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Laplace Transform Method for Solving Nonlinear Biochemical Reaction Model and Nonlinear Emden-Fowler System

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Abstract: In this study, we look forward to using the Laplace transform method in order to obtain the approximate solutions of both nonlinear biochemical reaction model and systems of nonlinear equations of Emden-Fowler type. This method and interesting of the easiest and simplest ways to obtain very accurate results to solve all differential equations whether linear or non-linear.

Key words: Emden-Fowler, biochemical reaction model, Laplace transform, non-linear systems equations, Iraq

INTRODUCTION

In the 19th century in 1913, Michael and Manten gave a simple idea to describe enzyme processes and the basic enzymatic model was given by the planner (Schnell and Mendoza, 1997; Khader, 2013):

$$Ry(t) = \sum_{m=0}^{\infty} Ay_m(t) \tag{1}$$

Where:

M = The enzyme
N = The substrate
L = The enzyme-substrate intermediate complex
R = The product from the law of mass action

Which states that reaction rates are proportional to the concentrations of the reactants, the time evolution of the scheme Eq. 1 can be determined from the solution of the system of coupled nonlinear ordinary differential equations (Khader, 2013; Sen 1988):

$$\frac{dN}{dt} = -k_1MN+k_{-1}L \tag{2}$$

$$\frac{dM}{dt} = -k_1MN+(k_{-1}+k_2)L \tag{3}$$

$$\frac{dL}{dt} = k_1MN-(k_{-1}+k_2)L \tag{4}$$

$$\frac{dR}{dt} = k_2L \tag{5}$$

With initial conditions:

$$N(0) = N_0, M(0) = M_0, L(0) = 0, R(0) = 0 \tag{6}$$

where the parameters k_1, k_{-1} and k_2 are positive rate constants for each reaction. Systems Eq. 2-5 can be shortened to only two equations for N and L and in dimensionless form of concentrations of substrate, u and intermediate complex between enzyme and substrate, ν, are given by Khader (2013) and Sen (1988):

$$\frac{du}{dt} = -u+(\beta-\alpha)v+uv \tag{7}$$

$$\frac{dv}{dt} = \frac{1}{\gamma}(u-\beta v-uv) \tag{8}$$

Subject to the initial conditions:

$$u(0) = 1, v(0) = 0 \tag{9}$$

where, α, β and γ are dimensionless parameters. For more details on mathematical formulation of Eq.7, 8 and an intrinsic knowledge of its analysis (Khader, 2013; Sen 1988). There are many methods that solve nonlinear differential equations such as Abassy *et al*. (2007), Biazar and Ghazvini (2007), Goha *et al*. (2010), He (2000), Sweilam and Khader (2010).

Many problems in the fields of mathematical physics and astrophysics are exppressed by the equation (Wazwaz, 2011, 2005a, b):

$$y''+\frac{\gamma}{x}y'+f(x)g(x) = h(x), x>0 \tag{10}$$

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Subject to:

$$y(0)=1,\ y'(0)=0 \tag{11}$$

where, $\gamma>0$ is a constant. For $f(x) = 1$, $g(y) = y^m$ and $h(x) = 0$, Eq. 10 is the standard Lane-Emden equation that has been used to model several phenomena in mathematical physics.

In this research, we study systems of nonlinear equations of Emden-Fowler type subject with the initial conditions given by the form Wazwaz (2011, 2005a, b):

$$u''+\frac{\alpha}{x}u'+f\left(u(x),v(x)\right)=h_1(x),\ \ x>0,\ \alpha>0 \tag{12}$$

$$v''+\frac{\beta}{x}v'+g\left(u(x),v(x)\right)=h_2(x),\ \ x>0,\ \ \beta>0 \tag{13}$$

Subject to:

$$u(0)=v(0)=1,\ u'(0)=v'(0)=0 \tag{14}$$

MATERIALS AND METHODS

Basic definitions of fractional calculus: In this study, we present the basic definitions and properties of the fractional calculus theory which are used further in this study.

Definition 1: A real function f(t), t>0 is said to be in the space $C\alpha$, $\alpha\epsilon R$, if there exists a real number $p>\alpha$ such that $f(t) = t^p f_1(t)$ where $f_1(t)\epsilon C[0, \infty)$ and it is said to be in the space C_α^m if $f^m \epsilon C_\alpha$, $m\epsilon N$.

Definition 2: The Laplace transform is defined over the set of functions (Jafari *et al*., 2011; Kumar *et al*., 2014):

$$A=\left\{f(t)\left|\exists M,\tau_1,\tau_2>0,\left|f(t)\right|<Me^{\frac{|t|}{\tau_j}},\text{if } t\in(-1)^j\times\left[0,\infty\right)\right.\right\} \tag{15}$$

By the following Eq. 16:

$$F(s)=L\left[f(t)\right]=\int_0^\infty e^{-st}f(t)\,dt \tag{16}$$

where some special properties of the sumudu transform are as follows (Jafari *et al*., 2011):

$$L\left[1\right]=\frac{1}{s};\ L\left[t^n\right]=\frac{n!}{s^{n+1}};\ L\left[e^{at}\right]=\frac{1}{s-a}$$

Definition 3: The laplace transform L[f (t)] of the order derivatives are defined as Jafari *et al*. (2011) and Kumar *et al*. (2014):

$$L\left[F^n(t)\right]=s^nF(s)-\sum_{k=0}^{n-1}s^{n-k-1}f^k(0)\text{ for } n\geq 1 \tag{17}$$

At very special case for n = 1:

$$L\left[F'(t)\right]=sF(s)-F(0) \tag{18}$$

This is very important to calculate approximate solution of the problems.

RESULTS AND DISCUSSION

Laplace decomposition method: In order to elucidate the solution procedure of this method, we consider a general fractional nonlinear differential equation of the form Jafari *et al*. (2011), Kumar *et al*. (2014), Fadaei (2011), Khan and Faraz (2011), Gaxiola (2017), Hussain and Khan (2010), Eltayeb (2017):

$$My(t)+Ny(t)+Ry(t)=q(t) \tag{19}$$

with $m-1<\alpha\leq m$ and subject to the initial condition:

$$y^j(0)=c_j,\ j=0,1,\ldots,m-1 \tag{20}$$

Where:

m = Lower order derivative
q(t) = The source term
N = The linear operator
R = The general nonlinear operator

Applying Laplace transform (denoted throughout this study by L on both sides of Eq. 19, we have:

$$L\left[My(t)\right]+L\left[Ny(t)+Ry(t)\right]=L\left[q(t)\right] \tag{21}$$

Using the property of the Laplace transform and the initial conditions in Eq. 20, we have:

$$s^nL\left[y(t)\right]-\sum_{k=0}^{n-1}s^{n-k-1}f^k(0)+L\left[Ny(t)+Ry(t)\right]=L\left[q(t)\right]$$

$$L\left[y(t)\right]=\frac{1}{s^n}\sum_{k=0}^{n-1}s^{n-k-1}f^k(0)+\frac{1}{s^n}L\left[q(t)\right]-\frac{1}{s^n}L\left[Ny(t)+Ry(t)\right] \tag{22}$$

Operating with the Sumudu inverse on both sides of Eq. 22, we get:

$$y(t) = Z(t)\text{-}L^{-1}\left[\frac{1}{s^n}L\left[Ny(t)+Ry(t)\right]\right] \tag{23}$$

where, Z(t) represents the term arising from the source term and the prescribed initial conditions. Now, applying the classical perturbation technique. And assuming that the solution of Eq. 23 is in the form:

$$y(t) = \sum_{m=0}^{\infty} y_m(t) \tag{24}$$

where, p∈[0, 1] is the homotopy parameter. The nonlinear term of Eq. 23 can be decomposed as:

$$Ry(t) = \sum_{m=0}^{\infty} Ay_m(t) \tag{25}$$

for some Adomian's polynomials A_m which can be calculated with the equation (Ghorbani, 2009):

$$A_m = \frac{1}{m!}\frac{d^m}{dp^m}\left[N\left(\sum_{i=0}^{\infty} p^i y_i(t)\right)\right]_{p=0},\ n = 0, 1, 2, \ldots, \tag{26}$$

Substituting Eq. 24 and 26 in Eq. 23, we get:

$$\sum_{m=0}^{\infty} y_m(t) = Z(t)\text{-}L^{-1}\left[\frac{1}{s^n}L\left[\begin{matrix}N\left(\sum_{m=0}^{\infty} y_m(t)\right)+\\ \sum_{m=0}^{\infty} A_m\end{matrix}\right]\right] \tag{27}$$

On comparing both sides of the Eq. 27, we get:

$$\begin{aligned} y_0(t) &= Z(t),\\ y_1(t) &= -L^{-1}\left[\frac{1}{s^n}L\left[Ny_0(t)+A_0\right]\right],\\ y_2(t) &= -L^{-1}\left[\frac{1}{s^n}L\left[Ny_1(t)+A_1\right]\right],\\ y_3(t) &= -L^{-1}\left[\frac{1}{s^n}L\left[Ny_2(t)+A_2\right]\right]\\ &\vdots \end{aligned} \tag{28}$$

In general the recursive relation is given by:

$$y_0(t) = Z(t),\ y_{m+1}(t) = -L^{-1}\left[\frac{1}{s^n}L\left[Ny_m(t)+A_m\right]\right]$$

Finally, we approximate the analytical solution by truncated series as:

$$y(t) = \lim_{M\to\infty}\sum_{m=0}^{M} y_m(t) \tag{29}$$

Applications: In this study, to illustrate the method and to show the ability of the method two examples are presented.

Example 1: The nonlinear biochemical reaction model of Eq. 7 and 8 with initial condition Eq. 9 at α = 0.375, β = 1 and γ = (Khader, 2013; Sen 1988). First by taken the Laplace transform to Eq. 7 and 8 as:

$$\begin{aligned} S\left[u^{\cdot}\right] &= sL(u)\text{-}u(0) = L\left[-u+(\beta\text{-}\alpha)v+uv\right]\\ S\left[v\right] &= sL(v)\text{-}v(0) = L\left[\frac{1}{\gamma}(u\text{-}\beta v\text{-}uv)\right] \end{aligned} \tag{30}$$

$$\begin{aligned} L(u) &= \frac{1}{s}u(0)+\frac{1}{s}L\left[-u+(\beta\text{-}\alpha)v+uv\right]\\ L(v) &= \frac{1}{s}v(0)+\frac{1}{s}L\left[\frac{1}{\gamma}(u\text{-}\beta v\text{-}uv)\right] \end{aligned} \tag{31}$$

Second by taken the inverse of Laplace transform to the Eq. 31 with the initial condition Eq. 9, we have :

$$\begin{aligned} u(t) &= 1+L^{-1}\left[\frac{1}{s}L\left[-u+(ß\text{-}a)v+uv\right]\right]\\ v(t) &= L^{-1}\left[\frac{1}{s}L\left[\frac{1}{\gamma}(u\text{-}ßv\text{-}uv)\right]\right] \end{aligned} \tag{32}$$

Third by assuming that the solution as infinite series of unknown functions:

$$u(t) = \sum_{m=0}^{\infty} u_m(t),\ v(t) = \sum_{m=0}^{\infty} v_m(t)$$

Then:

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(t) &= 1+L^{-1}\left[\frac{1}{s}L\left[-\sum_{n=0}^{\infty} u_n(t)+(\beta\text{-}\alpha)\sum_{n=0}^{\infty} v_n(t)+\sum_{n=0}^{\infty} A_n\right]\right]\\ \sum_{n=0}^{\infty} v_n(t) &= L^{-1}\left[\frac{1}{s}L\left[\frac{1}{\gamma}\left(\sum_{n=0}^{\infty} u_n(t)\text{-}\beta\sum_{n=0}^{\infty} v_n(t)\text{-}\sum_{n=0}^{\infty} A_n\right)\right]\right] \end{aligned} \tag{33}$$

where, A_n is adomian polynomials that refers to the nonlinear term and the first three components of the Adomian polynomials is given as follows:

$$A_0 = u_0v_0,\ A_1 = u_0v_1+u_1v_0,\ A_2 = u_0v_2+u_1v_1+u_2v_0$$

Then, we have:

$$\begin{aligned} u_0 = 1u_1 &= L^{-1}\left[\frac{1}{s}L\left[-u_0+(ß\text{-}a)v_0+A_0\right]\right]\\ u_{k+1} &= L^{-1}\left[\frac{1}{s}L\left[-u_k+(ß\text{-}a)v_k+A_k\right]\right] \end{aligned}$$

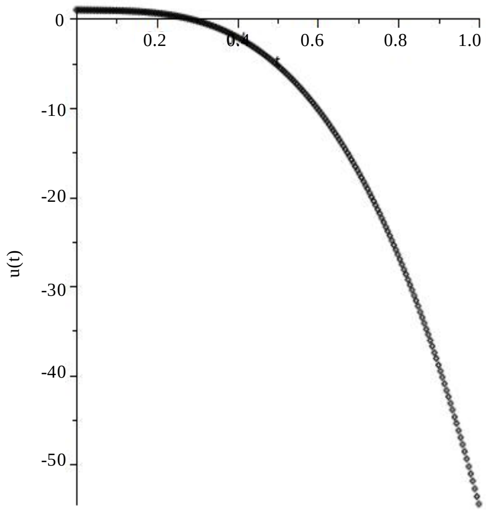

Fig. 1: The behavior of u(t)

And:

$$v_0 = 0 \quad v_1 = L^{-1}\left[\frac{1}{s}L\left[\frac{1}{\gamma}\left(u_0-\beta v_0-A_0\right)\right]\right]$$

$$v_{k+1} = L^{-1}\left[\frac{1}{s}L\left[\frac{1}{\gamma}\left(u_k-\beta v_k-A_k\right)\right]\right]$$

By using that $\alpha = 0.375$, $\beta = 1$, $\gamma = 1$, we have:

- $A_0 = 0$
- $u_1 = -t$
- $v_1 = 10t$
- $A_1 = 10t$
- $u_2 = 8.625t^2$
- $v_2 = -105t^2$
- $A_2 = -115t^2$
- $u_3 = -63.083t^3$
- $v_3 = 73.208t^3$

By continue, we get the solution as series:

$$u(t) = 1-t+8.625t^2-63.083t^3+, \ldots, \quad (34)$$

$$v(t) = 10t-105t^2+73.208t^3+, \ldots, \quad (35)$$

It is evident that the efficiency of this approach can dramatically enhance by computing further terms of u(t), v(t) when Laplace transform method is used to obtain the solutions of nonlinear biochemical reaction model. The results in Fig. 1-3 are in full agreement with the results obtained by Jafari *et al*. (2011) using Laplace transform method (Fig. 4).

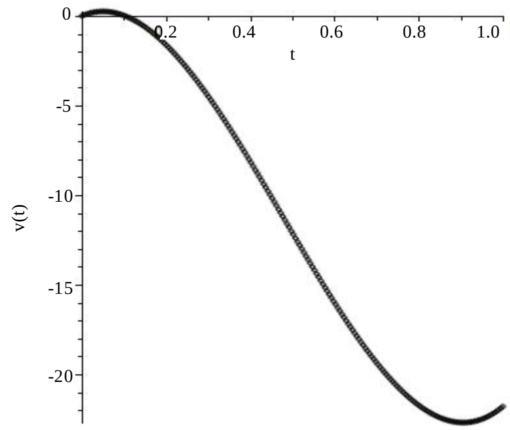

Fig. 2: The behavior of v(t)

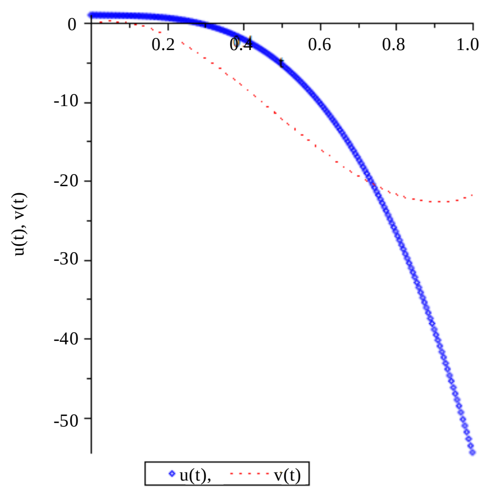

Fig. 3: The behavior of u(t) and v(t)

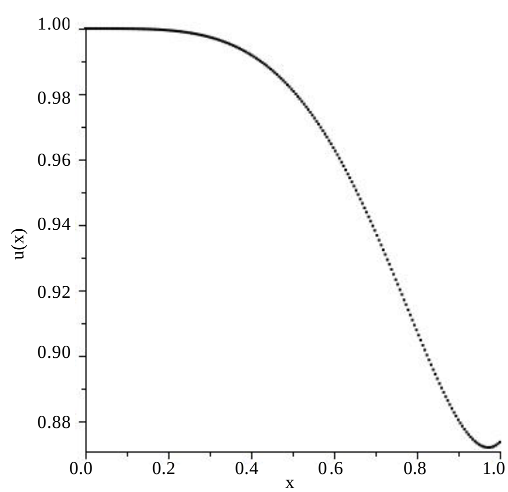

Fig. 4: The behavior of u(t)

Example 2: Consider the systems of nonlinear equations of Emden-Fowler type see (Gad-Allah and Elzaki, 2017; Eltayeb, 2017):

$$u''(x)+\frac{2}{x}u'+v^2(x)-u^2(x)+6v(x)=6+6x^2 \quad (36)$$

$$v''(x)+\frac{2}{x}v'+u^2(x)-v^2(x)-6v(x)=6-6x^2 \quad (37)$$

Subject to:

$$u(0)=1,\ u'(0)=0 \quad (38)$$

$$v(0)=-1,\ v'(0)=0 \quad (39)$$

By taking the Laplace transform on both sides of Eq. 36 and 37, thus, we get:

$$\begin{cases} L[u'']+L\left[\frac{2}{x}u'+v^2-u^2+6v\right]=\frac{6}{s}+\frac{12}{s^3} \\ L[v'']+L\left[\frac{2}{x}v'+u^2-v^2-6v\right]=\frac{6}{s}-\frac{12}{s^3} \end{cases}$$

$$\begin{cases} L[u'']=\frac{6}{s}+\frac{12}{s^3}-L\left[\frac{2}{x}u'+v^2-u^2+6v\right] \\ L[v'']=\frac{6}{s}-\frac{12}{s^3}-L\left[\frac{2}{x}v'+u^2-v^2-6v\right] \end{cases} \quad (40)$$

Using the property of the Laplace transform and the initial condition in Eq. 38-39, we have

$$\begin{cases} s^2L[u]=su(0)+u'(0)+\frac{6}{s}+\frac{12}{s^3}-L\left[\frac{2}{x}u'+v^2-u^2+6v\right], \\ s^2L[v]=sv(0)+v'(0)+\frac{6}{s}-\frac{12}{s^3}-L\left[\frac{2}{x}v'+u^2-v^2-6v\right] \end{cases}$$

$$\begin{cases} L[u]=\frac{1}{s}+\frac{6}{s^3}+\frac{12}{s^5}-\frac{1}{s^2}L\left[\frac{2}{x}u'+v^2-u^2+6v\right], \\ L[v]=-\frac{1}{s}+\frac{6}{s^3}-\frac{12}{s^5}-\frac{1}{s^2}L\left[\frac{2}{x}v'+u^2-v^2-6v\right] \end{cases} \quad (41)$$

Operating with the Sumudu inverse on both sides of Eq. 41, we get:

$$\begin{aligned} u &= 1+3x^2+\frac{1}{2}x^4-L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}u'+v^2-u^2+6v\right]\right], \\ v &= -1+3x^2-\frac{1}{2}x^4-L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}v'+u^2-v^2-6v\right]\right] \end{aligned} \quad (42)$$

By assuming that:

$$u(x)=\sum_{n=0}^{\infty}u_n(x),\ v(x)=\sum_{n=0}^{\infty}v_n(x) \quad (43)$$

By substituting Eq. 43 in Eq. 42, we have:

$$\sum_{n=0}^{\infty}u_n(x)=1+3x^2+\frac{1}{2}x^4-L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}\frac{d}{dx}\sum_{n=0}^{\infty}u_n(x)+\sum_{n=0}^{\infty}A_n(x)-\sum_{n=0}^{\infty}B_n(x)+6\sum_{n=0}^{\infty}v_n(x)\right]\right]$$

$$\sum_{n=0}^{\infty}v_n(x)=-1+3x^2-\frac{1}{2}x^4-L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}\frac{d}{dx}\sum_{n=0}^{\infty}v_n(x)+\sum_{n=0}^{\infty}B_n(x)-\sum_{n=0}^{\infty}A_n(x)-6\sum_{n=0}^{\infty}v_n(x)\right]\right]$$

where, A_n, B_n are Adomian polynomials that represent nonlinear term. So, Adomian polynomials are given as follows:

$$A_n(x)=v^2(x),\ B_n(x)=u^2(x)$$

The few components of the Adomian polynomials are given as follows:

- $A_0 = v^2_0$, $B_0 = u^2_0$
- $A_1 = 2v_0v_1$, $B_1 = 2u_0u_1$
- $A_2 = 2v_0v_2+v^2_1$, $B_2 = 2u_0u_2+u^2_1$

Then, we have:

$$u_0 = 1+3x^2+\frac{1}{2}x^4$$

$$v_0 = -1+3x^2-\frac{1}{2}x^4$$

$$A_0 = 1-6x^2+10x^4-3x^6+\frac{1}{4}x^8$$

$$B_0 = 1+6x^2+10x^4+3x^6+\frac{1}{4}x^8$$

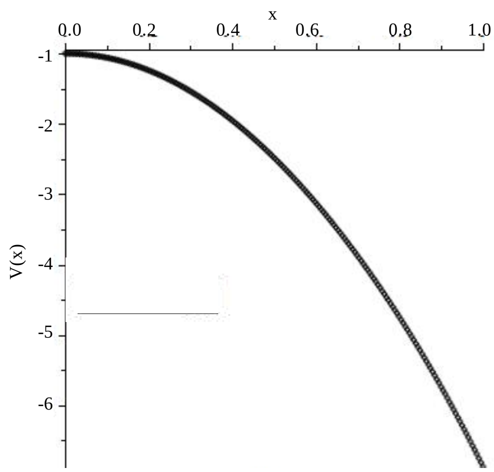

Fig. 5: The behavior of v(x)

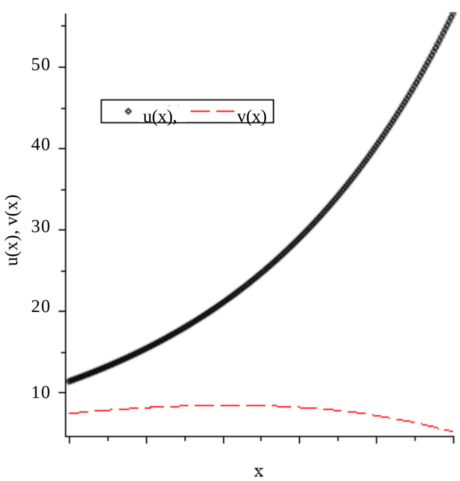

Fig. 6: The behavior of u(x) and v(x)

$$U_{k+1}(x) = -L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}\frac{d}{dx}u_k(x)+A_k(x)-B_k(x)+6v_k(x)\right]\right]$$

$$V_{k+1}(x) = -L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}\frac{d}{dx}v_k(x)+B_k(x)-A_k(x)-6v_k(x)\right]\right]$$

$$U_1(x) = -L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}\frac{d}{dx}u_0(x)+A_0(x)-B_0(x)+6v_0(x)\right]\right]$$

$$V_1(x) = -L^{-1}\left[\frac{1}{s^2}L\left[\frac{2}{x}\frac{d}{dx}v_0(x)+B_0(x)-A_0(x)-6v_0(x)\right]\right]$$

$$u_1 = \frac{3}{28}x^8+\frac{1}{10}x^6-\frac{5}{6}x^4-3x^2$$
$$v_1 = -\frac{3}{28}x^8-\frac{1}{10}x^6+\frac{5}{6}x^4-9x^2$$

Since:

$$u(x) = u_1+u_2+u_3+,\cdots,$$
$$v(x) = v_1+v_2+v_3+,\cdots,$$

Then:

$$u(x) = 1-\frac{1}{3}x^4+\frac{1}{10}x^6+\frac{3}{28}x^8+,\ldots, 44$$
$$v(x) = -1-6x^2+\frac{1}{3}x^4-\frac{1}{10}x^6-\frac{3}{28}x^8+,\ldots, \quad (44)$$

It is evident that the efficiency of this approach can dramatically enhance by computing further terms of u(x), v(x) when Laplace transform method is used to obtain the solutions for systems of nonlinear equations of Emden-Fowler type. The results in Fig. 4-6 are in full agreement with the results obtained by Wazwaz (2011) using Laplace transform method.

CONCLUSION

The main aim of this study is to know that the Laplace transform method is one of the most important and simplest methods used in solving linear and nonlinear differential equations. This method has been successfully applied to the nonlinear biochemical reaction model and for systems of nonlinear equations of Emden-Fowler type.in this method we do not need to do the difficult computation for finding the Adomian polynomials. Generally, speaking the proposed method is promising and applicable to a broad class of linear and nonlinear problems.

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