Journal of Engineering and Applied Sciences 13 (22): 9720-9724, 2018
ISSN: 1816-949X

Types of Moduli of Smoothness: As Tools for Peoples Working in Approximation Theory

Khawla Ali Abd Al-Hameed
Department of Soil Science and Water Resources, College of Agriculture,
AL-Qasim Green University, Babil, Hillah, Iraq

Abstract: Moduli of smoothness are tools for peoples working in approximation theory real analysis, numerical analysis, differential equations, functional analysis and statistical estimation. Differentiating the functions many times to know the number of its derivative is a too groud method. In approximation theory, A sutable tool for measuring smoothness of function is the modulus of smoothness. In Ditizian and Totik the failure of the elassical moduli of smoothness to solve some problems in characterizing the behavior of the degree of best approximation, make Ditizian and Totik introduced a tool for measuring smoothness. Called Ditizian and Totik modulus of smoothness in our paper we introduce some versions of moduli of smoothness and then we show that they are equivalent to Ditizian-Totik (DT) modulus of smoothness of function in $L_p[-1,1]$ spaces for $0<p\leq 1$.

Key words: Smoothness, approximation, functional, statistical, modulus, equivalent

INTRODUCTION

Let us start with a simple example. Suppose that A_p^a is the space of of all functions in $L_p[-1,1], 0<p\leq 1, \tau_k(f, \delta(t,.))_{q,p}$ is finite whenever $q\leq p$. How can we characterize this approximation spase? The answer is very well known by now. There are several approaches but the ones that became most that popular in recent decades involve Ivanov moduli of smoothness $\tau_k(f,\delta(t,.))_{q,p}$ (introduced in 1980-1981) and Ditzian-Totik $\omega_k^\varphi(f,t)_p$ (introduced around 1984). The Ditzian-Totik (DT) modulus is defined in Eq. 2: by letting r = 0 (Remark 1.1.1) and the Ivanov modulus for example, (Bhaya and Ahmed, 2018; Bhaya and Fadel, 2018; Bhaya, 2010; Bhaya and Al-Sammak, 2018; Bhaya and Anoon, 2018; Bhaya and Madlool, 2018; Kopotun *et al*., 2015) is given by:

$$t_k(f, d(t,.))_{q,p} = \|\omega_k(f,\ldots,d(t,\ldots,))_q\|_p$$

Where:

$$\omega_k(f,x,\delta(t,x))_q^q = ((1/2\delta(t,x)\int_{-d(t,x)}^{d(t,x)}|\Delta_u^k(f,x)|^q d\upsilon)^{1/q},$$
$$0<q<1$$

It turns out (Ditzian, 2007)) that $\omega_k^\varphi(f,t)_p \sim \tau_k(f,\Delta(t,.))_{p,p}$ with $\Delta(t,.) = t\varphi(.)+t^2$ but in Ditzian (2007), "The [Ivanov] moduli, … , are complicated method to describe smoothness than, … , [DT moduli] with difficult computation".

MATERIALS AND METHODS

As alluded to above, we are interested in the constructive characterization of the functions in $L_p[-1,1]$, $0<p\leq 1$ and c(-1,1) when p = 1. The first section are devoted to introducing the above mentioned DT moduli of smoothness in a new, equivalent form which is more transparent and simpler. We prove the equivalence via K-functionals. For p = 1, these moduli of smoothness were introduced by Theorem (1.1.4), however, no relations to weighted DT moduli were discussed. In the sequel, we will have constants C that may depend only on some of the parameters involved (p, k, r) but are independent of the function and of as the case may be. The constants c may be different even if they appear in the same line Define:

$$L_p = \left\{f : \left(\int_J |f(x)|^p dx\right)^{\frac{1}{p}} < \infty\right\}, 0<p\leq 1 \text{and}$$

$$\varphi(x) = \sqrt{1-x^2}$$

For, $k\in\mathbb{N}_0$, $h\geq 0$ an interval j and $f: J\to R$ let:

$$\Delta_h^k(f,x,J) = \left\{\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f\left(x+\left(i-\frac{k}{2}\right)h\right), \text{if } x\pm\frac{kh}{2}\in J\right.$$

otherwise, be the kth symmetric difference and let $\Delta_h^k(f,x) = \Delta_h^k(f,x,[-1,1])$.

Let, $0<p\leq 1$ and $r\in\mathbb{N}$. Then for $r\geq 1$, let

$$\Im_p^r := \{f \mid f^{(r-1)} \in AC_{loc}(-1,1) \text{ and } \|f^{(r)}\varphi^r\|_p < \infty\} \quad (1)$$

$$\text{and set } \Im_p^0 = L_p[-1,1]$$

(Recall that $AC_{loc}(-1, 1)$ denotes the set of functions which are locally absolutely continuous in (-1, 1)):

For, $f \in \Im_p^r$ define:

$$\omega_{k,r}^{f}\left(f^{(r)},t\right)_p := \sup_{0\le h\le t}\left\|W_{kh}^r(.)\Delta_{h\varphi(.)}^k\left(f^{(r)},.\right)\right\|_p \quad (2)$$

Where:

$$W_{\delta(x)} = \left(\left(1-x-\frac{\delta\varphi(x)}{2}\right)\left(1+x-\frac{\delta\varphi(x)}{2}\right)\right)^{1/2}$$

For $\delta>0$, let:

$$D_\delta := \{x \mid 1-\frac{\delta\varphi(x)}{2} \ge |x|\} \setminus \{\pm 1\}$$

$$= \left\{x \,\middle|\, |x| \le \frac{4-\delta^2}{4+\delta^2}\right\} = \left[-1+\mu(\delta), 1-\mu(\delta)\right]$$

Where:

$$\mu(\delta) = \frac{2\delta^2}{(4+\delta^2)}$$

Observe that, $D_\delta = \theta$ if $\delta>2$ and note that $\Delta_{h\varphi(x)}^k(f,x)$ is defined to be identically 0 if $x \notin D_{kh}$ and that W_δ is well defined on D_δ (in fact if $\delta\le 2$ then Dom $(W\delta) = D_\delta \cup \{\pm 1\}$ (Kopotun *et al*., 2015).

Hence:

$$\omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p := \sup_{0\le h\le t}\left\|W_{kh}^r(.)\Delta_{h\varphi(.)}^k\left(f^{(r)},.\right)_{L_p(D_{kh})}\right\|$$

And:

$$\omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p := \omega_{k,r}^{\varphi}\left(f^{(r)},\frac{2}{k}\right)_p, \text{ for } t \ge \frac{2}{k} \quad (3)$$

Remark 1.1.1: When, r = 0 in (1.1.3) $\omega_k^{\varphi}(f,t)_p$ reduces to the well known kth DT modulus of smoothness $\omega_k^{\varphi}(f,t)_p$:

Definition 1.1.2: Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0$ and $f \in \Im_p^r$ $0<p\le 1$. Then, the averaged modulus of smoothness is defined as:

$$\omega_{k,r}^{*f}\left(f^{(r)},t\right)_p = \left(\frac{1}{t}\int_0^t\int_{Dkt}\left|W_{kt}^r(x)\Delta_{tf(x)}^k\left(f^{(r)},x\right)\right|^p dx\,dt\right)^{1/p}$$

While, the modulus $\omega_{k,r}^{\varphi}(f,t)_p$ is obviously a non-decreasing function of t, the averaged modulus $\omega_{k,r}^{*\varphi}(f^{(r)},t)_p$ does not have to be non-decreasing. At the same time, it immediately follows from definition 1.1.2 that:

$$\omega_{k,r}^{*\varphi}\left(f^{(r)},t_1\right)_p \le \left(\frac{t_2}{t_1}\right)^{1/p}\omega_{k,r}^{*\varphi}\left(f^{(r)},t_2\right)_p, \text{ for } 0<t_1\le t_2 \quad (4)$$

The above moduli of smoothness are equivalent to the following K-functional.

Definition 1.1.3 (K-functional): For k, $r \in \mathbb{N}$, $r \in \mathbb{N}_0$, $0<p\le 1$ and $f \in \Im_p^r$:

$$K_{k,r}^{\varphi}\left(f^{(r)},t^k\right)_p = \inf_{g\in B_p^{k+r}}\left(\left\|(f^{(r)}-g^{(r)})\varphi^r\right\|_p + t^k\left\|g^{(k+r)}\varphi^{k+r}\right\|_p\right)$$

The following weighted DT moduli are defined in (Ditzian and Totik, 1987) (with D = (0,1)). For any $0<p\le 1$:

$$\omega_\psi^k(f,t)_{\omega,p} = \sup_{0<h\le t}\left\|\omega\Delta_{h\psi}^k f\right\|_{L_p[t_0^*,1-t_1^*]} + \sup_{0<h\le t_0^*}\left\|\omega\overrightarrow{\Delta}_h^k f\right\|_{L_p[0,12t_0^*]} + \sup_{0<h\le t_1^*}\left\|\omega\overleftarrow{\Delta}_h^k f\right\|_{L_p[1-12t_1^*,1]},$$

where, if $\psi=\sqrt{x(1-x)}$, , then $t_0^*=t_1^*=k^2t^2$. In Ditzian and Totik (1987) there is a result for such ψ and ω $\omega_\psi^k(f,t)_{\omega,p}$ is equivalent to the following weighted K-functional:

$$K_{k,\psi}\left(f,t^k\right)_{\omega,p} = \inf_{g^{(k-1)}\in AC_{loc}}\left(\left\|(f-g)\omega\right\|_{L_p(D)} + t^k\left\|(\omega\psi^k g^{(k)}\right\|_{L_p(D)}\right)$$

With a simple me dification we get:

$$\omega_\varphi^k(f,t)_{\varphi^r,p} = \sup_{0<h\le t}\left\|\varphi^r\Delta_{h\varphi}^k f\right\|_{L_p[-1+t^*,1-t^*]} + \sup_{0<h\le t^*}\left\|\varphi^r\overrightarrow{\Delta}_h^k f\right\|_{L_p[-1,-1+At^*]} + \sup_{0<h\le t_1^*}\left\|\varphi^r\overleftarrow{\Delta}_h^k f\right\|_{L_p[1-At^*,1]} \quad (5)$$

where, $t^* = 2k^2t^2$ and A is an absolute contant and note that we can prove that K-functional defined in Definition 1.1.3, satisfies:

$$K_{k,r}^{\varphi}\left(f,t^k\right)_p = K_{k,\varphi}\left(f,t^k\right)_{\varphi^r,p} = \inf_{g^{(k-1)}\in AC_{loc}}\left(\left\|(f-g)\varphi^r\right\|_p + t^k\left\|(\varphi^{k+r}g^{(k)}\right\|_p\right)$$

Theorem 6.1.1 in Ditzian and Totik (1987) implies:

$$M^{-1}\omega_f^k(f,t)_{\varphi^\tau,p} \le K_{k,r}^{\varphi}(f,t^k)_p \le M\omega_\varphi^k(f,t)_{\varphi^\tau,p} \quad (6)$$

$M>1$ and $0<t\le t_0$

A similar proof of 6.1.9 in Ditzian and Totik (1987), we have for $t^* = 2k^2t^2$:

$$\omega_\varphi^{*k}(f,t)_{\omega,p} = \left(\frac{1}{t}\int_0^t\int_{-1+t^*}^{1-t^*}\left|\omega(x)\Delta_{\tau\varphi(x)}^k(f,x)\right|^p dxdt\right)^{1/p} + \left(\frac{1}{t^*}\int_0^{t^*}\int_{-1}^{1-At^*}\left|\omega(x)\vec{\Delta}_u^k(f,x)\right|^p dxdu\right)^{1/p} + \left(\frac{1}{t^*}\int_0^{t^*}\int_{1-At^*}^{1}\left|\omega(x)\overleftarrow{\Delta}_u^k(f,x)\right|^p dxdu\right)^{1/p} \quad (7)$$

where, $0<p\le 1$

As a proof of a result in Ditzian and Totik (1987), we have for $t>0$ that:

$$K_{k,r}^{\varphi}(f,t^k)_p \le M_1\omega_\varphi^{*k}(f,t)_{\varphi^\tau,p} \quad (8)$$

Theorem 1.1.4:

$$cK_{k,r}^{\varphi}\left(f^{(r)},t^k\right)_p \le \omega_{k,r}^{*\varphi}\left(f^{(r)},t\right)_p \le \omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p \le cK_{k,r}^{\varphi}\left(f^{(r)},t^k\right)_p \quad (9)$$

C is constant depends in k, r and p.

Remark 1.1.5: Note that with an additional restriction that $t\le t_0$ in the case $r = 0$, Theorem 1.1.4 becomes [3, Theorem 2.1.1] (with $\varphi(x)=\sqrt{1-x^2}$) and that t_0 can be taken to be $(2k)^{-1}$ as was shown in [1, Theorem 6.6.2].

RESULTS AND DISCUSSION

Auxiliary results: In this section we shall introduce.

Proposition: (properties of $W_\delta(x)$ and D_δ) (DeVore and Lorentz, 1993):

- $W_\delta(x)\le\varphi(u)$, for $x\in D_\delta$ and $u\in\left[-|x|-\delta\varphi(x)/2, |x|+\delta\varphi(x)/2\right]$
- $W_\delta(x)\le\varphi(x)$, for $x\in D_\delta$
- $\varphi(x)\le 2W_\delta(x)$, for $x\in D_{2\delta}$
- $\delta|\varphi'(x)|\le 1$, for $x\in D_\delta$
- If $y(u)=x+\delta_1\varphi(x)/2$ and $|\delta_1|\le\delta$, then $1/2\le y'(x)\le 3/2$ for all $x\in D_\delta$
- If $\delta_1>\delta_2$, then $D_{1\delta}\subset D_{2\delta}$

Lemma 1.2.2: For any $0<p\le 1$, $r\in\mathbb{N}_0$, $\lambda\ge 0$ such that $\lambda>r-1$. If $g\in\Im_p^{r+1}$, then:

$$\left\|\varphi^\lambda g^{(r)}{}_p<\infty\right\|$$

Proof: Suppose that when $g\in\Im_p^{r+1}$. Since, $\varphi(u)\ge\varphi(x)$ for $|u|\le|x|$, we have:

$$\left\|\varphi^\lambda g^{(r)}\right\|_p^p = \int_{-1}^1\varphi^{\lambda p}(x)\left|\int_{-1}^1 g^{(r+1)}(u)du\right|^p dx \le \int_{-1}^1\varphi^{\lambda p-r-1}(x)\left|\left(\int_0^x\varphi^{r+1}\left|g^{(r+1)}(u)\right|^p du\right)^{1/p}\right|^p dx \le \left\|\varphi^{r+1}g^{(r+1)}\right\|_p^p\int_{-1}^1\varphi^{\lambda p-r-1}(x)dx \le c_p\left\|\varphi^{r+1}g^{(r+1)}\right\|_p^p \le c_p\left\|\varphi^{r+1}g^{(r+1)}\right\|_p^p<\infty$$

As a direct consequence of above lemma we get the following corollary.

Corollary 1.2.3: For any $0<p\le 1$, $r\in\mathbb{N}_0$ if $g\in\Im_p^r$.

Using a method as in (Kopotun *et al*., 2015)), we can prove the following:

Lemma 1.2.4: For any $k\in\mathbb{N}$, $0<p\le 1$, $r\in\mathbb{N}_0$ and $f\in\Im_p^r$. Then

$$\omega_\varphi^{*k}\left(f^{(r)},t\right)_{\varphi^\tau,p} \le c_{(p)}(k,r)\omega_{k,r}^{*k}\left(f^{(r)},c(k)t\right)_p,\quad 0<t\le c(k)$$

Proof: We estimate each of the three terms in Definition (1.1.7) separately. First, recall that $t^* = 2k^2t^2$ and note that $[-1+t^*, 1-t^*]\subset D_{2kt}$ and so using (iii) of Proposition 1.2.1 to get

$$\frac{1}{t}\int_0^t\int_{-1+t^*}^{1-t^*}\left|\varphi^r(x)\Delta_{\tau\varphi(x)}^k\left(f^{(r)},x\right)\right|^p dxd\tau \le \frac{2^{rp}}{t}\int_0^t\int_{-1+t^*}^{1-t^*}\left|W_{kt}^r(x)\Delta_{\tau\varphi(x)}^k\left(f^{(r)},x\right)\right|^p dxd\tau \le \frac{2^{rp}}{t}\int_0^t\int_{D2kt}\left|W_{kt}^r(x)\Delta_{\tau\varphi(x)}^k\left(f^{(r)},x\right)\right|^p dxd\tau \le 2^{rp}\omega_{k,r}^{*\varphi}(f^{(r)},t)_p^p$$

Let us now take the second term and the third term is the same If $t\le(2k\sqrt{A+K/2})^{-1}$, then using the fact:

$$\varphi(y-kh\varphi(y)/2)\le\sqrt{2}W_{kh}(y) \text{ if } 0\le h\le 2(y+1)/(k\varphi(y)) \text{ and } y\le -1/2$$

$$\frac{1}{t^*}\int_0^{t^*}\int_{-1}^{-1+At^*}\left|\varphi^r(x)\vec{\Delta}_u^k\left(f^{(r)},x\right)\right|^p dxdu =$$

$$\frac{1}{t^*}\int_0^{t^*}\int_{-1}^{-1+At^*}\left|\varphi^r(x)\Delta_u^k\left(f^{(r)},x+ku/2\right)\right|^p dxdu \le$$

$$\frac{1}{t^*}\int_0^{t^*}\int_{-1+ku/2}^{-1+(A+k/2)t^*}\left|\varphi^r(y-ku/2)\Delta_u^k\left(f^{(r)},y\right)\right|^p dydu \le$$

$$\frac{1}{t^*}\int_{-1}^{-1+(A+k/1t^*}\int_0^{2(y+1)/k}\left|\varphi^r(y-ku/2)\Delta_u^k\left(f^{(r)},y\right)\right|^p dudy =$$

$$\frac{1}{t^*}\int_{-1}^{-1+(A+k/1t^*}\int_0^{2(y+1)/(k\varphi(y))}\varphi(y)\left|\varphi^r(y-kh\varphi(y)/2)\Delta_{h\varphi(y)}^k\left(f^{(r)},y\right)\right|^p dhdy \le$$

$$c\frac{1}{t^*}\int_{-1}^{-1+(A+k/1t^*}\int_0^{2(y+1)/(k\varphi(y))}\varphi(y)\left|W_{kh}^r(y)\Delta_{h\varphi(y)}^k\left(f^{(r)},y\right)\right|^p dhdy \le$$

$$c\frac{1}{\sqrt{t^*}}\int_{-1}^{-1+(A+k/1t^*}\int_0^{2(y+1)/(k\varphi(y))}\left|W_{kh}^r(y)\Delta_{h\varphi(y)}^k\left(f^{(r)},y\right)\right|^p dhdy \le$$

$$c\frac{1}{\sqrt{t^*}}\int_0^{c\sqrt{t^*}}\int_{Dkh\cap[-1,-1+(A+k/2)t^*]}\left|W_{kh}^r(y)\Delta_{h\varphi(y)}^k\left(f^{(r)},y\right)\right|^p dydh \le$$

$$c\omega_{k,r}^{*\varphi}(f^{(r)}c(k),t)_p^p$$

Proof of Theorem 1.1.4: We have:

$$\omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p \le c(k,r,p)\,K_{k,r}^{\varphi}\left(f^{(r)},t^k\right)_p,\ \text{for all } t>0$$

Using (1.1.3) and sinc K-functional is monotme with respect to t, we may assume that $t\le 2/k$. Take any $g\in\Im_p^{r+1}$ Corollary 1.2.3 implies that $g\in\Im_p^{r+1}$, so:

$$\omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p \le \omega_{k,r}^{\varphi}\left(f^{(r)}-g^{(r)},t\right)_p+\omega_{k,r}^{\varphi}\left(g^{(r)},t\right)_p$$

Let $0<h\le t$:

Let $0\le i\le k$, $y_i(x) = x+(i-k/2)h\varphi(x)$. Using (v) proposition 1.2.1 to get $Y_i'(x)\ge 1/2$ for $x\in D_{kh}$ and so, we have:

$$\left\|\varphi^r(y_i)(f^{(r)}(y_i)-g^{(r)}(y_i))\right\|_{L_p(D_{kh})} =$$

$$\left(\int_{Dkh}\varphi^{rp}(y_i(x))\left|(f^{(r)}(y_i(x))-g^{(r)}(y_i(x)))\right|^p dx\right)^{1/p} \le$$

$$2^{1/P}\left(\int_{-1}^{1}\varphi^{rp}(y)\left|(f^{(r)}(y)-g^{(r)}(y)\right|^p dy\right)^{1/p} =$$

$$2^{1/p}\left\|\varphi^r(f^{(r)}-g^{(r)}\right\|_p$$

For the term $\omega_{k,r}^{\varphi}(g^{(r)},t)_p$ using the relation:

$$\Delta_h^k(f,x) = \int_{-h/2}^{h/2},\ldots,\int_{-h/2}^{h/2}f^{(k)}(x+u_1+,\ldots,+u_k)du_1,\ldots,du_k$$

We obtain:

$$\omega_{k,r}^{\varphi}\left(g^{(r)},t\right)_p = \sup_{0<h\le t}\left\|W_{kh}^r\Delta_{h\varphi}^k\left(g^{(r)},.\right)\right\|_{L_p(D_{kh})} =$$

$$\sup_{0<h\le t}\left\|W_{kh}^r\int_{-h\varphi/2}^{h\varphi/2}\int_{-h\varphi/2}^{h\varphi/2}g^{(k+r)}(.+u_1+,\ldots,+u_k)du_1,\ldots,du_k\right\|_{L_p(D_{kh})}$$

For any u satisfying $-1<x+u-hf(x)/2<x+u+h\varphi(x)/2<1$ and using (ii) of Proposition 1.2.1, we get:

$$\omega_{k,r}^{\varphi}\left(g^{(r)},t\right)_p \le$$

$$c\frac{h\varphi}{2}\int_{Dkh}\left|\int_{-h\varphi(x)/2}^{h\varphi(x)/2},\ldots,\int_{-h\varphi(x)/2}^{h\varphi(x)/2}\varphi^r(x)g^{(k+r)}(x+u_2+,\ldots,+u_k)du_1,\ldots,du_k\right|dx \le$$

$$c\left(\frac{h\varphi}{2}\right)^2\int_{Dkh}\left|\int_{-h\varphi(x)/2}^{h\varphi(x)/2},\ldots,\int_{-h\varphi(x)/2}^{h\varphi(x)/2}\varphi^r(x)g^{(k+r)}(x+u_3+,\ldots,+u_k)du_1,\ldots,du_k\right|dx \le$$

$$\int_{Dkh}c\left(\frac{h\varphi}{2}\right)^3\left|\int_{-h\varphi(x)/2}^{h\varphi(x)/2},\ldots,\int_{-h\varphi(x)/2}^{h\varphi(x)/2}\varphi^r(x)g^{(k+r)}(x+u_4+,\ldots,+u_k)du_1,\ldots,du_k\right|dx \le$$

$$c\left(\frac{h\varphi}{2}\right)^k\int_{Dkh}\left|\int_{-h\varphi(x)/2}^{h\varphi(x)/2}\varphi^r(x)g^{(k+r)}(x+u_k)du_k\right|dx \le c\left(\frac{h\varphi}{2}\right)^k\left\|\varphi^r g^{(k+r)}\right\|_p$$

If $0<p\le 1$:

$$\int_{-hf/2}^{hf/2} = \int_{-hf(x)/2}^{hf(x)/2}+\int_{-hf(x)/2}^{hf(x)/2}+\int_{-hf(x)/2}^{hf(x)/2}+,\ldots,$$
$$I(p) = I_1(p)\cup I_2(p)\cup I_3(p)U,\ldots,$$

Hence:

$$\omega_{k,r}^{\varphi}\left(f^{(r)},t\right)_p \le c(k,r,p)K_{k,r}^{\varphi}\left(f^{(r)},t^k\right)_p$$

Now show:

$$cK_{k,r}^{\kappa}\left(f^{(r)},t^k\right)_p \le \omega_{k,r}^{*\varphi}\left(f^{(r)},t\right)_p,$$

Estimates (1.1.6) and (1.1.8) with Lemma 1.2.4 imply that, for $f\in\Im_p^r$, $0<p\le 1$:

$$K_{k,r}^{\varphi}\left(f^{(r)},t^k\right)_p \le c\omega_{k,r}^{*\varphi}\left(f^{(r)},c_1t\right)_p, 0<t\le c_2,$$

where, $c_1 = c_1(k)$ and $c_2 = c_2(k)$ are positive constants

Suppose, $0<t\le 2/k$ and let $\mu = \max\{1, c_1, 2/(kc_2)\}$. Then, since, $t/\mu\le c_2$ using (1.1.4), we get:

$$K_{k,r}^{\varphi}\left(f^{(r)},t^k\right)_p \le \mu^k K_{k,r}^{\varphi}\left(f^{(r)},\left(t/\mu\right)^k\right)_p \le c\omega_{k,r}^{*\varphi}\left(f^{(r)},c_1t/\mu\right)_p \le$$

$$c(\mu/c_1)^{1/2}\omega_{k,r}^{*\varphi}\left(f^{(r)},t\right)_p$$

Thus:

$$cK_{k,r}^{\varphi}\left(f^{(r)},t^k\right)_p \le \omega_{k,r}^{*\varphi}\left(f^{(r)},t\right)_p$$

CONCLUSION

We can define new modulus of smoothness for functions in $L_p[-1,1]$ quasinormed spaces for $0<p\leq 1$. This modulus of smoothness equivalent to Ditzian-Totik modulus of smoothness.

REFERENCES

Bhaya, E.S. and D.B. Ahmed, 2018. On monotone rational approximation. J. Univ. Babylon, 26: 41-53.

Bhaya, E.S. and O.A. Al-Sammak, 2018. The essential order of (L_p, p<1) approximation using regular neural networks. J. Univ. Babylon, 26: 77-83.

Bhaya, E.S. and Q.A. Madlool, 2018. On the mixed rth modulus of smoothness. J. Babylon Univ. Pure Appl. Sci., 26: 2424-2430.

Bhaya, E.S. and Z.M. Fadel, 2018. Nearly exponential approximation for neural networks. J. Univ. Babylon, 26: 103-113.

Bhaya, E.S., 2010. Interpolating operators for multiapproximation. J. Math. Stat., 6: 240-245.

Bhaya, E.S.and A. Anoon, 2018. Spherical approximation on unit sphere. J. Babylon Univ. Pure Appl. Sci., 26: 61-68.

DeVore, R.A. and G.G. Lorentz, 1993. Fundamental Principles of Mathematical Sciences. Vol. 303, Springer, Berlin, Germany,.

Ditzian, Z. and V. Totik, 1987. Moduli of Smoothness. Springer, Berlin, Germany, ISBN:9780387965369, Pages: 225.

Ditzian, Z., 2007. Polynomial approximation and ùr ? (f, t) twenty years later. Surv. Approx. Theory, 3: 106-151.

Kopotun, K.A., D. Leviatan and I.A. Shevchuk, 2015. New moduli of smoothness: Weighted DT moduli revisited and applied. Constructive Approximation, 42: 129-159