseteq$((q(X):B). Therefore, (q(X):q(B)) = (q(X):B). On the opposite side: for all, B$\in$S we have (q(X):h(B)) = (q(X):B). To prove q is $V_{CL}$ operation. Put = B, then (q(X):q(X)) = R. Therefore, X$\sqsubseteq$q(X) for all X$\in$S.

Now, put = q(X), then (q(X):q(q(X)) = (q(X):q(X)) Therefore, q(q(X)) = q(X) for all X$\in$S. Next if $\sqsubseteq$X, then (q(X):q(B)) = (q(X):(q(X):B)$\sqsupseteq$(X:B) = R and hence, q(X)$\sqsupseteq$q(B). In the last, we have X$\sqsubseteq$q(X) but X is visible submodule, then IX = q(IX) for each a nonzero ideal I of R. Therefore, (q(IX):q(X)) = (q(IX):X) = (q(X:X)$\sqsupseteq$(X:X) = R (since, X is visible submodule) form (Eq. 1), thus, (q(IX):q(X) = R. And hence, q(X)$\sqsubseteq$q(IX)(X is visible submodule, then q(X) is also visible submodule). This lead to Iq(X)$\sqsubseteq$q(IX).

**Conversely:** From (Eq. 1), we get (X)$\sqsubseteq$q(X). Then I(X)$\sqsubseteq$q(X). For each a nonzero ideal I of R. And hence, q(IX)$\sqsubseteq$q(q(X)) = q(X). Therefore, q(IX)$\sqsubseteq$Iq(X) (since (q(X) is visible submodule). Thus, we obtain a(IX). Finally, we get h is $V_{CL}$ operation.

**Proposition (3.12):** Let $h_\lambda$:S→S where ($\lambda\in\wedge$) be a family of $V_{CL}$ operation and h(W) = $\cap_{\lambda\in\wedge}h_\lambda(W)$ for all W$\in$S. Then h:S→S is a $V_{CL}$ operation.

**Proof:** We have W$\sqsubseteq h_\lambda$(W) for all, then W = $\cap_{\lambda\in\wedge}h_\lambda(W)$ and hence, W$\sqsubseteq$(W). In particular h(W)$\sqsubseteq$h(h(W). And the opposite:

$$h_\lambda(W)=h_\lambda(h_\lambda(A)\supseteq h_\lambda(\bigcap_{\lambda\in\wedge}h_\lambda)=$$
$$h_\lambda(h(W)\supseteq\bigcap_{\lambda\in\wedge}h_\lambda(W)=h(h(W)$$

Therefore, h(h(W)$\sqsubseteq$h(W). And hence, h(h(W = h(W). Now, if $\sqsubseteq$K, then $h_\lambda(W)\sqsupseteq h_\lambda(K)$ implies, h(W)$\sqsupseteq$h(K). In the end Ih(A) = I$\cap_{\lambda\in\wedge}h_\lambda(A)$ = $\cap I_{\lambda\in\wedge}h_\lambda(A)$ by proposition (3.12) but $Ih_\lambda(A)$ = $h_\lambda$(IA)($h_\lambda$(A) is $V_{CL}$ operation. Therefore, Ih(A) = $\cap_{\lambda\in\wedge}h_\lambda(IA)$ = h(IA). This complete the proof.

**Proposition (3.13):** Let h:S→S be a $V_{CL}$ operation. Then:

- h($\cap_{\lambda\in\wedge}A_\lambda$)$\sqsubseteq$($\cap_{\lambda\in\wedge}h(A_\lambda)$ = h($\cap_{\lambda\in\wedge}h(A_\lambda)$)
- $\sum_\lambda h(A_\lambda)\sqsubseteq h(\sum_\lambda A_\lambda)$ = h ($\sum_\lambda h(A_\lambda)$)
- h(A:I)$\sqsubseteq$h(A):I = h(h(A):I)

**Proof:** Since, $\cap W_\lambda$ for all, so, h($\cap_\lambda W_\lambda$) for all $\lambda$ and h($\cap_\lambda W_\lambda$)$\sqsubseteq\cap_\lambda h(W_\lambda)\sqsubseteq$h($\cap_\lambda h(W_\lambda)$. Then h($\cap_\lambda W_\lambda$)$\sqsubseteq\cap_\lambda h(W_\lambda)$) = $\cap_\lambda h(W_\lambda)$. $W_\lambda\sqsubseteq\sum W_\lambda$, so, $W_\lambda\sqsubseteq h(W_\lambda)\sqsubseteq h(\sum_\lambda W_\lambda)$ for all $\lambda$.

And $\sum_{\lambda} W_{\lambda} \sqsubseteq \sum_{\lambda} h(W_{\lambda}) \sqsubseteq h(\sum_{\lambda} W_{\lambda})$. Therefore, $h(\sum_{\lambda} W_{\lambda}) \sqsubseteq h(\sum_{\lambda} h(W_{\lambda}) \sqsubseteq h(h(\sum_{\lambda} W_{\lambda}) = h(\sum_{\lambda} W_{\lambda})$. Since, $\sqsupseteq I(A:I)$, then $h(A) \sqsupseteq h(I(A:I))$. Now, $h(A:I) \sqsubseteq (h(A:I) \sqsubseteq h(h(A):I \sqsubseteq (h(h(A)):I) = (h(A):I$. Therefore, $(h(A):I) \sqsubseteq (h(A):I)$ and $(h(A):I) = (h(A):I)$ and (Eq. 3) follows.

**Proposition (3.14):** Let X be an R-module and $h:S \rightarrow S$ such that $h(N) = VradN$ for every $N \in S$ and N is visible radical submodule of X. Then h is $V_{CL}$ operator.

**Proof:** Form proposition (3.4), we get (Eq. 1-3) which are conditions of definition of closure operation. It remains to achieve the last condition we have $Vrad_x N = Vrad_x(AN)$ for every a nonzero ideal A of R but N is visible radical submodule that is $VradN = N$. Then $Vrad_x(AN) = Vrad_x N = N = AN = AVrad_x N$. Therefore, h is $V_{CL}$ operator.

**Corollary (3.15):** X is a module over R and h is defined in proposition (3.14). Let, L be submodule of X and A is a nonzero ideal of. Then:

- $(Vrad_x N{:}Vrad_x L) = (Vrad_x N{:}L)$
- $Vrad_x(N{:}A) \sqsubseteq Vrad_x N{:}A =$

**Proof:** We have $h(N) = Vrad_x N$. Then $(Vrad_x N{:}Vrad_x L) = h(N){:}h(L)$ but from proposition (3.13), $h(N){:}h(L) = h(N){:}L$. Therefore, $(Vrad_x N{:}Vrad_x L) = h(N){:}L = = (Vrad_x N{:}L)$. $Vrad_x(N{:}A) = h(N{:}A) \sqsubseteq h(N){:}A = Vrad_x N{:}A$. And $h(N){:}A = h(h(N){:}A) = Vrad_x(Vrad_x(N){:}A)$. Therefore, (Eq. 2) holds.

## CONCLUSION

During this study, we are dealing with commutative rings that contain an identity element as well as all the modules here are unitary.

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